Chapter 3

# More Algorithmic Simplification Techniques

The algebraic methods developed in Chapter 2 allow us, in theory, to simplify any function. However, there are a number of problems with that approach. There is no formal method, such as first apply Property 10, then P14, etc. The approach is totally heuristic, depending heavily on experience. After manipulating a function, we often cannot be sure whether or not it is a minimum. We may not always find the minimum, even though it appears that there is nothing else to do. Furthermore, it gets rather difficult to do algebraic simplification with more than four or five variables. Finally, it is easy to make copying mistakes as we rewrite the equations. ■

In this chapter we will examine two approaches that are easier to implement. First, we will examine the *Karnaugh map* (sometimes referred to as a K-map). This is a graphical approach to finding suitable product terms for use in sum of product expressions. (The product terms that are "suitable" for use in minimum sum of products expressions are referred to as *prime implicants*. We will define that term shortly.) The map is useful for problems of up to six variables and is particularly straightforward for most problems of three or four variables. Although there is no guarantee of finding a minimum solution, the methods we will develop nearly always produce a minimum. We will adapt the approach (with no difficulty) to finding minimum product of sums expressions, to problems with don't cares, and to multiple output problems.

The second approach we will consider is referred to as *iterated consensus*. It uses the consensus operation to find all prime implicants.

It then uses a tabular method to select a minimum set of prime implicants. This method has been computerized and can be used for larger problems than for which the map is suitable. Nevertheless, it is still limited to relatively small problems, particularly for hand computation. Another approach (that we will not include) that is often used to find all of the prime implicants is the Quine-McCluskey method. It, too, is an algorithmic method for finding the prime implicants and utilizes the same tabular approach to finding a minimum set.

## 3.1 THE KARNAUGH MAP

We introduced the Karnaugh map in Section 2.6. In this chapter, we will develop techniques to find minimum sum of product expressions using the map. We will start with three- and four-variable maps and will include five- and six-variable maps later.

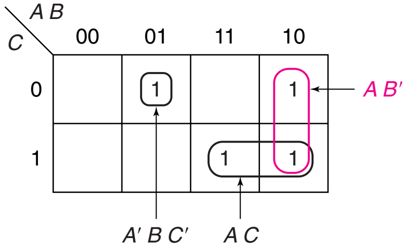
We can plot any function on the map. Either, we know the minterms, and use that form of the map (as we did earlier), or we put the function in sum of products form and plot each of the product terms.

**EXAMPLE 3.1**

Map

$$F = AB' + AC + A'BC'$$

The map for $F$ is shown below, with each of the product terms circled. Each of the two-literal terms corresponds to two squares on the map (since one of the variables is missing). The $AB'$ term is in the 10 column. The $AC$ term is in the $C = 1$ row and in the 11 and 10 columns (with a common 1 in the $A$ position). Finally, the minterm $A'BC'$ corresponds to one square, in the 01 ($A'B$) column and in the $C = 0$ row.

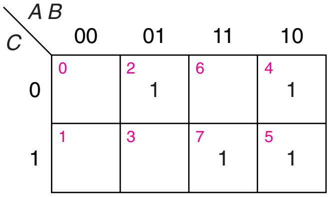



We could have obtained the same map by first expanding $F$ to minterm form algebraically, that is,

$$\begin{aligned} F &= AB'(C' + C) + AC(B' + B) + A'BC' \\ &= AB'C' + AB'C + AB'C + ABC + A'BC' \\ &= m_4 + m_5 + m_5 + m_7 + m_2 \\ &= m_2 + m_4 + m_5 + m_7 \end{aligned}$$

(removing duplicates and reordering)

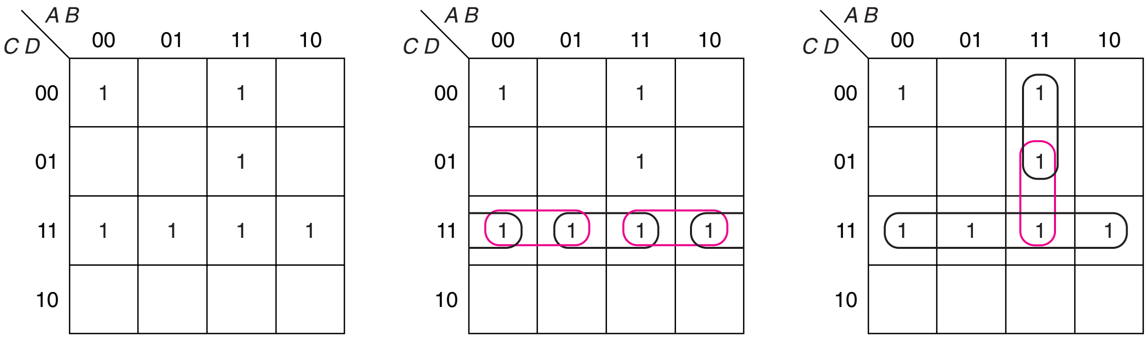
We can then use the numeric map and produce the same result.




We are now ready to define some terminology related to the Karnaugh map. An *implicant* of a function is a product term that can be used in a sum of products expression for that function, that is, the function is 1 whenever the implicant is 1 (and maybe other times, as well). From the point of view of the map, an implicant is a rectangle of 1, 2, 4, 8, . . . (any power of 2) 1's. That rectangle may not include any 0's.

Consider the function, *F*, of Map 3.1. The second map shows the first four groups of 2; the third map shows the other group of 2 and the group of 4.

**Map 3.1** A function to illustrate definitions.



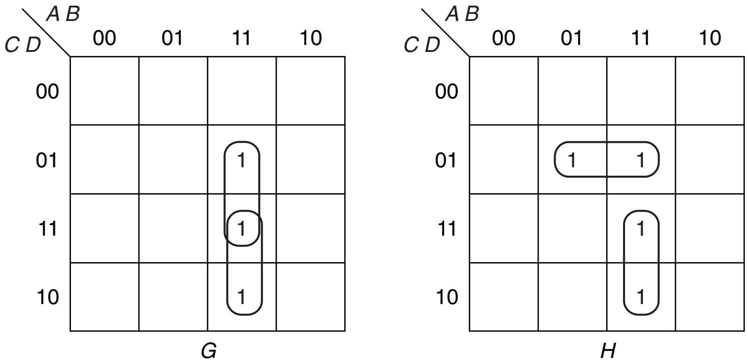

The implicants of *F* are

| *Minterms* | *Groups of 2* | *Groups of 4* |
|---|---|---|
| *A′B′C′D′* | *A′CD* | *CD* |
| *A′B′CD* | *BCD* | |
| *A′BCD* | *ACD* | |
| *ABC′D′* | *B′CD* | |
| *ABC′D* | *ABC′* | |
| *ABCD* | *ABD* | |
| *AB′CD* | | |

Any sum of products of expression for $F$ must be a sum of implicants. Indeed, we must choose enough implicants such that each of the 1's of $F$ are included in at least one of these implicants. Such a sum of products expression is sometimes referred to as a *cover* of $F$ and we sometimes say that an implicant *covers* certain minterms (for example, $ACD$ covers $m_{11}$ and $m_{15}$).

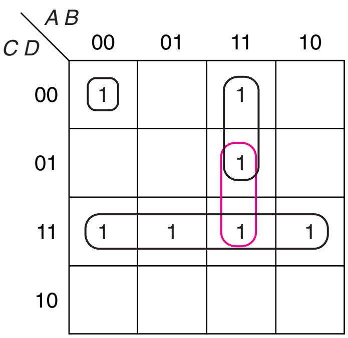
Implicants must be rectangular in shape and the number of 1's in the rectangle must be a power of 2. Thus, neither of the functions whose maps are shown in Example 3.2 are covered by a single implicant, but rather by the sum of two implicants each (in their simplest form).

**EXAMPLE 3.2**




*G* consists of three minterms, *ABC′D*, *ABCD*, and *ABCD′*, in the shape of a rectangle. It can be reduced no further than is shown on the map, namely, to *ABC* + *ABD*, since it is a group of three 1's, not two or four. Similarly, *H* has the same three minterms plus *A′BC′D*; it is a group of four, but not in the shape of a rectangle. The minimum expression is, as shown on the map, *BC′D* + *ABC*. (Note that *ABD* is also an implicant of *G*, but it includes 1's that are already included in the other terms.)

**Map 3.2** Prime implicants.




A *prime implicant* is an implicant that (from the point of view of the map) is not fully contained in any one other implicant. For example, it is a rectangle of two 1's that is not part of a single rectangle of four 1's. On Map 3.2, all of the prime implicants of $F$ are circled. They are $A'B'C'D'$, $ABC'$, $ABD$, and $CD$. Note that the only minterm that is not part of a larger group is $m_0$ and that the other four implicants that are groups of two 1's are all part of the group of four.

From an algebraic point of view, a prime implicant is an implicant such that if any literal is removed from that term, it is no longer an implicant. From that viewpoint, $A'B'C'D'$ is a prime implicant because

$B'C'D'$, $A'C'D'$, $A'B'D'$, and $A'B'C'$ are not implicants (that is, if we remove any literal from that term, we get a term that is 1 for some input combinations for which the function is to be 0). However, $ACD$ is not a prime implicant since when we remove $A$, leaving $CD$, we still have an implicant. (Surely, the graphical approach of determining which implicants are prime implicants is easier than the algebraic method of attempting to delete literals.)

The purpose of the map is to help us find minimum sum of products expressions (where we defined minimum as being minimum number of product terms (implicants) and among those with the same number of implicants, the ones with the fewest number of literals. However, the only product terms that we need consider are prime implicants. Why? Say we found an implicant that was not a prime implicant. Then, it must be contained in some larger implicant, a prime implicant, one that covers more 1's. But that larger implicant (say four 1's rather than two) has fewer literals. That alone makes a solution using the term that is not a prime implicant not a minimum. (For example, $CD$ has two literals, whereas, $ACD$ has three.) Furthermore, that larger implicant covers more 1's, which often will mean that we need fewer terms.

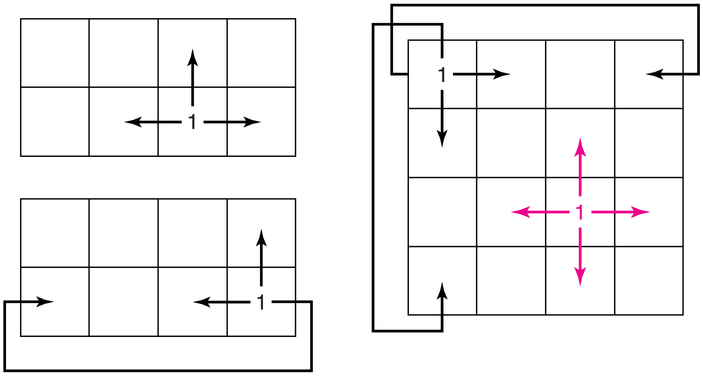
An *essential prime implicant* is a prime implicant that includes at least one 1 that is not included in any other prime implicant. (If we were to circle all of the prime implicants of a function, the essential prime implicants are those that circle at least one 1 that no other prime implicant circles.) In the example of Map 3.2, $A'B'C'D'$, $ABC'$, and $CD$ are essential prime implicants; $ABD$ is not. The term "essential" is derived from the idea that we must use that prime implicant in any minimum sum of products expression. A word of caution is in order. There will often be a prime implicant that is used in a minimum solution (even in all minimum solutions when more than one equally good solution exists) that is not "essential." That happens when each of the 1's covered by this prime implicant could be covered in other ways. We will see examples of that in Section 3.1.1.

### 3.1.1 Minimum Sum of Product Expressions Using the Karnaugh Map

In this section, we will describe two methods for finding minimum sum of products expressions using the Karnaugh map. Although these methods involve some heuristics, we can all but guarantee that they will lead to a minimum sum of products expression (or more than one when multiple solutions exist) for three- and four-variable problems. (They also work for five- and six-variable maps, but our visualization in three dimensions is more limited. We will discuss this in detail in Section 3.1.5.)

In the process of finding prime implicants, we will be considering each of the 1's on the map starting with the most *isolated* 1's. By isolated, we mean that there are few (or no) adjacent squares with a 1 in it. In an **n**-variable map, each square has **n** adjacent squares. Examples for three- and four-variable maps are shown in Map 3.3.

**Map 3.3** Adjacencies on three- and four-variable maps.




### *Map Method 1*

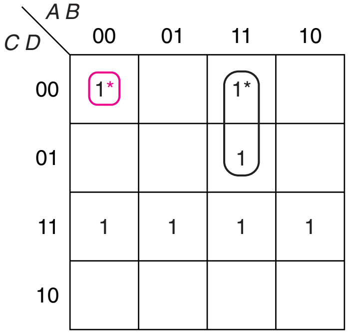
1. Find all essential prime implicants. Circle them on the map and mark the minterm(s) that make them essential with an asterisk (*). Do this by examining each 1 on the map that has not already been circled. It is usually quickest to start with the most isolated 1's, that is, those that have the fewest adjacent squares with 1's in them.
2. Find *enough* other prime implicants to cover the function. Do this using two criteria:
   *a.* Choose a prime implicant that covers as many new 1's (that is, those not already covered by a chosen prime implicant).
   *b.* Avoid leaving isolated uncovered 1's.

It is often obvious what "enough" is. For example, if there are five uncovered 1's and no prime implicants cover more than two of them, then we need at least three more terms. Sometimes, three may not be sufficient, but it usually is.

We will now look at a number of examples to demonstrate this method. First, we will look at the example used to illustrate the definitions.

**EXAMPLE 3.3**

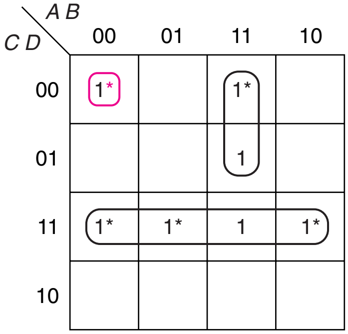
As noted, $m_0$ has no adjacent 1's; therefore, it ($A'B'C'D'$) is a prime implicant. Indeed, it is an essential prime implicant, since no other prime implicant covers this 1. (That is always the case when minterms are prime implicants.) The next place that we look is $m_{12}$, since it has only one adjacent 1. Those 1's are covered by prime implicant $ABC'$. Indeed, no other prime implicant covers $m_{12}$, and thus $ABC'$ is essential. (Whenever we have a 1 with only one adjacent 1, that group of two is an essential prime implicant.) At this point, the map has become




and

$$F = A'B'C'D' + ABC' + \cdots$$

Each of the 1's that have not yet been covered are part of the group of four, $CD$. Each has two adjacent squares with 1's that are part of that group. That will always be the case for a group of four. (Some squares, such as $m_{15}$ may have more than two adjacent 1's.) $CD$ is essential because no other prime implicant covers $m_3$, $m_7$, or $m_{11}$. However, once that group is circled, as shown below, we have covered the function:




resulting in

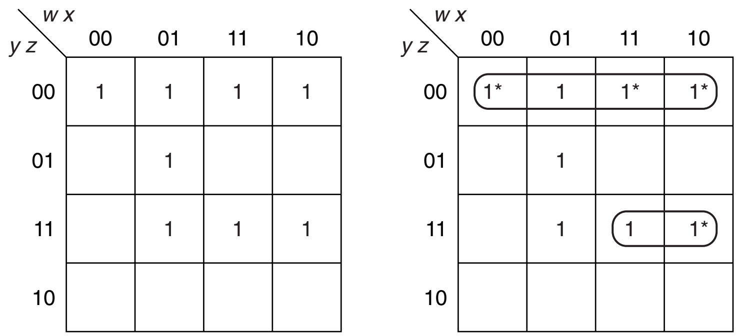
$$F = A'B'C'D + ABC' + CD$$

In this example, once we have found the essential prime implicants, we are done; all of the 1's have been covered by one (or more) of the essential prime implicants. We do not need step 2. There may be other prime implicants that were not used (such as $ABD$ in this example).

Another function that is covered using only essential prime implicants is shown in Example 3.4.

**EXAMPLE 3.4**

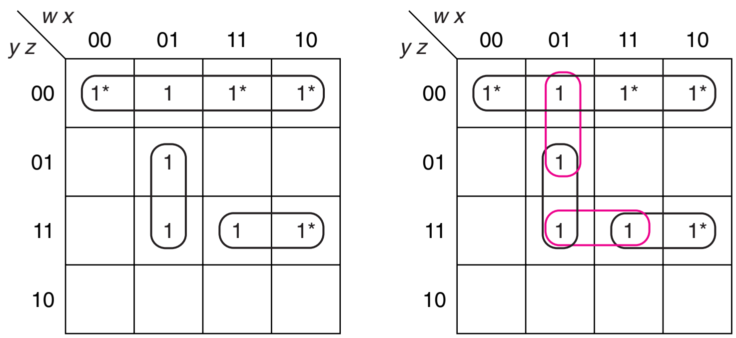
We start looking at the most isolated 1, $m_{11}$. It is covered only by the group of two shown, *wyz*. The other essential prime implicant is *y′z′*, because of $m_0$, $m_8$, or $m_{12}$. None of these are covered by any other prime implicant; each makes that prime implicant essential. The second map shows these two terms circled.




That leaves two 1's uncovered. Each of these can be covered by two different prime implicants; but the only way to cover them both with one term is shown on the first map below.

Thus, the minimum sum of product solution is

$$f = y'z' + wyz + w'xz$$



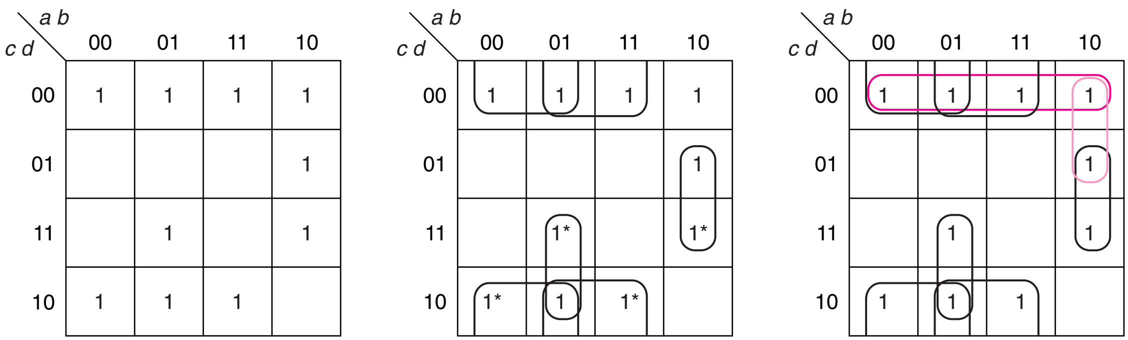

The other two prime implicants are *w′xy′* and *xyz*, circled in red on the second map. They are redundant, however, since they cover no new 1's. Even though *w′xz* must be used in a minimum solution, it does not meet the definition of an essential prime implicant; each of the 1's covered by it can be covered by other prime implicants.

Sometimes, after selecting all of the essential prime implicants, there are two choices for covering the remaining 1's, but only one of these produces a minimum solution, as in Example 3.5.

**EXAMPLE 3.5**

$f(a, b, c, d) = \Sigma m(0, 2, 4, 6, 7, 8, 9, 11, 12, 14)$

The first map shows the function and the second shows all essential prime implicants circled. In each case, one of the 1's (as indicated with an asterisk, *) can be covered by only that prime implicant. (That is obvious from the last map, where the remaining two prime implicants are circled.)



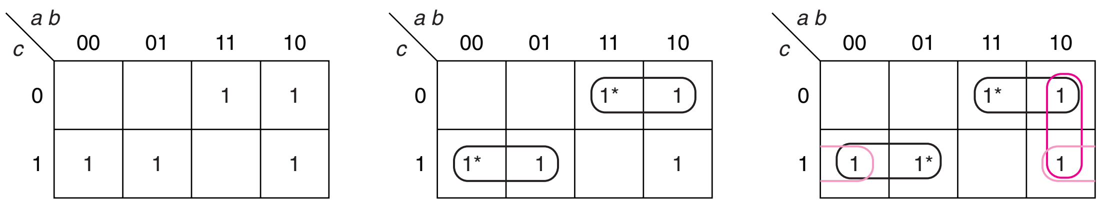

Only one 1 ($m_8$) is not covered by an essential prime implicant. It can be covered in two ways, by a group of four (in red) and a group of two (pink). Clearly, the group of four provides a solution with one less literal, namely,

$$f = a'd' + bd' + a'bc + ab'd + c'd'$$

We will now consider some examples with multiple minimum solutions, starting with a three-variable function.

**EXAMPLE 3.6**

There are two essential prime implicants, as shown on the following maps:




After finding the two essential prime implicants, $ac'$ and $a'c$, as shown on the center map, $m_5$ is still uncovered. As can be seen from the map on the right, there are two ways to cover that term, yielding two, equally good, minimum solutions:

$$f = ac' + a'c + ab'$$
$$= ac' + a'c + b'c$$

As an aside, we can show that these two solutions are mathematically equal. We can take the first expression and add to it the consensus of the last two terms, $a'c$ ¢ $ab' = b'c$, leaving

$$f = ac' + a'c + ab' + b'c$$

Notice that the consensus term is the third term of the second expression. We could do the same thing with the first and third terms of the second expression, $ac'$ ¢ $b'c = ab'$ and add that to the second expression, obtaining

$$f = ac' + a'c + b'c + ab'$$

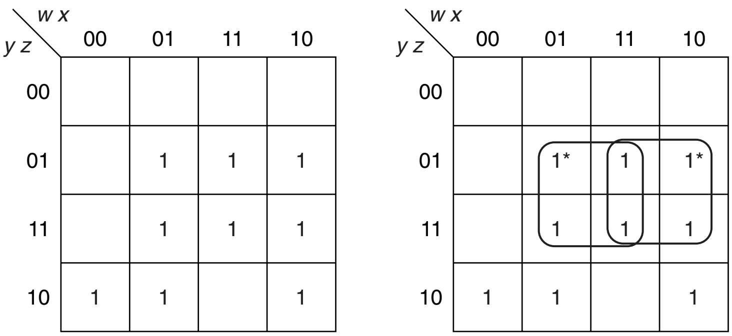
These two expressions are indeed the same set of terms in a different order.

**EXAMPLE 3.7**

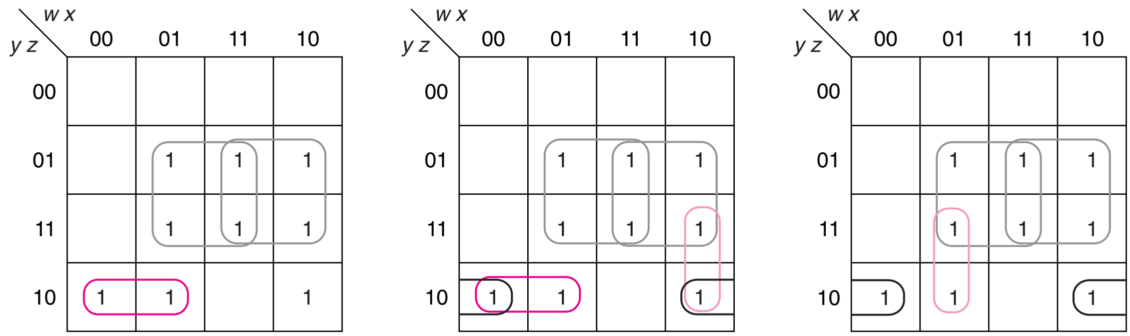
$$g(w, x, y, z) = \Sigma m(2, 5, 6, 7, 9, 10, 11, 13, 15)$$

The function is mapped first, and the two essential prime implicants are shown on the second map, giving

$$g = xz + wz + \cdots$$




Although $m_2$ looks rather isolated, it can indeed be covered by $w'yz'$ (with $m_6$) or by $x'yz'$ (with $m_{10}$). After choosing the essential prime implicants, the remaining three 1's can each be covered by two different prime implicants. Since there are three 1's left to be covered (after choosing the essential prime implicants), and since all the remaining prime implicants are groups of two and thus have three literals, we need at least two more of these prime implicants. Indeed, there are three ways to cover the remaining 1's with two more prime implicants. Using the first criteria, we choose one of the prime implicants that covers two new 1's, $w'yz'$, as shown on the left map below.



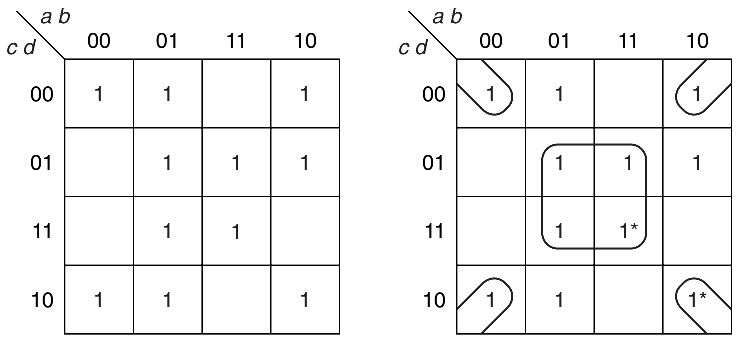

Then, only $m_{10}$ remains and it can be covered either by $wx'y$ or by $x'yz'$, as shown on the center map. Similarly, we could have started with $x'yz'$, in which case we could use $w'xy$ to complete the cover, as on the third map. (We could also have chosen $w'yz'$, but that repeats one of the answers from before.) Thus, the three solutions are

$$g = xz + wz + w'yz' + wx'y$$
$$g = xz + wz + w'yz' + x'yz'$$
$$g = xz + wz + x'yz' + w'xy$$

All three minimum solutions require four terms and 10 literals.

At this point, it is worth stating the obvious. If there are multiple minimum solutions (as was true in this example), all such minimums have the same number of terms and the same number of literals. Any solution that has more terms or more literals is not minimum!

**EXAMPLE 3.8**

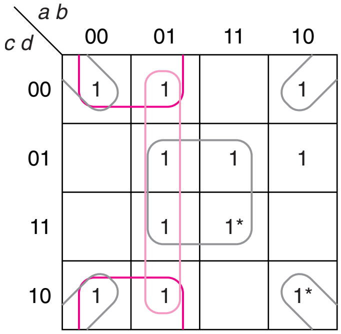


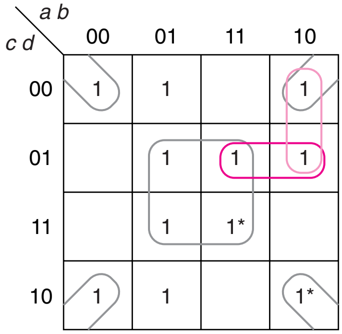

Once again there are two essential prime implicants, as shown on the right map. The most isolated 1's are $m_{10}$ and $m_{15}$. Each has only two adjacent 1's. But all of the 1's in groups of four have at least two adjacent 1's; if there are only two, then that minterm will make the prime implicant essential.

(Each of the other 1's in those groups of four has at least three adjacent 1's.) The essential prime implicants give us

$$f = b'd' + bd + \cdots$$

There are three 1's not covered by the essential prime implicants. There is no single term that will cover all of them. However, the two in the 01 column can be covered by either of two groups of four, as shown on the map on the left (one circled in red, the other in pink). And, there are two groups of two that cover $m_9$ (also one circled in red, the other in pink), shown on the map to the right.




We can choose one term from the first pair and (independently) one from the second pair. Thus, there are four solutions. We can write the solution as shown, where we take one term from within each bracket

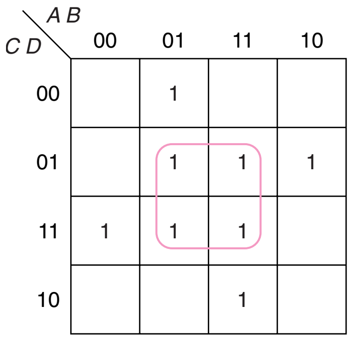
$$f = b'd' + bd + \begin{Bmatrix} a'd' \\ a'b \end{Bmatrix} + \begin{Bmatrix} ac'd \\ ab'c' \end{Bmatrix}$$

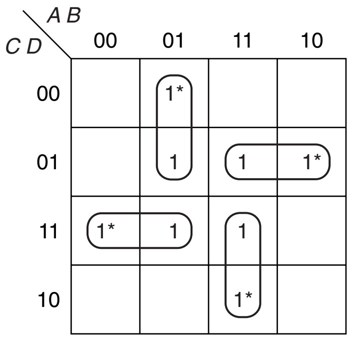
or we can write out all four expressions

$$\begin{aligned} f &= b'd' + bd + a'd' + ac'd \\ &= b'd' + bd + a'd' + ab'c' \\ &= b'd' + bd + a'b + ac'd \\ &= b'd' + bd + a'b + ab'c' \end{aligned}$$

**EXAMPLE 3.9**

This example is one we call "don't be greedy."

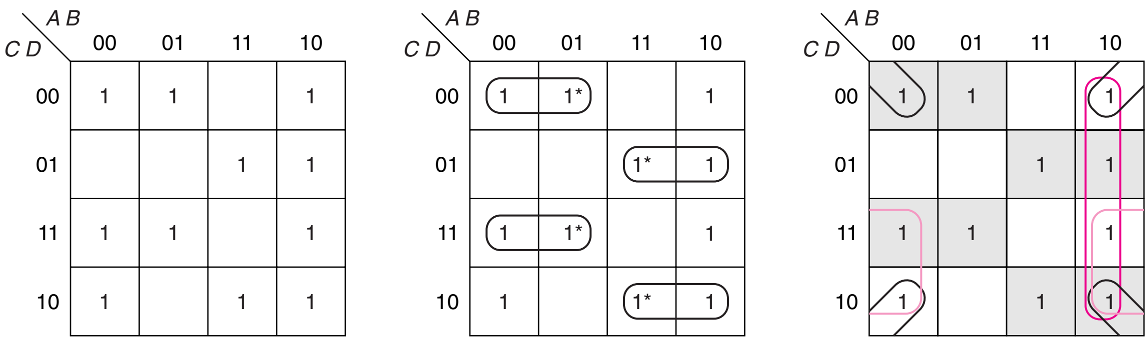


At first glance, one might want to take the only group of four (circled in pink). However, that term is not an essential prime implicant, as is obvious once we circle all of the essential prime implicants and find that the four 1's in the center are covered. Thus, the minimum solution is

$$G = A'BC' + A'CD + ABC + AC'D$$

**EXAMPLE 3.10**




The four essential prime implicants are shown on the second map, leaving three 1's to be covered:

$$F = A'C'D' + AC'D + A'CD + ACD' + \cdots$$

These squares are shaded on the third map. The three other prime implicants, all groups of four, are also shown on the third map. Each of these covers two of the remaining three 1's (no two the same). Thus any two of $B'D'$, $AB'$, and $B'C$ can be used to complete the minimum sum of products expression. The resulting three equally good answers are

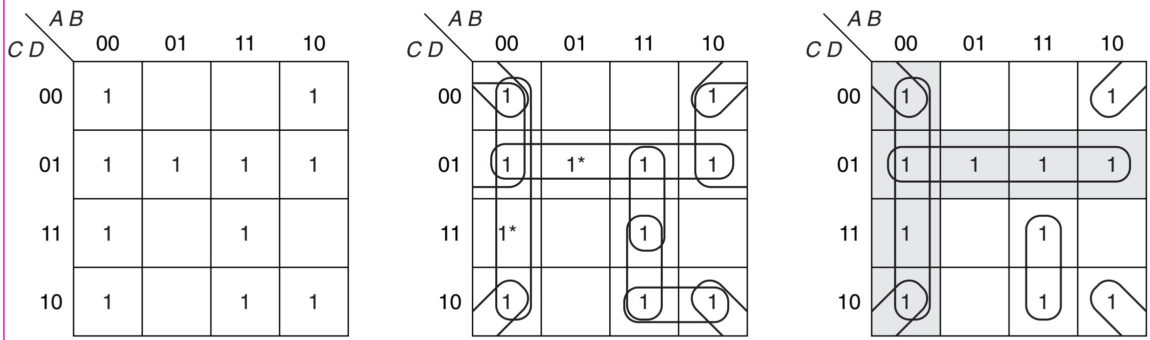
$$F = A'C'D' + AC'D + A'CD + ACD' + B'D' + AB'$$
$$F = A'C'D' + AC'D + A'CD + ACD' + B'D' + B'C$$
$$F = A'C'D' + AC'D + A'CD + ACD' + AB' + B'C$$

Before doing additional (more complex) examples, we will introduce a somewhat different method for finding minimum sum of products expressions.

### *Map Method 2*

1. Circle all of the prime implicants.
2. Select all essential prime implicants; they are easily identified by finding 1's that have only been circled once.
3. Then choose enough of the other prime implicants (as in Method 1). Of course, these prime implicants have already been identified in step 1.

## EXAMPLE 3.11




All of the prime implicants have been circled on the center map. Note that $m_0$ has been circled three times and that several minterms have been circled twice. However, $m_3$ and $m_5$ have only been circled once. Thus, the prime implicants that cover them, $A'B'$ and $C'D$ are essential. On the third map, we have shaded the part of the map covered by essential prime implicants to highlight what remains to be covered. There are four 1's, each of which can be covered in two different ways, and five prime implicants not used yet. No prime implicant covers more than two new 1's; thus, we need at least two more terms. Of the groups of four, only $B'D'$ covers two new 1's; $B'C'$ covers only one. Having chosen the first group, we must use $ABC$ to cover the rest of the function, producing

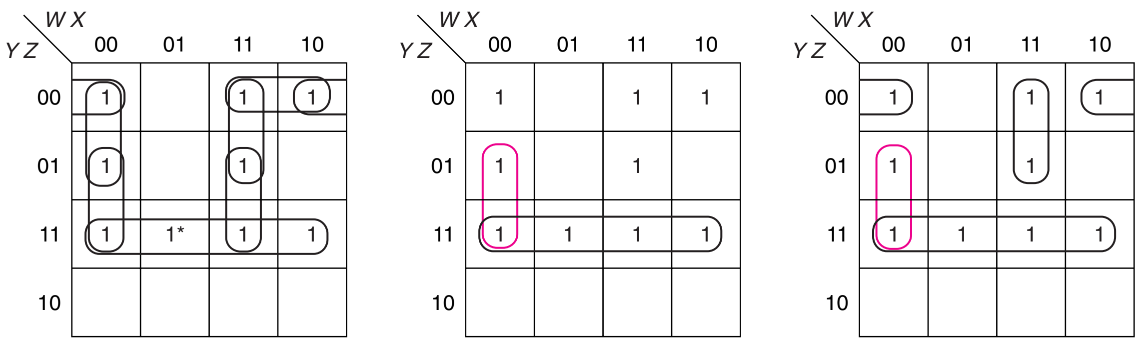
$$F = A'B' + C'D + B'D' + ABC$$

Notice that this is the only set of four prime implicants (regardless of size) that covers the function.

## EXAMPLE 3.12

$$G(A, B, C, D) = \Sigma m(0, 1, 3, 7, 8, 11, 12, 13, 15)$$

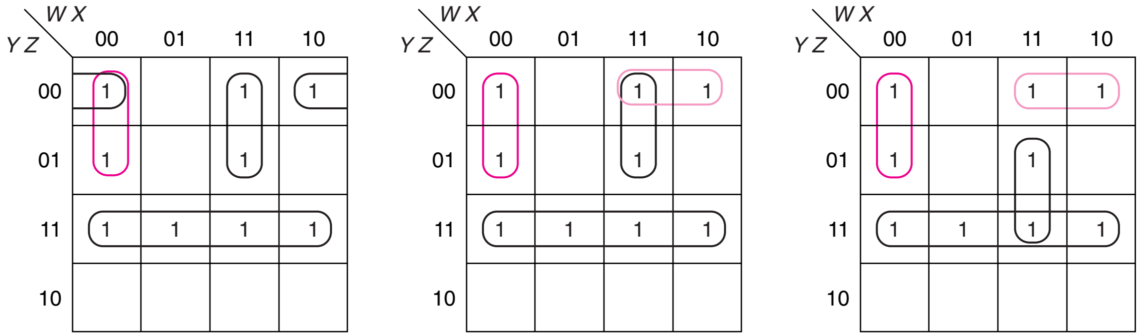
This is a case with more 1's left uncovered after finding the essential prime implicant. The first map shows all the prime implicants circled. The only essential prime implicant is $YZ$; there are five 1's remaining to be covered. Since all of the other prime implicants are groups of two, we need three more prime implicants. These 1's are organized in a chain, with each prime implicant linked to one on either side. If we are looking for just one solution, we should follow the guidelines from Method 1, choosing two terms that



each cover new 1's and then select a term to cover the remaining 1. One such example is shown on the third map, starting with *WXY′* and *X′Y′Z′*. If we wish to find all of the minimum solutions, one approach is to start at one end of the chain (as shown in the second map). (We could have started at the other end, with $m_{13}$, and achieved the same results.) To cover $m_1$, we must either use *W′X′Z*, as shown in red above, or *W′X′Y′* (as shown on the maps below). Once we have chosen *W′X′Z*, we have no more freedom, since the terms shown on the third map above are the only way to cover the remaining 1's in two additional terms. Thus, one solution is

$$F = YZ + W'X'Z + X'Y'Z' + WXY'$$

The next three maps show the solutions using *W′X′Y′* to cover $m_0$.




After choosing *W′X′Y′*, there are now three 1's to be covered. We can use the same last two terms as before (left) or use *WY′Z′* to cover $m_8$ (right two maps). The other three solutions are thus

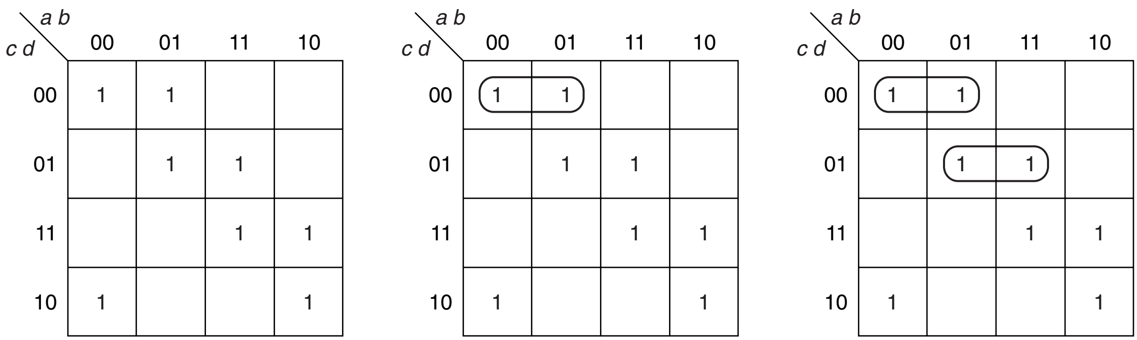
$$F = YZ + W'X'Y' + X'Y'Z' + WXY'$$

$$F = YZ + W'X'Y' + WY'Z' + WXY'$$

$$F = YZ + W'X'Y' + WY'Z' + WXZ$$

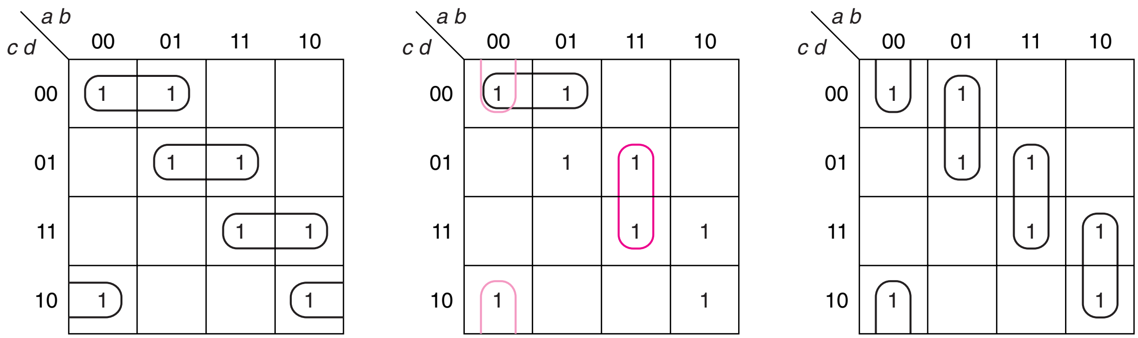
We will now look at some examples with no essential prime implicants. A classic example of such a function is shown in Example 3.13.

**EXAMPLE 3.13**



There are eight 1's; all prime implicants are groups of two. Thus, we need at least four terms in a minimum solution. There is no obvious place to start; thus, in the second map, we arbitrarily chose one of the terms, $a'c'd'$. Following the guidelines of step 2, we should then choose a second term that covers two new 1's, in such a way as not to leave an isolated uncovered 1. One such term is $bc'd$, as shown on the third map. Another possibility would be $b'cd'$ (the group in the last row). As we will see, that group will also be used. Repeating that procedure, we get the cover on the left map below,

$$f = a'c'd' + bc'd + acd + b'cd'$$




Notice, that if, after starting with $a'c'd'$, we chose one of the prime implicants not included in this solution above, such as $abd$, shown on the middle map, we leave an isolated uncovered 1 (which would require a third term) plus three more 1's (which would require two more terms). A solution using those two terms would require five terms (obviously not minimum since we found one with four). Another choice would be a term such as $a'b'd'$, which covers only one new 1, leaving five 1's uncovered. That, too, would require at least five terms.

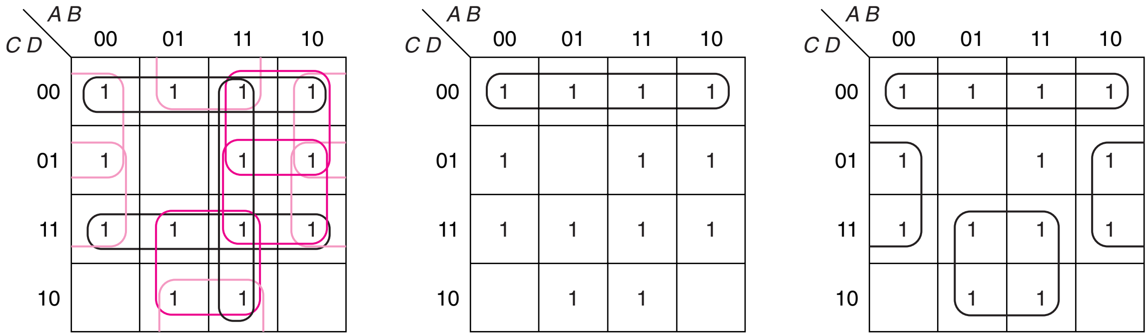
The other solution to this problem starts with $a'b'd'$, the only other prime implicant to cover $m_0$. Using the same process, we obtain the map on the right and the expression

$$f = a'b'd' + a'bc' + abd + ab'c$$

**EXAMPLE 3.14**

$$G(A, B, C, D) = \Sigma m(0, 1, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15)$$

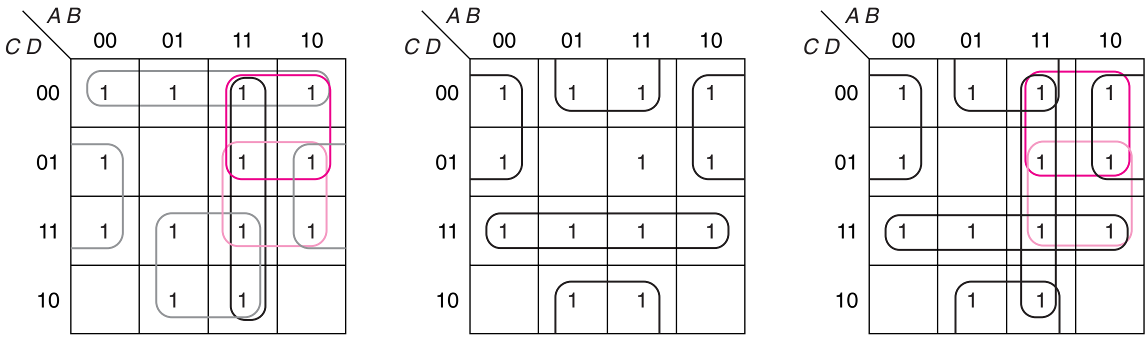
All of the prime implicants are groups of four. Since there are 13 1's, we need at least four terms. The first map shows all of the prime implicants circled; there are nine. There are no 1's circled only once, and thus, there are no essential prime implicants.



As a starting point, we choose one of the minterms covered by only two prime implicants, say $m_0$. On the second map, we used $C'D'$ to cover it. Next, we found two additional prime implicants that cover four new 1's each, as shown on the third map. That leaves just $m_{13}$ to be covered. As can be seen on the fourth map (shown below), there are three different prime implicants that can be used. Now, we have three of the minimum solutions.

$$F = C'D' + B'D + BC + \{AB \quad \text{or} \quad AC' \quad \text{or} \quad AD\}$$

If, instead of using $C'D'$ to cover $m_0$, we use $B'C'$ (the only other prime implicant that covers $m_0$), as shown on the next map, we can find two other groups of four that each cover four new 1's and leave just $m_{13}$ to be covered. Once again, we have three different ways to complete the cover (the same three terms as before).



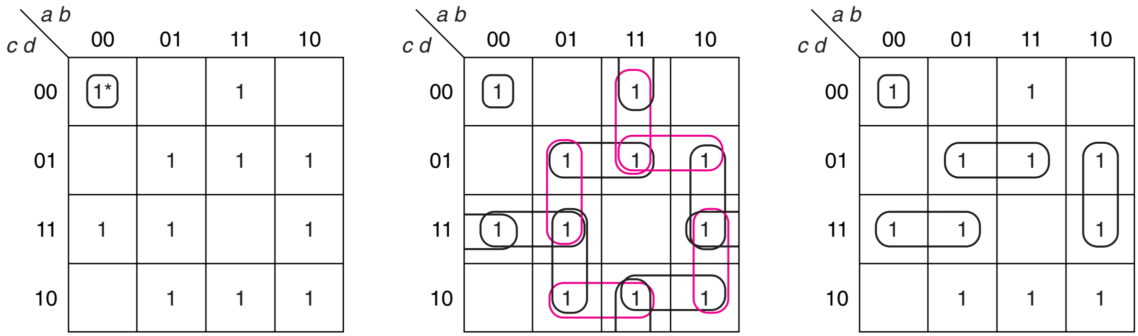
Thus, there are six equally good solutions

$$F = \begin{Bmatrix} C'D' + B'D + BC \\ B'C' + BD' + CD \end{Bmatrix} + \begin{Bmatrix} AB \\ AC' \\ AD \end{Bmatrix}$$

where one group of terms is chosen from the first bracket and an additional term from the second. We are sure that there are no better solutions, since each uses the minimum number of prime implicants, four. Although it may not be obvious without trying other combinations, there are no additional minimum solutions.

A number of other examples are included in Solved Problems 1 and 2. Example 3.15 is one of the most complex four-variable problems, requiring more terms than we might estimate at first.

**EXAMPLE 3.15**




This function has one essential prime implicant (a minterm) and ten other 1's. All of the other prime implicants are groups of two. The second map shows all 13 prime implicants. Note that every 1 (other than $m_0$) can be covered by two or three different terms.

Since there are ten 1's to be covered by groups of two, we know that we need at least five terms, in addition to $a'b'c'd'$. The third map shows the beginnings of an attempt to cover the function. Each term covers two new 1's without leaving any isolated uncovered 1. (The 1 at the top could be combined with $m_{14}$.) The four 1's that are left require three additional terms. After trying several other groupings, we can see that it is not possible to cover this function with less than seven terms. There are 32 different minimum solutions to this problem. A few of the solutions are listed below. The remainder are left as an exercise (Ex 1p).

$$
\begin{aligned}
f &= a'b'c'd' + a'cd + bc'd + ab'd + abc' + a'bc + acd' \\
&= a'b'c'd' + a'cd + bc'd + ab'd + abd' + bcd' + ab'c \\
&= a'b'c'd' + b'cd + a'bd + ac'd + abd' + acd' + bcd' \\
&= a'b'c'd' + b'cd + abc' + bcd' + a'bd + ab'c + ab'd
\end{aligned}
$$

*[SP 1, 2; EX 1, 2, 3]*

## 3.1.2 Don't Cares

Finding minimum solutions for functions with don't cares does not significantly change the methods we developed in the last section. We need to modify slightly the definitions of a prime implicant and clarify the definition of an essential prime implicant.

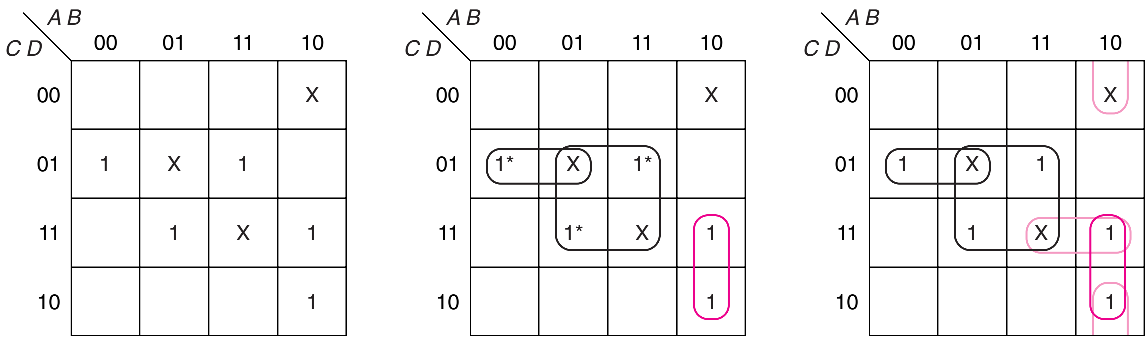
> A *prime implicant* is a rectangle of 1, 2, 4, 8, . . . 1's or X's not included in any one larger rectangle. Thus, from the point of view of finding prime implicants, X's (don't cares) are treated as 1's.

An *essential prime implicant* is a prime implicant that covers at least one 1 not covered by any other prime implicant (as always). Don't cares (X's) do not make a prime implicant essential.

Now, we just apply either of the methods of the last section. When we are done, some of the X's may be included and some may not. But we *don't care* whether or not they are included in the function.

**EXAMPLE 3.16**

$F(A, B, C, D) = \Sigma m(1, 7, 10, 11, 13) + \Sigma d(5, 8, 15)$




We first mapped the function, entering a 1 for those minterms included in the function and an X for the don't cares. We found two essential prime implicants, as shown on the center map. In each case, the 1's with an asterisk cannot be covered by any other prime implicant. That left the two 1's circled in red to cover the rest of the function. That is not an essential prime implicant, since each of the 1's could be covered by another prime implicant (as shown in pink on the third map). However, if we did not use $AB'C$, we would need two additional terms, instead of one. Thus, the only minimum solution is

$$F = BD + A'C'D + AB'C$$

and terms $AB'D'$ and $ACD$ are prime implicants not used in the minimum solution. Note that if all of the don't cares were made 1's, we would need a fourth term to cover $m_8$, making

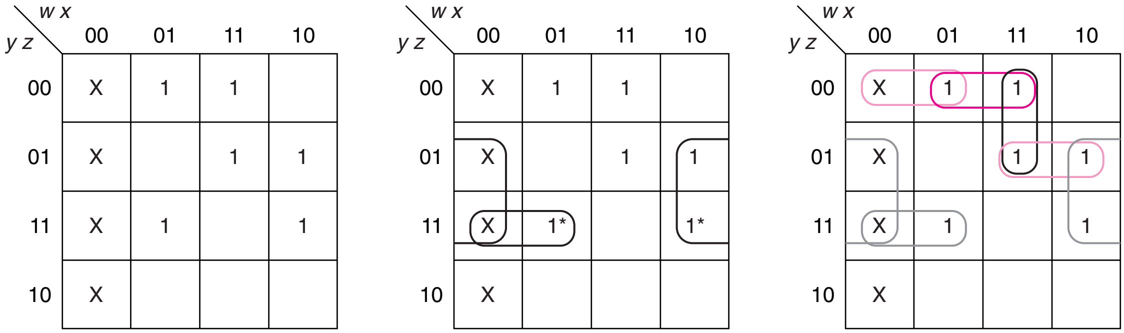
$$F = BD + A'C'D + AB'C + AB'D' \quad \text{or}$$
$$F = BD + A'C'D + ACD + AB'D'$$

and that if all of the don't cares were 0's, the function would become

$$F = A'B'C'D + A'BCD + ABC'D + AB'C$$

In either case, the solution is much more complex then when we treated those terms as don't cares (and made two of them 1's and the other a 0).

## EXAMPLE 3.17

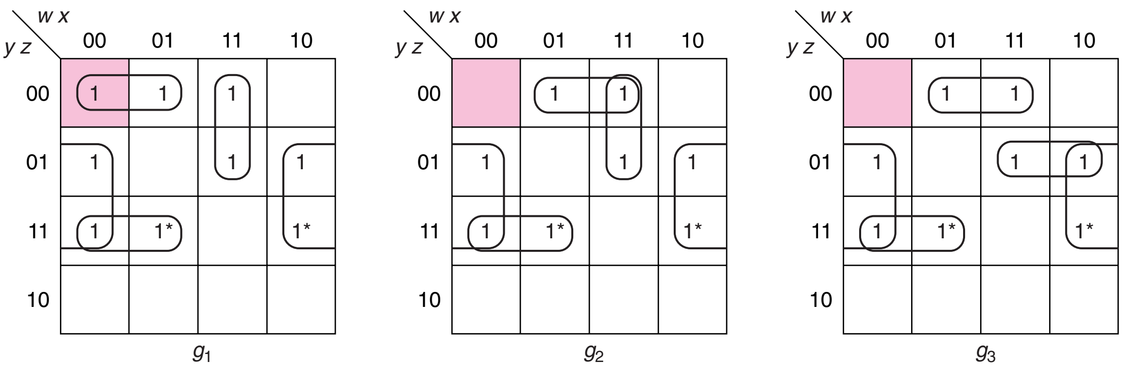


There are two essential prime implicants, as shown on the center map, $x'z$ and $w'yz$. The group of four don't cares, $w'x'$, is a prime implicant (since it is a rectangle of four 1's or X's) but it is not essential (since it does not cover any 1's not covered by some other prime implicant). Surely, a prime implicant made up of all don't cares would never be used, since that would add a term to the sum without covering any additional 1's. The three remaining 1's require two groups of two and thus there are three equally good solutions, each using four terms and 11 literals:

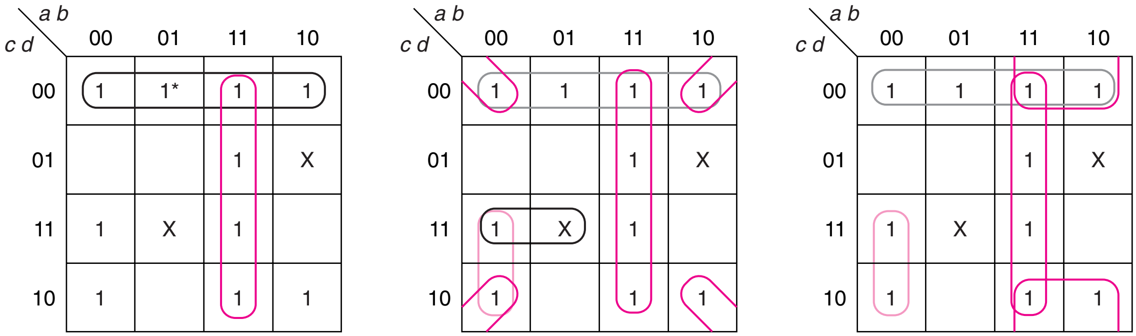
$$g_1 = x'z + w'yz + w'y'z' + wxy'$$
$$g_2 = x'z + w'yz + xy'z' + wxy'$$
$$g_3 = x'z + w'yz + xy'z' + wy'z$$

An important thing to note about Example 3.17 is that the three algebraic expressions are not all equal. The first treats the don't care for $m_0$ as a 1, whereas the other two (which are equal to each other) treat it as a 0. This will often happen with don't cares. They must treat the specified part of the function (the 1's and the 0's) the same, but the don't cares may take on different values in the various solutions. The maps of Map 3.4 show the three functions.

**Map 3.4** The different solutions for Example 3.17.



**EXAMPLE 3.18**




On the first map, we have shown the only essential prime implicant, $c'd'$, and the other group of four that is used in all three solutions, $ab$. (This must be used since the only other prime implicant that would cover $m_{15}$ is $bcd$, which requires one more literal and does not cover any 1's that are not covered by $ab$.) The three remaining 1's require two terms, one of which must be a group of two (to cover $m_3$) and the other must be one of the groups of four that cover $m_{10}$. On the second map, we have shown two of the solutions, those that utilize $b'd'$ as the group of four. On the third map, we have shown the third solution, utilizing $ad'$. Thus, we have

$$g_1 = c'd' + ab + b'd' + a'cd$$
$$g_2 = c'd' + ab + b'd' + a'b'c$$
$$g_3 = c'd' + ab + ad' + a'b'c$$

We can now ask if these solutions are equal to each other. We can either map all three solutions as we did for Example 3.17 or we can make a table of the behavior of the don't cares—one column for each don't care and one row for each solution.

| | $m_7$ | $m_9$ |
|---|---|---|
| $g_1$ | 1 | 0 |
| $g_2$ | 0 | 0 |
| $g_3$ | 0 | 0 |

From the table, it is clear that $g_2 = g_3$, but neither is equal to $g_1$. A more complex example is found in the solved problems.

Don't cares provide us with another approach to solving map problems for functions with or without don't cares.

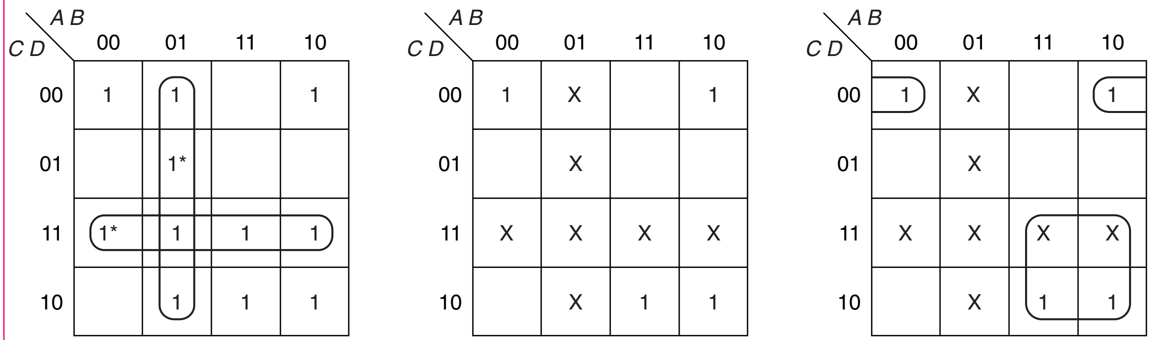
***Map Method 3***

1. Find all essential prime implicants using either Map Method 1 or 2).
2. Replace all 1's covered by the essential prime implicants with X's. This highlights the 1's that remain to be covered.
3. Then choose enough of the other prime implicants (as in Methods 1 and 2).

Step 2 works because the 1's covered by essential prime implicants may be used again (as part of a term covering some new 1's), but need not be. Thus, once we have chosen the essential prime implicants, these minterms are, indeed, don't cares.

**EXAMPLE 3.19**

$$F(A, B, C, D) = \Sigma m(0, 3, 4, 5, 6, 7, 8, 10, 11, 14, 15)$$




We first found the two essential prime implicants, $A'B$ and $CD$. On the second map, we converted all of the 1's covered to don't cares. Finally, we can cover the remaining 1's with $AC$ and $B'C'D'$, producing

$$F = A'B + CD + AC + B'C'D'$$

Replacing covered minterms by don't cares accomplishes the same thing as the shading that we did a few examples ago; it highlights the 1's that remain to be covered.

*[SP 3, 4; EX 4, 5]*

## 3.1.3 Product of Sums

Finding a minimum product of sums expression requires no new theory. The following approach is the simplest:

1. Map the complement of the function. (If there is already a map for the function, replace all 0's by 1's, all 1's by 0's and leave X's unchanged.)

2. Find the minimum sum of products expression for the complement of the function (using the techniques of the last two sections).
3. Use DeMorgan's theorem (P11) to complement that expression, producing a product of sums expression.

Another approach, which we will not pursue here, is to define the dual of prime implicants (referred to as prime implicates) and develop a new method.

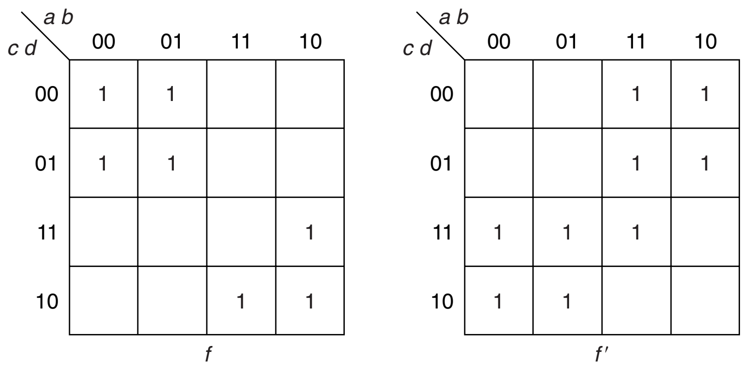
**EXAMPLE 3.20**

$$f(a, b, c, d) = \Sigma m(0, 1, 4, 5, 10, 11, 14)$$

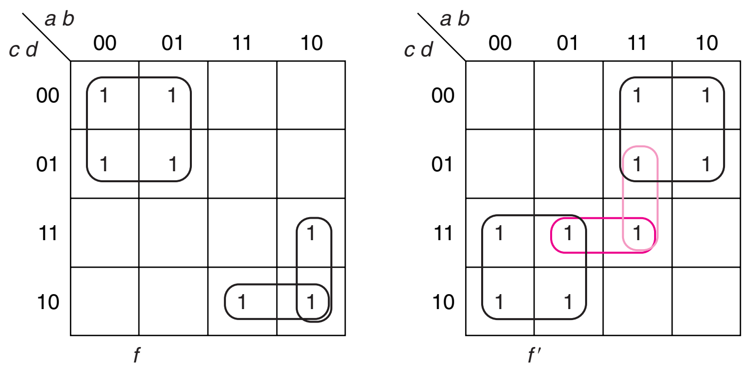
Since all minterms must be either minterms of $f$ or of $f'$, then, $f'$ must be the sum of all of the other minterms, that is

$$f'(a, b, c, d) = \Sigma m(2, 3, 6, 7, 8, 9, 12, 13, 15)$$

Maps of both $f$ and $f'$ are shown below




We did not need to map $f$, unless we wanted both the sum of products expression and the product of sums expression. Once we mapped $f$, we did not need to write out all the minterms of $f'$; we could have just replaced the 1's by 0's and 0's by 1's. Also, instead of mapping $f'$, we could look for rectangles of 0's on the map of $f$. This, function is rather straightforward. The maps for the minimum sum of product expressions for both $f$ and $f'$ are shown below:



There is one minimum solution for $f$ and there are two equally good solutions for the sum of products for $f'$:

$$f = a'c' + ab'c + acd' \qquad f' = ac' + a'c + abd$$
$$f' = ac' + a'c + bcd$$

We can then complement the solutions for $f'$ to get the two minimum product of sums solutions for $f$:

$$f = (a' + c)(a + c')(a' + b' + d')$$
$$f = (a' + c)(a + c')(b' + c' + d')$$

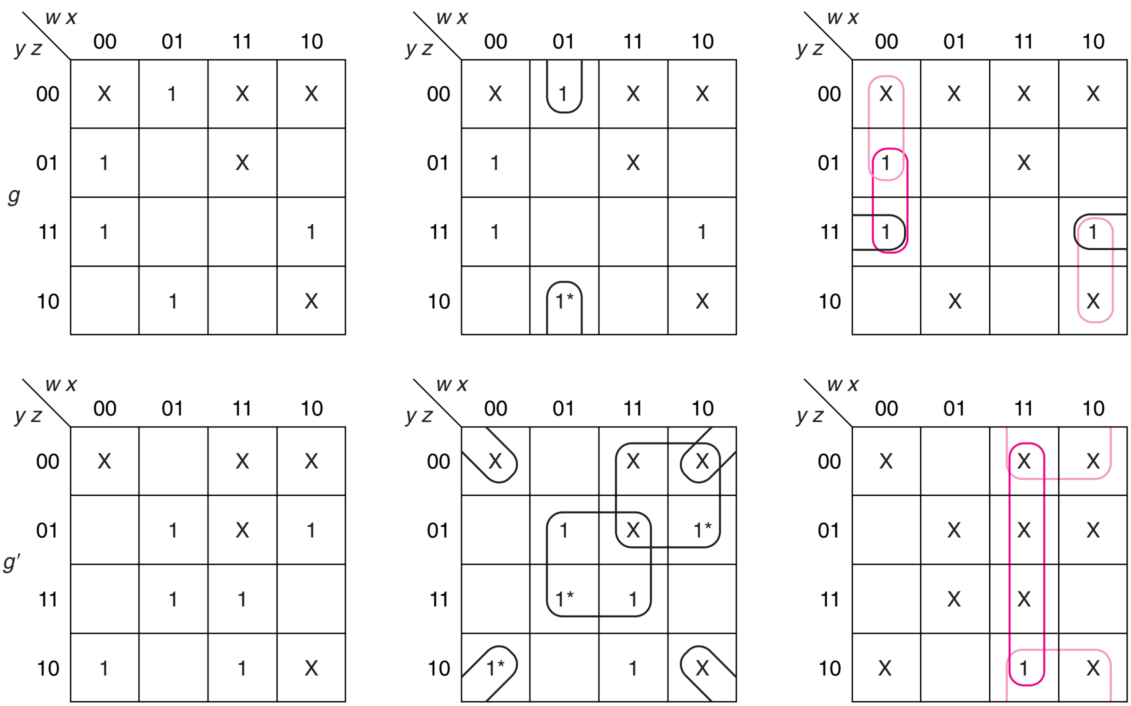
The minimum sum of products solution has three terms and eight literals; the minimum product of sums solutions have three terms and seven literals. (There is no set pattern; sometimes the sum of products solution has fewer terms or literals, sometimes the product of sums does, and sometimes they have the same number of terms and literals.)

**EXAMPLE 3.21**

Find all of the minimum sum of products and all minimum product of sums solutions for

$$g(w, x, y, z) = \Sigma m(1, 3, 4, 6, 11) + \Sigma d(0, 8, 10, 12, 13)$$

We first find the minimum sum of products expression by mapping $g$. However, before complicating the map by circling prime implicants, we also map $g'$ (below $g$). Note that the X's are the same on both maps.



For $g$, the only essential prime implicant, $w'xz'$ is shown on the center map. The 1's covered by it are made don't cares on the right map and the remaining useful prime implicants are circled. We have seen similar examples before, where we have three 1's to be covered in groups of two. There are three equally good solutions.

$$g = w'xz' + \begin{Bmatrix} w'x'y' + x'yz \\ w'x'z + x'yz \\ w'x'z + wx'y \end{Bmatrix}$$

For $g'$, there are three essential prime implicants, as shown on the center map. Once all of the 1's covered by them have been made don't cares, there is only one 1 left; it can be covered in two ways as shown on the right map:

$$g' = x'z' + xz + wy' + \begin{Bmatrix} wx \\ wz' \end{Bmatrix}$$

$$g = (x + z)(x' + z')(w' + y)\begin{Bmatrix} (w' + x') \\ (w' + z) \end{Bmatrix}$$

Note that in this example, the sum of product solutions each require only three terms (with nine literals), whereas the product of sums solutions each require four terms (with eight literals).

Finally, we want to determine which, if any, of the five solutions are equal. The complication (compared to this same question in the last section) is that when we treat a don't care as a 1 for $g'$, that means that we are treating it as a 0 of $g$. Labeling the three sum of product solutions as $g_1$, $g_2$, and $g_3$, and the two product of sums solutions as $g_4$ and $g_5$, we produce the following table

| | **0** | **8** | **10** | **12** | **13** |
|---|---|---|---|---|---|
| $g_1$ | 1 | 0 | 0 | 0 | 0 |
| $g_2$ | 0 | 0 | 0 | 0 | 0 |
| $g_3$ | 0 | 0 | 1 | 0 | 0 |
| $g_4'$ | 1 | 1 | 1 | 1 | 1 |
| $g_4$ | 0 | 0 | 0 | 0 | 0 |
| $g_5'$ | 1 | 1 | 1 | 1 | 1 |
| $g_5$ | 0 | 0 | 0 | 0 | 0 |

The product of sum solutions treat all of the don't cares as 1's of $g'$ since each is circled by the essential prime implicants of $g'$. (Thus, they are 0's of $g$.) We then note that the three solutions that are equal are

$$g_2 = w'xz' + w'x'z + x'yz$$

$$g_4 = (x + z)(x' + z')(w' + y)(w' + x')$$

$$g_5 = (x + z)(x' + z')(w' + y)(w' + z)$$

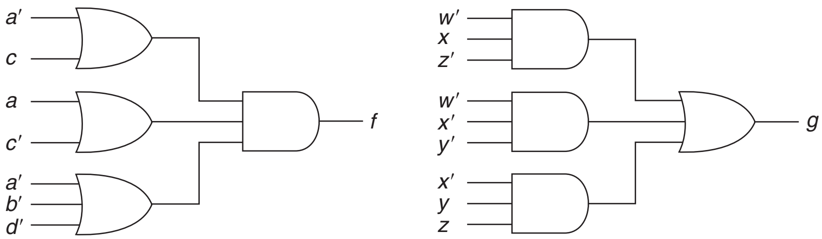
*[SP 5, 6; EX 6, 7]*

### 3.1.4 Minimum Cost Gate Implementations

We are now ready to take another look at implementing functions with various types of gates. In this section, we will limit our discussion to two-level solutions for systems where all inputs are available both uncomplemented and complemented. (In Section 2.10, we examined multilevel circuits.) The minimization criteria is minimum number of gates, and among those with the same number of gates, minimum number of gate inputs. (Other criteria, such as minimum number of integrated circuit packages, were also discussed in Section 2.10 and will be examined further in Chapter 4.) The starting point is almost always to find the minimum sum of products solutions and/or the minimum product of sums solutions. That is because each term (other than single literal ones) corresponds to a gate. Then, unless the function has only one term, there is one output gate. Minimizing the number of literals minimizes the number of inputs to these gates.

First, we will look for solutions using AND and OR gates. We must look at both the minimum sum of products and minimum product of sums solutions. In Examples 3.19 and 3.20 from the last section, the product of sums solutions for *f* had one less gate input than the sum of products solution and the sum of products solutions for *g* had one less gate than the product of sums solutions. One of the minimum cost solutions for each is shown in Figure 3.1. (There are three equally good ones for *f* and two equally good ones for *g*.)

**Figure 3.1** Minimum cost AND/OR implementations.




For a two-level solution using NAND gates, we need to start with a minimum sum of products solution. Thus, for *g* we can use the solution we obtained for AND and OR, but for *f*, we must use the sum of products solution, the one with one more gate input, as shown in Figure 3.2.

Similarly, for a two-level solution with NOR gates, we use a minimum product of sums solution, resulting in the circuits of Figure 3.3. Note that the NOR gate solution for *g* uses one more gate than the NAND gate solution.

If we are not limited to two levels, we have one additional option for implementing NAND gate solutions (or NOR gate solutions) beyond the algebraic manipulation of Section 2.8. We could find a minimum sum of

**Figure 3.2** NAND gate implementations.

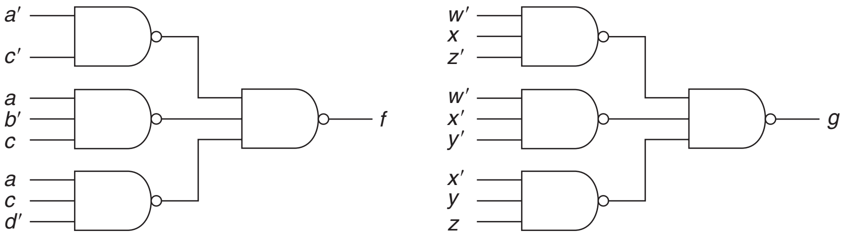
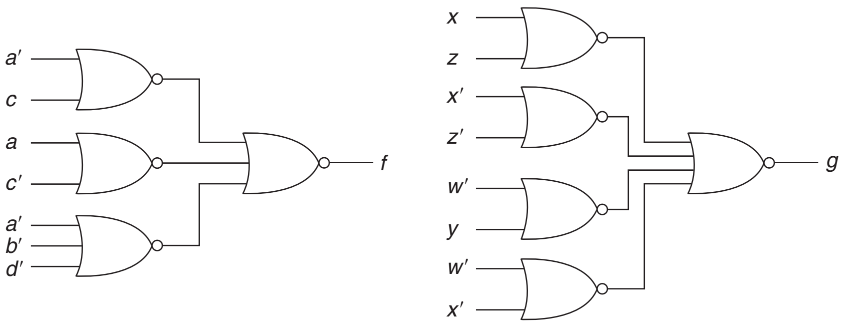



**Figure 3.3** NOR gate implementations.




products expression for $f'$ and implement that with NAND gates. We would then place a NOT gate at the output to produce $f$.

**EXAMPLE 3.22**

$$G(A, B, C, D) = \Sigma m(0, 1, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15)$$

In Example 3.14, we found six equally good minimum sum of products solutions, each of which has four terms and eight literals. These solutions would require five gates. One of them is

$$G = C'D' + B'D + BC + AB$$

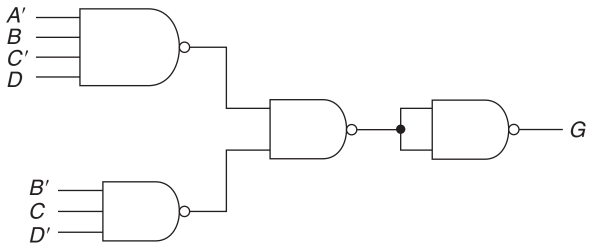
The map for $G'$ is shown below

| CD \ AB | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | | | | |
| 01 | | 1 | | |
| 11 | | | | |
| 10 | 1 | | | 1 |

and thus

$$G' = A'BC'D + B'CD'$$

We could implement $G'$ with three NAND gates and then use a NOT gate (or a two-input NAND with the inputs tied together) on the output as shown below:



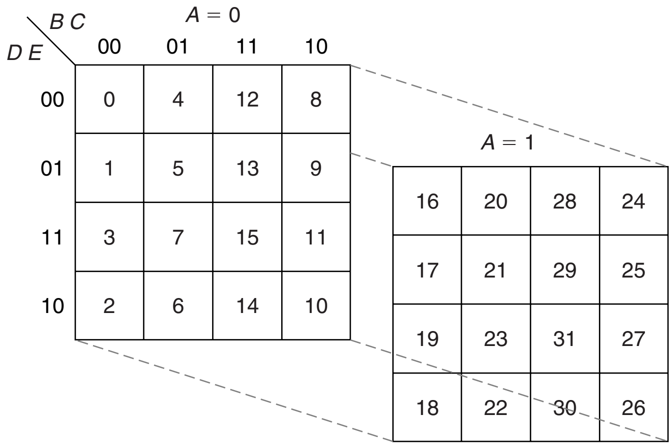

This requires only four gates compared to the sum of products solution which required five gates. (Either would require two 7400 series packages.)

*[SP 7; EX 8]*

## 3.1.5 Five- and Six-Variable Maps

A five-variable map consists of $2^5 = 32$ squares. Although there are several arrangements that have been used, we prefer to look at it as two layers of 16 squares each. The top layer (on the left below) contains the squares for the first 16 minterms (for which the first variable, $A$, is 0) and the bottom layer contains the remaining 16 squares, as pictured in Map 3.5:

**Map 3.5** A five-variable map.




Each square in the bottom layer corresponds to the minterm numbered 16 more than the square above it. Product terms appear as rectangular solids of 1, 2, 4, 8, 16, . . . 1's or X's. Squares directly above and below each other are adjacent.

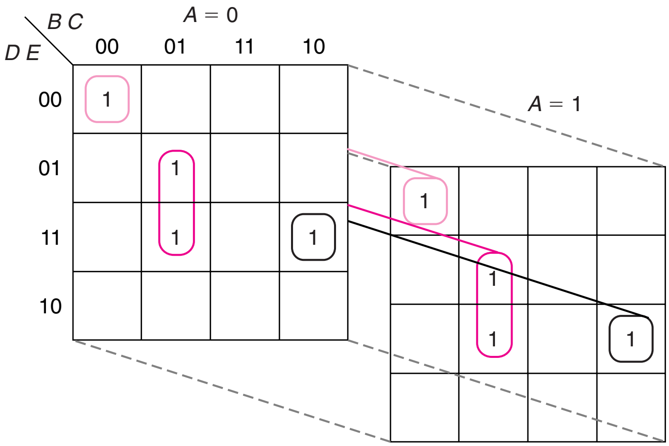
**EXAMPLE 3.23**

$m_0 + m_{16} = A'B'C'D'E' + AB'C'D'E' = B'C'D'E'$

$m_{13} + m_{29} = A'BCD'E + ABCD'E = BCD'E$

$m_5 + m_7 + m_{21} + m_{23} = B'CE$

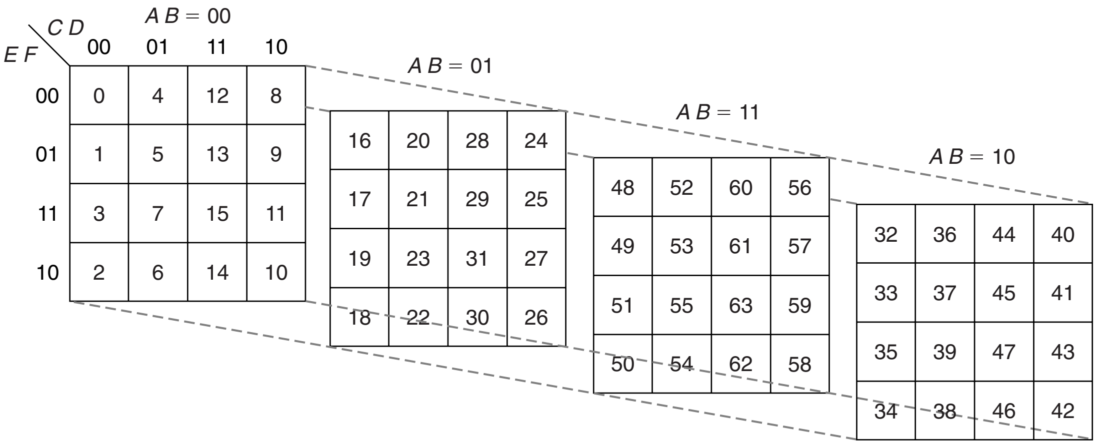
These terms are circled on the map below.




In a similar manner, six-variable maps are drawn as four layers of 16-square maps, where the first two variables determine the layer and the other variables specify the square within the layer. The layout, with minterm numbers shown, is given in Map 3.6. Note that the layers are ordered in the same way as the rows and the columns, that is 00, 01, 11, 10.

In this section, we will concentrate on five-variable maps, although we will also do an example of six-variable maps at the end. The

**Map 3.6** A six-variable map.



techniques are the same as for four-variable maps; the only thing new is the need to visualize the rectangular solids. Rather than drawing the maps to look like three dimensions, we will draw them side by side.

The approach for five-variable maps is the same as for three- and four-variable maps. The function, $f$, is mapped in Map 3.7.

$$F(A, B, C, D) = \Sigma m(4, 5, 6, 7, 9, 11, 13, 15, 16, 18, 27, 28, 31)$$

**Map 3.7** A five-variable problem.

A = 0

| DE \ BC | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | | 1 | | |
| 01 | | 1 | 1 | 1 |
| 11 | | 1 | 1 | 1 |
| 10 | | 1 | | |

A = 1

| DE \ BC | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | | 1 | |
| 01 | | | | |
| 11 | | | 1 | 1 |
| 10 | 1 | | | |

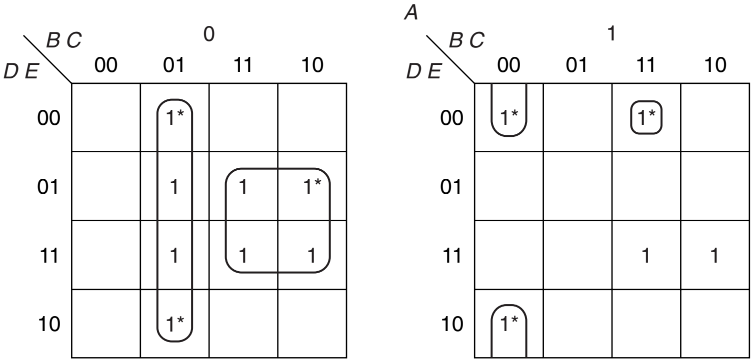
As always, we first look for the essential prime implicants. A good starting point is to find 1's on one layer for which there is no 1 in the corresponding square on an adjoining layer. Prime implicants that cover that 1 are contained completely on that layer (and thus, we really only have a four-variable map problem). In this example, $m_4$ meets this criteria (since there is a 0 in square 20 below it). Thus, the only prime implicants covering $m_4$ must be on the first layer. Indeed, $A'B'C$ is an essential prime implicant. (Note that the $A'$ comes from the fact that this group is contained completely on the $A = 0$ layer of the map and the $B'C$ from the fact that this group is in the second column.) Actually, all four 1's in this term have no counterpart on the other layer and $m_6$ would also make this prime implicant essential. (The other two 1's in that term are part of another prime implicant, as well.) We also note that $m_9$, $m_{16}$, $m_{18}$, and $m_{28}$ also fit this criteria. Thus Map 3.8 shows each of these circled, highlighting the prime implicants that are contained on one layer.

So far, we have

$$F = A'B'C + A'BE + AB'C'E' + ABCD'E' + \cdots$$

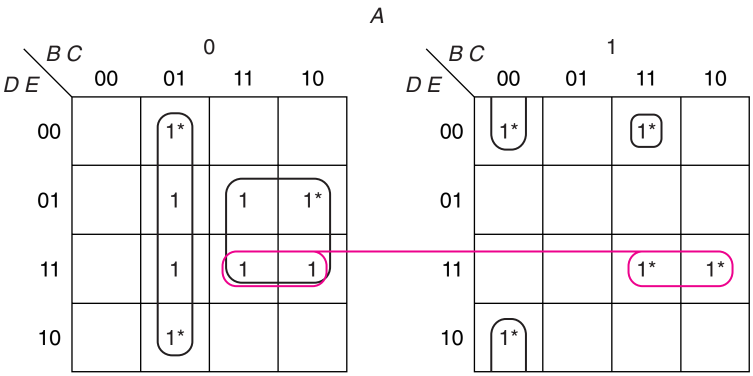
The two 1's remaining uncovered do have counterparts on the other layer. However, the only prime implicant that covers them is $BDE$, as shown on Map 3.9 in red. It, too, is an essential prime implicant. (Note that prime implicants that include 1's from both layers do not have the variable A in them. Such prime implicants must, of course, have

**Map 3.8** Essential prime implicants on one layer.




the same number of 1's on each layer; otherwise, they would not be rectangular.)

**Map 3.9** A prime implicant covering 1's on both layers.




The complete solution is thus

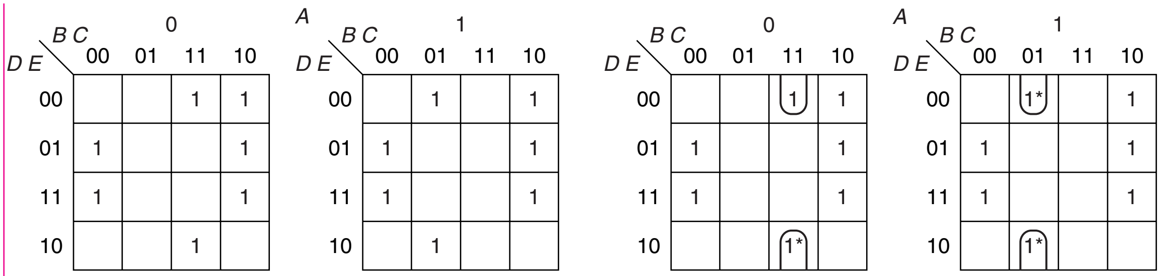
$$F = A'B'C + A'BE + AB'C'E' + ABCD'E' + BDE$$

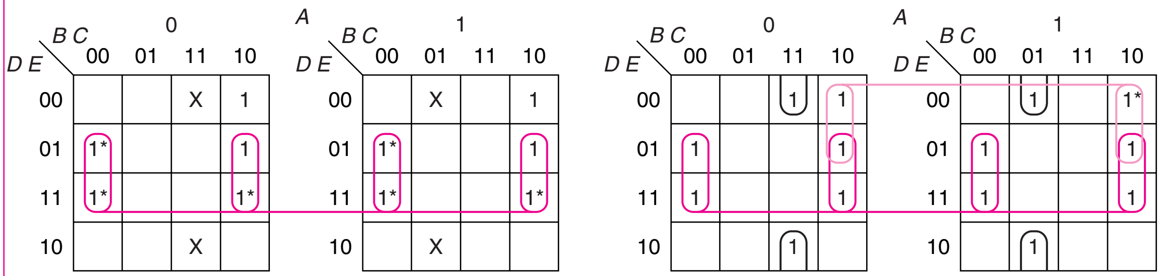
Groups of eight 1's are not uncommon in five-variable problems, as illustrated in Example 3.24.

**EXAMPLE 3.24**

$$G(A, B, C, D, E) = \Sigma m(1, 3, 8, 9, 11, 12, 14, 17, 19, 20, 22, 24, 25, 27)$$

The first map shows a plot of that function. On the second map, to the right, we have circled the two essential prime implicants that we found by considering 1's on one layer that were not on the other. The group of eight 1's, $C'E$ (also an essential prime implicant), is shown in *red* on the third map (where the essential prime implicants found on the second map are shown as don't cares). Groups of eight have three literals missing (leaving only two). At this point, only two 1's are left uncovered; that requires the essential prime implicant, $BC'D'$, shown on the fourth map in pink.



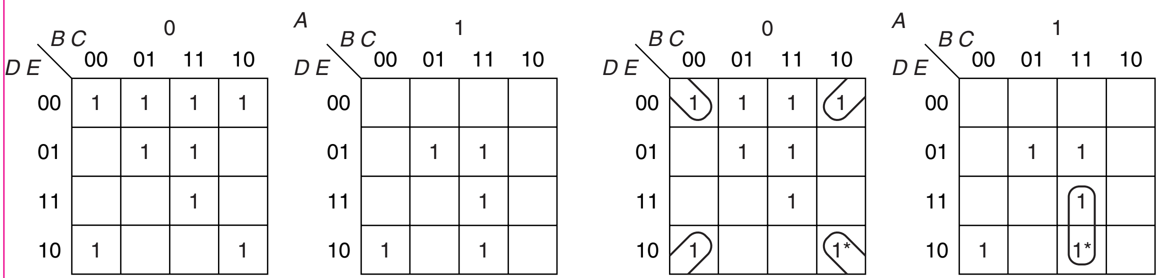

The solution is thus

$$G = A'BCE' + AB'CE' + C'E + BC'D'$$

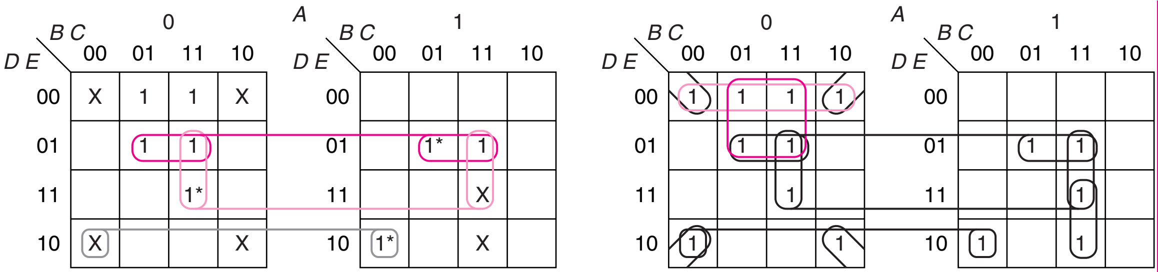
Note that there is only one other prime implicant in this function, $A'BD'E'$; it covers no 1's not already covered.

**EXAMPLE 3.25**

The next problem is shown on the maps below. Once again, we start by looking for 1's that are on one layer, with no corresponding 1 on the other layer. Although there are several such 1's on the $A = 0$ layer, only $m_{10}$ makes a prime implicant essential. Similarly, on the $A = 1$ layer, $m_{30}$ is covered by an essential prime implicant. These terms, $A'C'E'$ and $ABCD$, are shown on the second map. The 1's covered are shown as don't cares on the next map.




Three other essential prime implicants include 1's from both layers of the map; they are $CD'E$, $BCE$ and $B'C'DE'$, as shown on the left map below. These were found by looking for isolated 1's, such as $m_{15}$, $m_{18}$, and $m_{21}$.



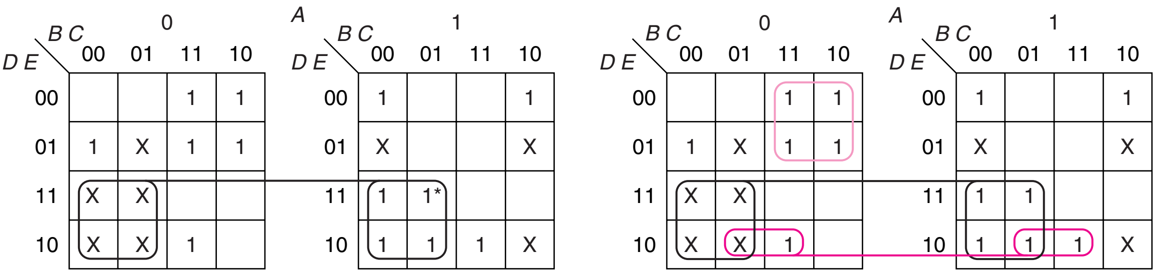

Finally, the remaining two 1's ($m_4$ and $m_{12}$) can be covered in two ways, as shown on the right map above, $A'CD'$ and $A'D'E'$. Thus, the two solutions are

$$F = A'C'E' + ABCD + CD'E + BCE + B'C'DE' + A'CD'$$
$$F = A'C'E' + ABCD + CD'E + BCE + B'C'DE' + A'D'E'$$

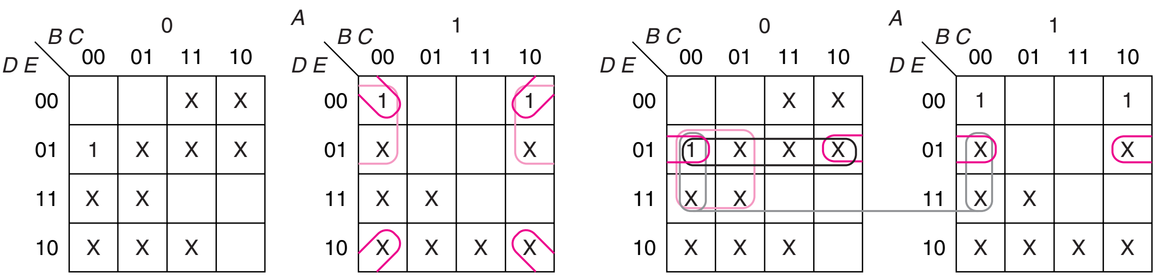
**EXAMPLE 3.26**

$$H(A, B, C, D, E) = \Sigma m(1, 8, 9, 12, 13, 14, 16, 18, 19, 22, 23, 24, 30) + \Sigma d(2, 3, 5, 6, 7, 17, 25, 26)$$

A map of $H$ is shown below on the left with the only essential prime implicant, $B'D$, (a group of eight, including four 1's and four don't cares) circled.




Next, we choose $CDE'$, since otherwise separate terms would be needed to cover $m_{14}$ and $m_{30}$. We also chose $A'BD'$ since it covers four new 1's. Furthermore, if that were not used, a group of two ($A'BCE'$) would be needed to cover $m_{12}$. That leaves us with three 1's ($m_1$, $m_{16}$, and $m_{24}$) to be covered. On the maps below, we have replaced all covered 1's by don't cares (X's) to highlight the remaining 1's. No term that covers $m_1$ also covers either of the other terms. However, $m_{16}$ and $m_{24}$ can be covered with one term in either of two ways ($AC'E'$ or $AC'D'$) as shown on the first map below and $m_1$ can



be covered by four different groups of four, as shown on the second map ($A'D'E$, $A'B'E$, $B'C'E$, or $C'D'E$), yielding the eight solutions shown.

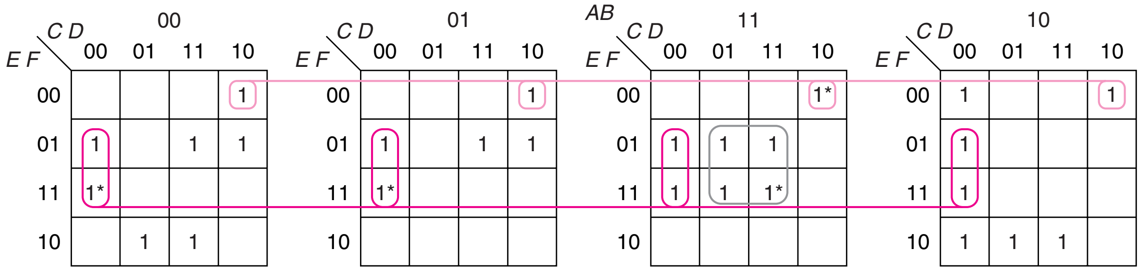
$$H = B'D + CDE' + A'BD' + \begin{Bmatrix} AC'E' \\ AC'D' \end{Bmatrix} + \begin{Bmatrix} A'D'E \\ A'B'E \\ B'C'E \\ C'D'E \end{Bmatrix}$$

Finally, we will look at one example of a six-variable function.

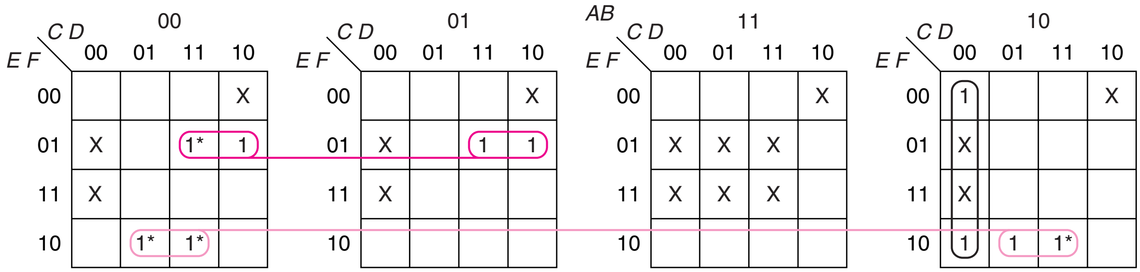
**EXAMPLE 3.27**

$$G(A, B, C, D, E, F) = \Sigma m(1, 3, 6, 8, 9, 13, 14, 17, 19, 24, 25, 29, 32, 33, 34, 35, 38, 40, 46, 49, 51, 53, 55, 56, 61, 63)$$

The map is drawn horizontally, with the first two variables determining the 16-square layer (numbered, of course 00, 01, 11, 10).




The first map shows three of the essential prime implicants. The only one that is confined to one layer is on the third layer, $ABDF$. The 1's in the upper right corner of each layer form another group of four (without the first two variables), $CD'E'F'$. The red squares form a group of eight, $C'D'F$. The next map shows 1's covered by the first three prime implicants as don't cares.




The other two essential prime implicants are $A'CE'F$ and $B'DEF'$. (Remember that the top and bottom layers are adjacent.) Finally, $m_{32}$ and $m_{34}$ (on the fourth layer) remain uncovered; they are covered by the term, $AB'C'D'$. (Each of them could have been covered by a group of two; but that would take two terms.) Thus, the minimum expression is

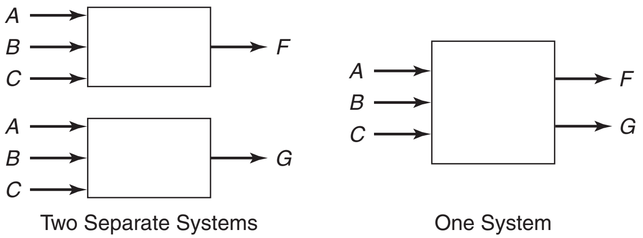
*[SP 8, 9; EX 9, 10]*

$$G = ABDF + CD'E'F' + C'D'F + A'CE'F + B'DEF' + AB'C'D'$$

## 3.1.6 Multiple Output Problems

Many real problems involve designing a system with more than one output. If, for example, we had a problem with three inputs, $A$, $B$, and $C$ and two outputs, $F$ and $G$, we could treat this as two separate problems (as shown on the left in Figure 3.4). We would then map each of the functions, and find minimum solutions. However, if we treated this as a single system with three inputs and two outputs (as shown on the right), we may be able to economize by sharing gates.

**Figure 3.4** Implementation of two functions.



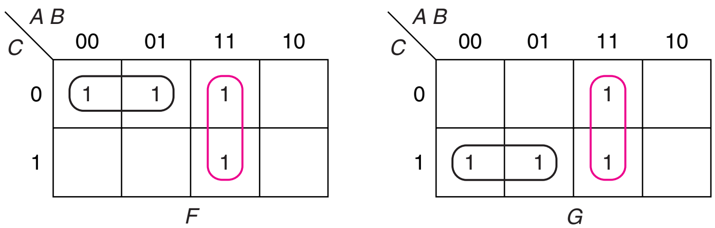

In this section, we will illustrate the process of obtaining two-level solutions using AND and OR gates (sum of products solutions), assuming all variables are available both uncomplemented and complemented. We could convert each of these solutions into NAND gate circuits (using the same number of gates and gate inputs). We could also find product of sums solutions (by minimizing the complement of each of the functions and then using DeMorgan's theorem).

We will illustrate this by first considering three very simple examples.

**EXAMPLE 3.28**

$F(A, B, C) = \Sigma m(0, 2, 6, 7) \qquad G(A, B, C) = \Sigma m(1, 3, 6, 7)$

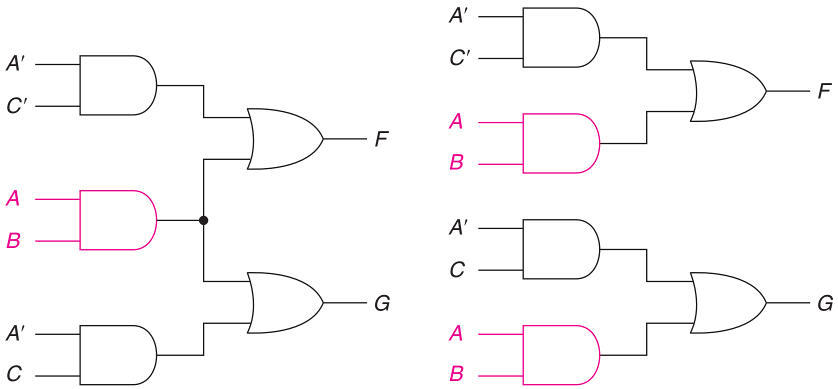
If we map each of these and solve them separately,




we obtain

$F = A'C' + AB \qquad G = A'C + AB$

Looking at the maps, we see that the same term ($AB$) is circled on both. Thus, we can build the circuit on the left, rather than the two circuits on the right.



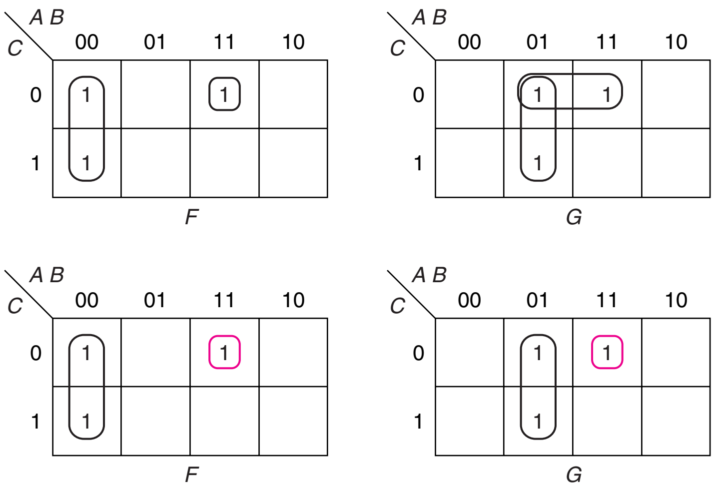

Obviously, the version on the left requires only five gates, whereas the one on the right uses six.

This example is the simplest. Each of the minimum sum of product expressions contains the same term. It would take no special techniques to recognize this and achieve the savings.

Even when the two solutions do not have a common prime implicant, we can share as illustrated in the following example:

**EXAMPLE 3.29**

$F(A, B, C) = \Sigma m(0, 1, 6) \qquad G(A, B, C) = \Sigma m(2, 3, 6)$



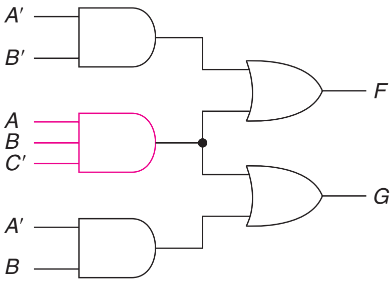

In the top maps, we considered each function separately and obtained

$F = A'B' + ABC' \qquad G = A'B + BC'$

This solution requires six gates (four ANDs and two ORs) with 13 inputs. However, as can be seen from the second pair of maps, we can share the term *ABC′* and obtain

$$F = A'B' + ABC' \qquad G = A'B + ABC'$$

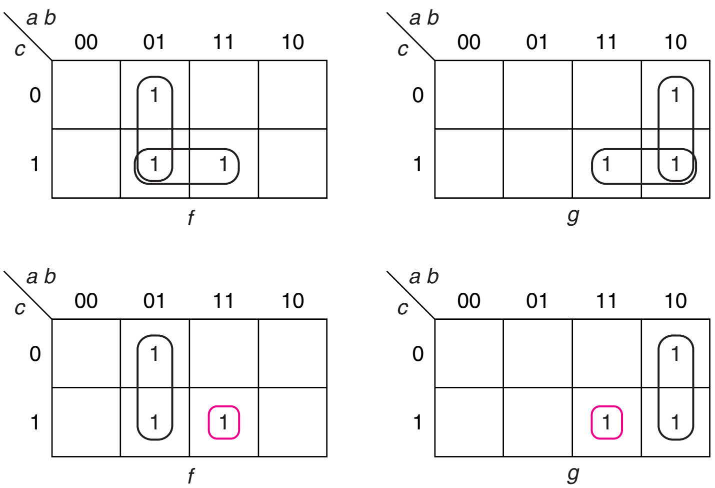
(To emphasize the sharing, we have shown the shared term in red, and will do that in other examples that follow.) As can be seen from the circuit below, this only requires five gates with 11 inputs.




This example illustrates that a shared term in a minimum solution need not be a prime implicant. (In Example 3.29, *ABC′* is a prime implicant of *F* but not of *G*; in Example 3.30, we will use a term that is not a prime implicant of either function.)

**EXAMPLE 3.30**

$$F(A, B, C) = \Sigma m(2, 3, 7) \qquad G(A, B, C) = \Sigma m(4, 5, 7)$$




In the first pair of maps, we solved this as two problems. Using essential prime implicants of each function, we obtained

$$f = a'b + bc \qquad g = ab' + ac$$

However, as can be seen in the second set of maps, we can share the term *abc*, even though it is not a prime implicant of either function, and once again get a solution that requires only five gates:

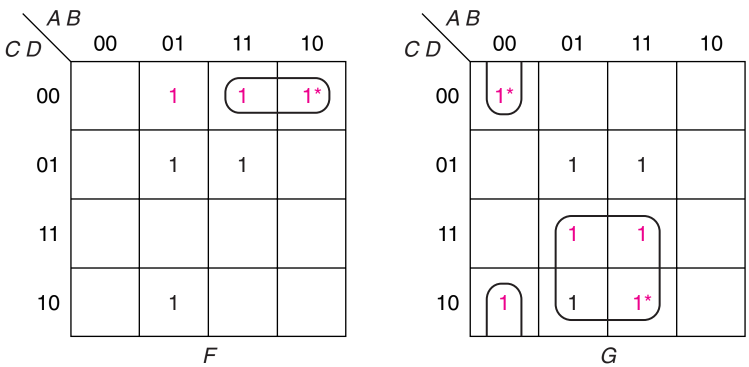
$$f = a'b + abc \qquad g = ab' + abc$$

The method for solving this type of problem is to begin by looking at the 1's of each function that are not 1's of the other function. They must be covered by prime implicants of that function. Only the shared terms need not be prime implicants. In this last example, we chose $a'b$ for $f$ since $m_2$ makes that an essential prime implicant of $F$ and we chose $AB'$ for $g$ since $m_4$ makes that an essential prime implicant of $g$. That left just one 1 uncovered in each function—the same 1—which we covered with *abc*. We will now look at some more complex examples.

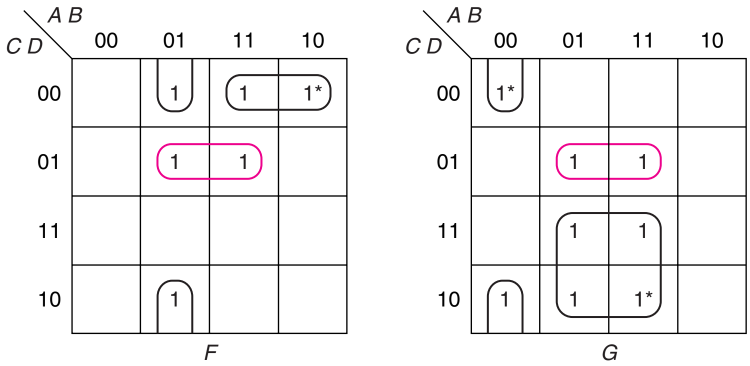
**EXAMPLE 3.31**

$$F(A, B, C, D) = \Sigma m(4, 5, 6, 8, 12, 13)$$
$$G(A, B, C, D) = \Sigma m(0, 2, 5, 6, 7, 13, 14, 15)$$

The maps of these functions are shown below. In them, we have shown in red the 1's that are included in one function and not the other.




We then circled each of those prime implicants that was made essential by a red 1. The only red 1 that was not circled in $F$ is $m_4$ because that can be covered by two prime implicants. Even though one of the terms would have fewer literals, we must wait. Next, we will use $A'BD'$ for $F$. Since $m_6$ was covered by an essential prime implicant of $G$, we are no longer looking for a term to share. Thus, $m_6$ will be covered in $F$ by the prime implicant, $A'BD'$. As shown on the maps below, that leaves $m_4$ and $m_{12}$ to be covered in both functions, allowing us to share the term $BC'D$, as shown on the following maps circled in red.




leaving

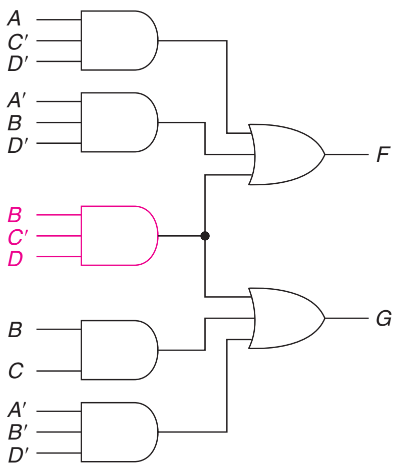
$$F = AC'D' + A'BD' + BC'D$$
$$G = A'B'D' + BC + BC'D$$

for a total of seven gates with 20 gate inputs. Notice that if we had minimized the functions individually, we would have used two separate terms for the third term in each expression, resulting in

$$F = AC'D' + A'BD' + BC'$$
$$G = A'B'D' + BC + BD$$

for a total of eight gates with 21 gate inputs. Clearly, the shared circuit costs less.

The shared version of the circuit is shown below.

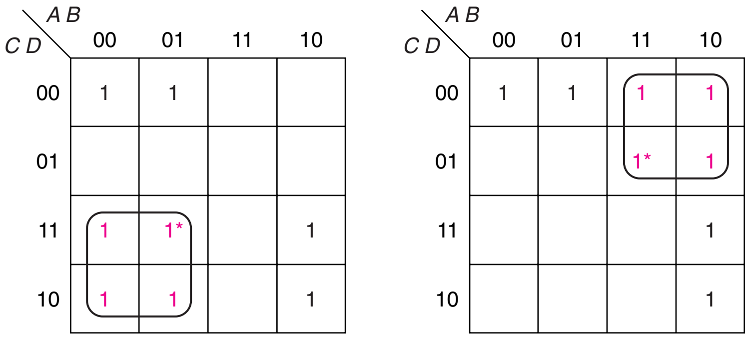


**EXAMPLE 3.32**

$$F(A, B, C, D) = \Sigma m(0, 2, 3, 4, 6, 7, 10, 11)$$

$$G(A, B, C, D) = \Sigma m(0, 4, 8, 9, 10, 11, 12, 13)$$

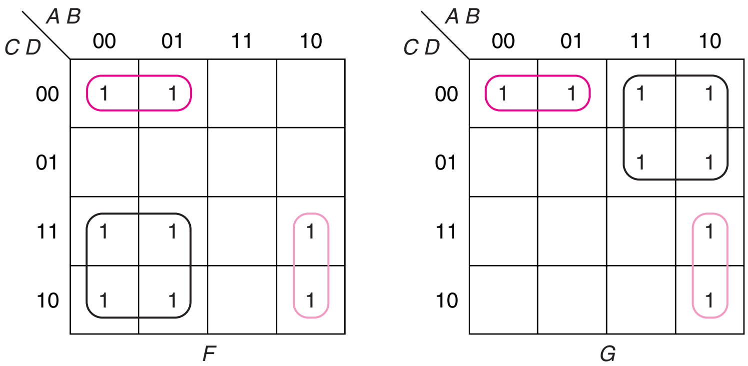
Once again the maps are shown with the unshared 1's in red and the prime implicants made essential by one of those 1's circled.




Each of the functions can be solved individually with two more groups of four, producing

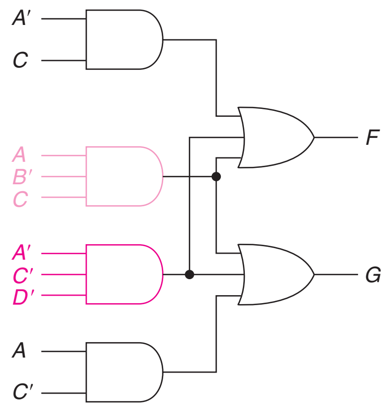
$$F = A'C + A'D' + B'C \qquad G = AC' + C'D' + AB'$$

That would require eight gates with 18 gate inputs. However, sharing the groups of two as shown on the next set of maps reduces the number of gates to six and the number of gate inputs to 16.




leaving the equations and the resulting AND/OR circuit.

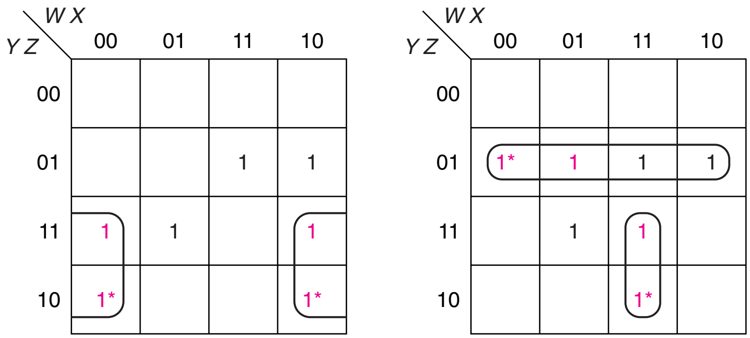
$$F = A'C + A'C'D' + AB'C \qquad G = AC' + A'C'D' + AB'C$$




**EXAMPLE 3.33**

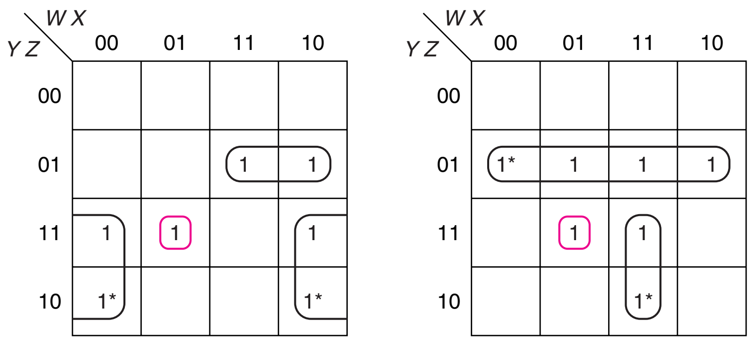
$$F(W, X, Y, Z) = \Sigma m(2, 3, 7, 9, 10, 11, 13)$$
$$G(W, X, Y, Z) = \Sigma m(1, 5, 7, 9, 13, 14, 15)$$

On the maps below, the 1's that are not shared are shown in red and the essential prime implicants that cover these 1's are circled.




*F* *G*

Now, there are three 1's left in *F*. Since $m_9$ and $m_{13}$ have been covered in *G* by an essential prime implicant, no sharing is possible for these terms in *F*. Thus, *WY′Z*, a prime implicant of *F*, is used in the minimum cover. Finally, there is one uncovered 1 in each function, $m_7$; it can be covered by a shared term, producing the solution




*F* *G*

$$F = X'Y + WY'Z + W'XYZ$$
$$G = Y'Z + WXY + W'XYZ$$

This requires seven gates and 20 inputs, compared to the solution we obtain by considering these as separate problems

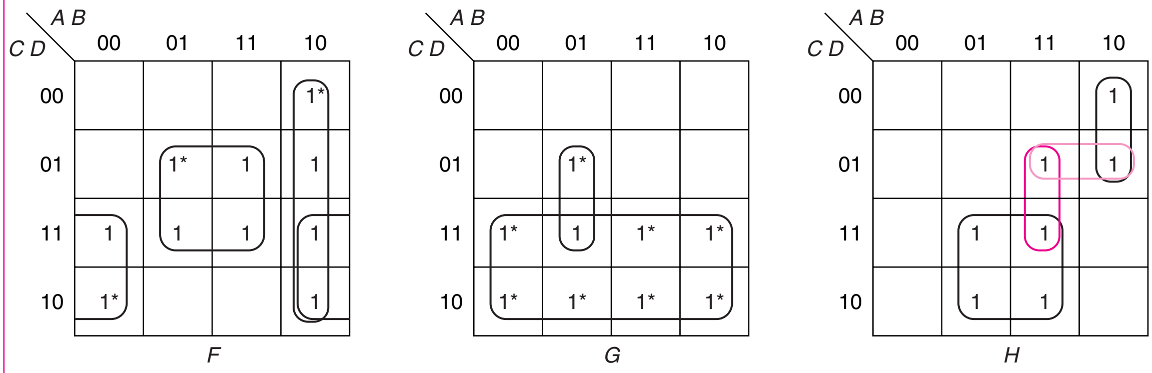
$$F = X'Y + WY'Z + W'YZ$$
$$G = Y'Z + WXY + XZ$$

which requires eight gates with 21 inputs.

The same techniques can be applied to problems with three or more outputs.

**EXAMPLE 3.34**

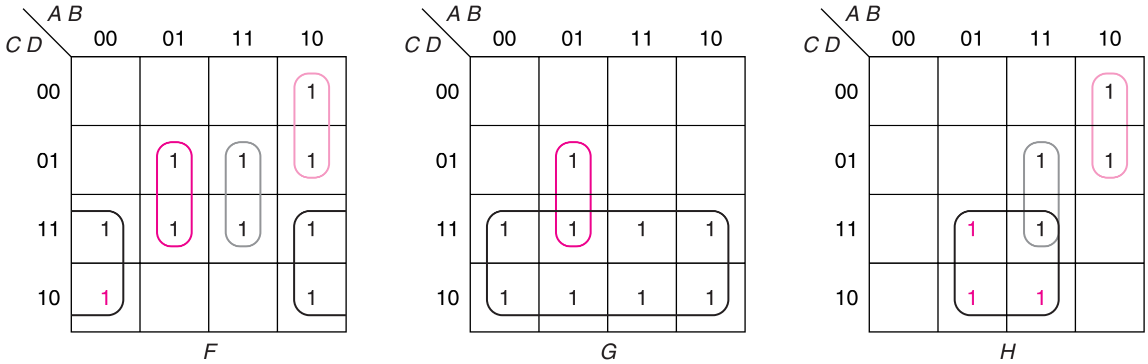
First, we show the solution obtained if we considered them as three separate problems.




$$F = AB' + BD + B'C$$
$$G = C + A'BD$$
$$H = BC + AB'C' + (ABD \text{ or } AC'D)$$

This solution requires 10 gates and 25 gate inputs. (Note that the term $C$ in function $G$ does not require an AND gate.)

The technique of first finding 1's that are only minterms of one of the functions does not get us started for this example, since each of the 1's is a minterm of at least two of the functions. The starting point, instead, is to choose $C$ for function $G$. The product term with only one literal does not require an AND gate and uses only one input to the OR gate. Any other solution, say sharing $B'C$ with $F$ and $BC$ with $H$, requires at least two inputs to the OR gate. Once we have made that choice, however, we must then choose $B'C$ for $F$ and $BC$ for $H$, because of the 1's shown in red on the following maps. There is no longer any sharing possible for those 1's and they make those prime implicants essential in $F$ and $H$.




The term $AB'C'$ (circled in pink) was chosen next for $H$ since it is an essential prime implicant of $H$ and it can be shared (that is, all of the 1's in that term are also 1's of $F$, the only place where sharing is possible). $AB'C'$ is also used for $F$, since it covers two 1's and we would otherwise require an additional term, $AB'$, to cover $m_8$. In a similar fashion, the term $A'BD$ is used for $G$ (it is the only way to cover $m_5$) and can then be shared with $F$. Finally, we can finish covering $F$ and $H$ with $ABD$ (a prime implicant of $H$, one of the choices for covering $H$ when we treated that as a separate problem). It would be used also for $F$, rather than using another AND gate to create the prime implicant $BD$. The solution then becomes

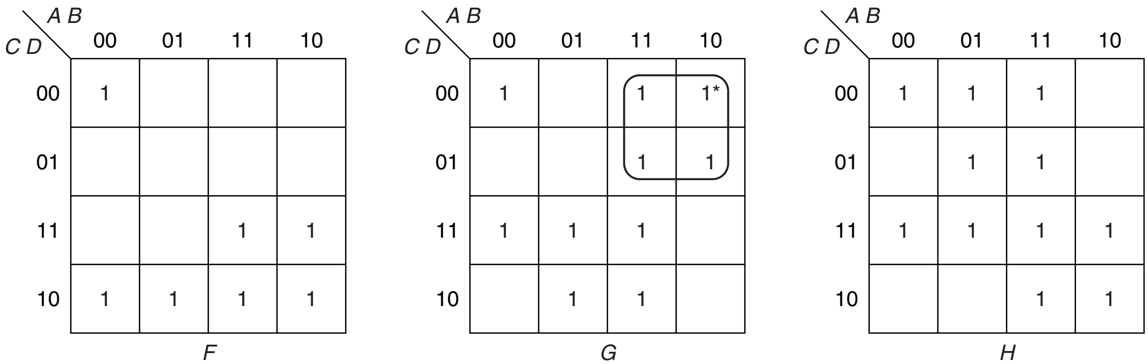
$$F = B'C + AB'C' + A'BD + ABD$$
$$G = C + A'BD$$
$$H = BC + AB'C' + ABD$$

which requires only eight gates and 22 gate inputs (a savings of two gates and three-gate inputs).

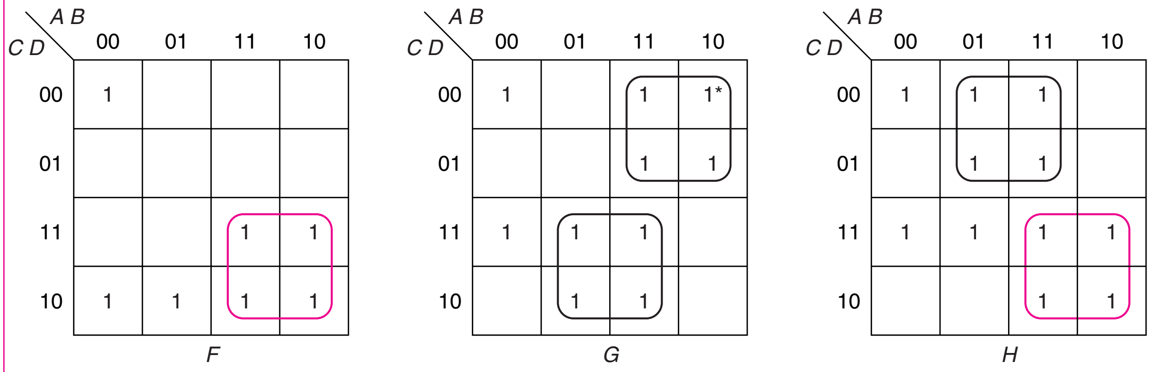
**EXAMPLE 3.35**

$$F(A, B, C, D) = \Sigma m(0, 2, 6, 10, 11, 14, 15)$$
$$G(A, B, C, D) = \Sigma m(0, 3, 6, 7, 8, 9, 12, 13, 14, 15)$$
$$H(A, B, C, D) = \Sigma m(0, 3, 4, 5, 7, 10, 11, 12, 13, 14, 15)$$

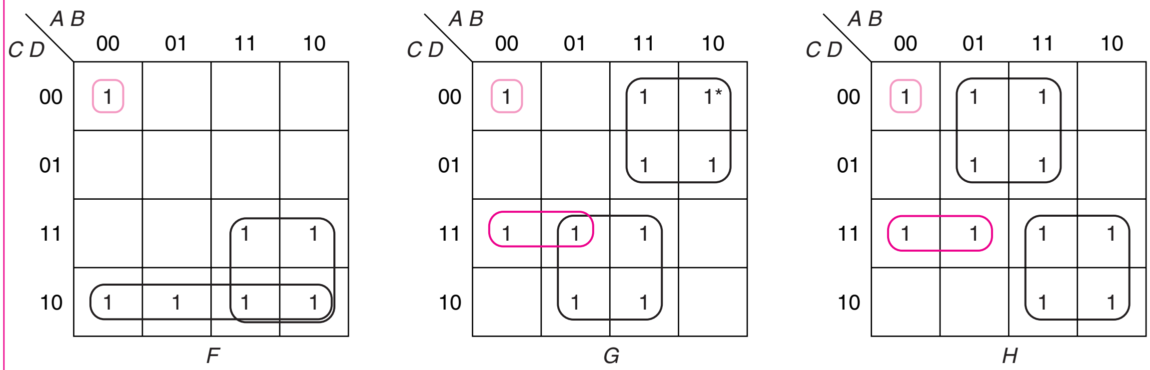
The map below shows these functions; the only 1 that is not shared and makes a prime implicant essential is $m_{11}$ in $G$. That prime implicant, $AC'$, is shown circled.



Next, we note that $AC$ is an essential prime implicant of $F$ (because of $m_{11}$ and $m_{15}$) and of $H$ (because of $m_{10}$). Furthermore, neither $m_{10}$ nor $m_{11}$ are 1's of $G$. Thus, that term is used for both $F$ and $H$. Next, we chose $BC'$ for $H$ and $BC$ for $G$; each covers four new 1's, some of which can no longer be shared (since the 1's that correspond to other functions have already been covered).




At this point, we can see that $A'B'C'D'$ can be used to cover $m_0$ in all three functions; otherwise, we would need three different three-literal terms. $A'CD$ can be used for $G$ and $H$, and, finally, $CD'$ is used for $F$, producing the following map and algebraic functions.




$$F = AC + A'B'C'D' + CD'$$
$$G = AC' + BC + A'B'C'D' + A'CD$$
$$H = AC + BC' + A'B'C'D' + A'CD$$

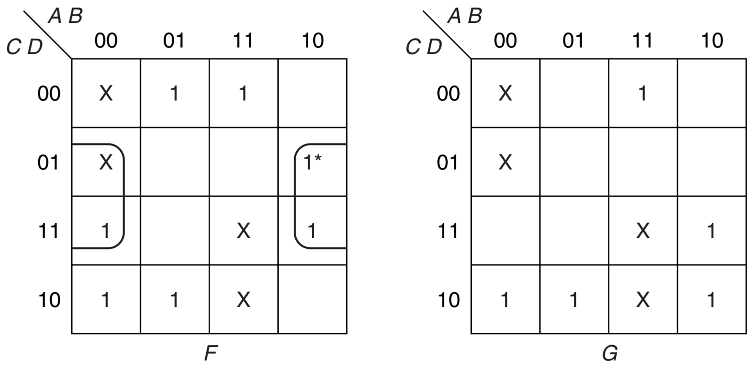
This solution requires 10 gates with 28 inputs, compared to 13 gates and 35 inputs if these were implemented separately.

**EXAMPLE 3.36**

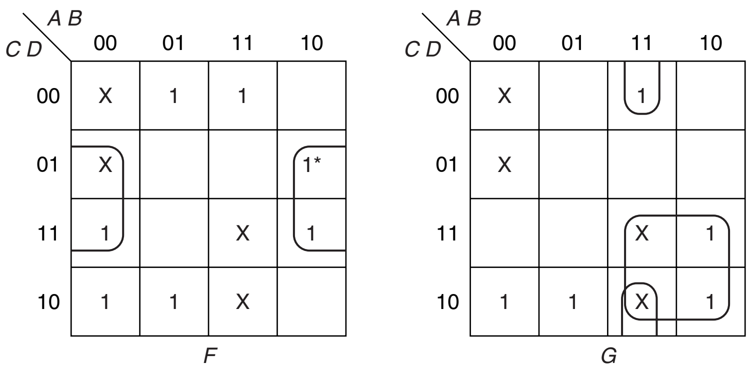
Finally, we will consider an example of a system with don't cares:

$$F(A, B, C, D) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$$
$$G(A, B, C, D) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$$

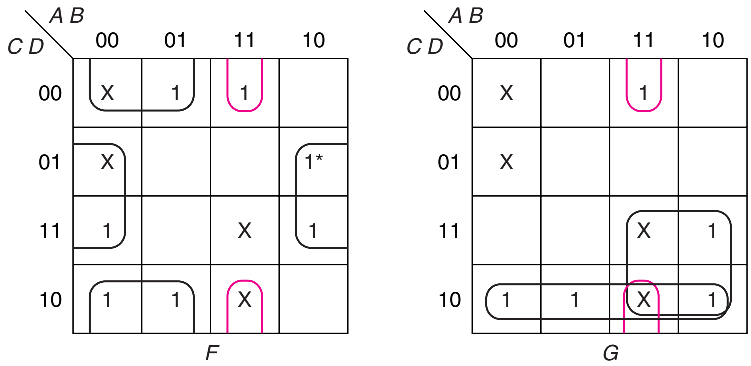
A map of the functions, with the only prime implicant made essential by a 1 that is not shared circled, $B'D$, is shown below.




Since $m_{11}$ has now been covered in $F$, we must use the essential prime implicant of $G$, $AC$, to cover $m_{11}$ there. Also, as shown on the next maps, $ABD'$ is used for $G$, since that is an essential prime implicant of $G$ and the whole term can be shared. (We will share it in the best solution.)



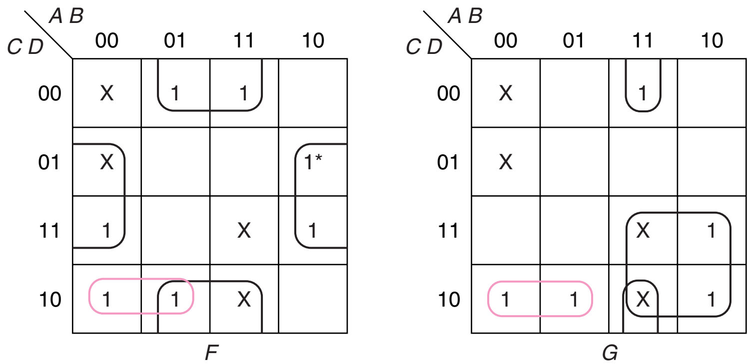

Since we need the term $ABD'$ for $G$, one approach is to use it for $F$ also. (That only costs a gate input to the OR gate.) If we do that, we could cover the rest of $F$ with $A'D'$ and the rest of $G$ with $CD'$, yielding the map and equations that follow.




$$F = B'D + ABD' + A'D'$$

$$G = AC + ABD' + CD'$$

That solution uses seven gates and 17 inputs. Another solution using the same number of gates but one more input shares $A'CD'$. That completes $G$ and then the cover of $F$ is completed with $BD'$. The maps and equations are thus:




$$F = B'D + A'CD' + BD'$$

$$G = AC + ABD' + A'CD'$$

That, too, requires seven gates, but using a three-input AND gate instead of a two-input one, bringing the total number of inputs to 18.

*[SP 10; EX 11, 12]*

# 3.2 AN ALGORITHMIC MINIMIZATION TECHNIQUE

There are two algorithmic approaches for finding all of the prime implicants of a function or of a set of functions. We will develop the method referred to as *iterated consensus* that uses the consensus operator. (See Property 13 in Chapter 2.) Once all of the prime implicants are found, *prime implicant tables* are used to find a minimum sum of products expression (or all of the minimum solutions).

### 3.2.1 Iterated Consensus for One Output

To simplify the discussion, we will first define the relationship *included in*.

> Product term $t_1$ is *included in* product term $t_2$ (written $t_1 \leq t_2$) if $t_2$ is 1 whenever $t_1$ is 1 (and elsewhere, too, if the two terms are not equal).[1]

All this really means for product terms is that either $t_1 = t_2$, or $t_1 = xt_2$ where $x$ is a literal or a product of literals. From the perspective of the map, it means that $t_1$ is a subgroup of $t_2$. If an implicant, $t_1$, is included in another implicant, $t_2$, then $t_1$ is not a prime implicant since

$$t_1 + t_2 = xt_2 + t_2 = t_2 \qquad \textbf{[P12a]}$$

The iterated consensus algorithm for single functions is as follows:

1. Find a list of product terms (implicants) that cover the function. Make sure that no term is equal to or included in any other term on the list. (These terms could be prime implicants or minterms or any other set of implicants. However, the rest of the algorithm proceeds more quickly if we start with prime implicants.)
2. For each pair of terms, $t_i$ and $t_j$, (including terms added to the list in step 3), compute $t_i ¢ t_j$.
3. If the consensus is defined, and the consensus term is not equal to or included in a term already on the list, add it to the list.
4. Delete all terms that are included in the new term added to the list.
5. The process ends when all possible consensus operations have been performed. The terms remaining on the list are ALL of the prime implicants.

Consider the following function (Example 3.4 from earlier in the chapter):

$$f(w, x, y, z) = \Sigma m(0, 4, 5, 7, 8, 11, 12, 15)$$

We chose as a starting point a set of product terms that cover the function; they include some prime implicants and a minterm, as well as other implicants.

| | |
|---|---|
| $A$ | $w'x'y'z'$ |
| $B$ | $w'xy'$ |
| $C$ | $wy'z'$ |
| $D$ | $xyz$ |
| $E$ | $wyz$ |

We numbered the terms for reference and go in the order, $B ¢ A$, $C ¢ B$, $C ¢ A$, $D ¢ C$, . . . , omitting any computation when the term has been

[1] The relationship *included in* is also applied to more complex functions than product terms, but that will not be important here.

removed from the list. When a term is removed, we cross it out. The first consensus, *B* ¢ *A*, produces $w'y'z'$; A is included in that term and can thus be removed. After the first step, the list becomes

| | |
|---|---|
| ~~*A*~~ | ~~$w'x'y'z'$~~ |
| *B* | $w'xy'$ |
| *C* | $wy'z'$ |
| *D* | $xyz$ |
| *E* | $wyz$ |
| *F* | $w'y'z'$ |

We next find *C* ¢ *B*, which creates term *G*, $xy'z'$; it is not included in any other term and no other term is included in it. There is no need to compute *C* ¢ *A*, since term A has already been removed from the list.

The complete computation is shown in Table 3.1, where each possible consensus is listed on a separate line.

The terms that remain, *B*, *D*, *E*, *H*, and *J*, that is, $w'xy$, $xyz$, $wyz$, $w'xz$, and $y'z'$, are all the prime implicants. The minimum sum of product expression(s) will use some of these, typically not all of them.

The process can be simplified by using a numeric representation of the terms. As in the truth table, a 0 represents a complemented variable, and a 1 represents an uncomplemented variable. If a variable is missing from a term, a dash (–) is used in its place so that each term has four entries. A consensus exists if there is a 1 for exactly one variable in one term and a 0 for that variable in the other. The consensus term has a 1 for a variable if one term has a 1 and the other either a 1 or a –; it has a 0 if one term has a 0 and the other a 0 or a –, and a – if one term has a 0 and the other a 1 or if both terms have a –. For the function of Table 3.1, the process becomes that of Table 3.2 (where we have not left lines for consensus operations that are undefined).

The five terms remaining in Table 3.2 are the same as those in Table 3.1.

**Table 3.1** Computing the prime implicants.

| | | | |
|---|---|---|---|
| ~~*A*~~ | ~~$w'x'y'z'$~~ | | |
| *B* | $w'xy'$ | | |
| ~~*C*~~ | ~~$wy'z'$~~ | | |
| *D* | $xyz$ | | |
| *E* | $wyz$ | | |
| ~~*F*~~ | ~~$w'y'z'$~~ | *B* ¢ *A* ≥ *A* (remove *A*) | |
| ~~*G*~~ | ~~$xy'z'$~~ | *C* ¢ *B* | |
| | | *D* ¢ *C* | undefined |
| *H* | $w'xz$ | *D* ¢ *B* | |
| | | *E* ¢ *D* | undefined |
| | | *E* ¢ *C* | undefined |
| | | *E* ¢ *B* | undefined |
| | | *F* ¢ *E* | undefined |
| | | *F* ¢ *D* | undefined |
| *J* | $y'z'$ | *F* ¢ *C* ≥ *G*, *F*, *C* (remove G, F, C) | |
| | | *H* ¢ *E* = *D* (do not add) | |
| | | *H* ¢ *D* | undefined |
| | | *H* ¢ *B* | undefined |
| | | *J* ¢ *H* = *B* (do not add) | |
| | | *J* ¢ *E* | undefined |
| | | *J* ¢ *D* | undefined |
| | | *J* ¢ *B* | undefined |

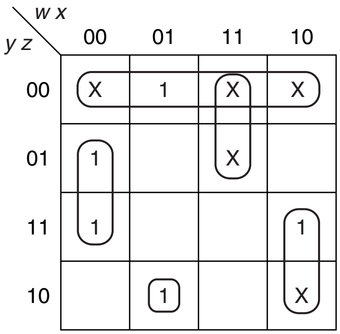
**Table 3.2** Numeric computation of prime implicants.

| | | | | | | |
|---|---|---|---|---|---|---|
| ~~*A*~~ | ~~0~~ | ~~0~~ | ~~0~~ | ~~0~~ | | |
| *B* | 0 | 1 | 0 | – | | |
| ~~*C*~~ | ~~1~~ | ~~–~~ | ~~0~~ | ~~0~~ | | |
| *D* | – | 1 | 1 | 1 | | |
| *E* | 1 | – | 1 | 1 | | |
| ~~*F*~~ | ~~0~~ | ~~–~~ | ~~0~~ | ~~0~~ | *B* ¢ *A* ≥ *A* | |
| ~~*G*~~ | ~~–~~ | ~~1~~ | ~~0~~ | ~~0~~ | *C* ¢ *B* | |
| *H* | 0 | 1 | – | 1 | *D* ¢ *B* | (*D* ¢ *C* undefined) |
| | | | | | (*E* ¢ *D*, *E* ¢ *C*, *E* ¢ *B*, *F* ¢ *E*, *F* ¢ *D* undefined) | |
| *J* | – | – | 0 | 0 | *F* ¢ *C* ≥ *G*, *F*, *C* | |
| | | | | | (*H* ¢ *E* = *D*; *H* ¢ *D*, *H* ¢ *B* undefined; *J* ¢ *H* = *B*; *J* ¢ *E*, *J* ¢ *D*, *J* ¢ *B* undefined) | |

If there are don't cares in the function, all of them must be included in at least one of the terms to start the process. The resulting list of prime implicants will then include all possible prime implicants (including possibly some that are made up of only don't cares). The prime implicant table will then allow us to choose the minimum cover.

**EXAMPLE 3.37**

$g(w, x, y, z) = \Sigma m(1, 3, 4, 6, 11) + \Sigma d(0, 8, 10, 12, 13)$




Using the map above, we chose the following list of implicants as a starting point

| | | | | | |
|---|---|---|---|---|---|
| *A* | *y′z′* | – | – | 0 | 0 |
| *B* | *w′x′z* | 0 | 0 | – | 1 |
| *C* | *w′xyz′* | 0 | 1 | 1 | 0 |
| *D* | *wxy′* | 1 | 1 | 0 | – |
| *E* | *wx′y* | 1 | 0 | 1 | – |

All of these, except the third, are prime implicants. It does not matter what set of terms we start with (as long as all of the 1's and don't cares are included in at least one term); we will get the same result. By choosing a pretty good cover, we will create few if any extraneous terms. The process then proceeds

| | | | | | |
|---|---|---|---|---|---|
| *A* | – | – | 0 | 0 | |
| *B* | 0 | 0 | – | 1 | |
| ~~*C*~~ | ~~0~~ | ~~1~~ | ~~1~~ | ~~0~~ | |
| *D* | 1 | 1 | 0 | – | |
| *E* | 1 | 0 | 1 | – | |
| *F* | 0 | 0 | 0 | – | *B* ¢ *A*<br>*C* ¢ *B* undefined |
| *G* | 0 | 1 | – | 0 | *C* ¢ *A* ≥ *C*<br>*D* ¢ *B*, *D* ¢ *A*, *E* ¢ *D* undefined |

*H* – 0 1 1 *E* ¢ *B*
*J* 1 0 – 0 *E* ¢ *A*

*F* ¢ *E*, *F* ¢ *D*, *F* ¢ *B*, *F* ¢ *A* undefined, *G* ¢ *F* = 0 – 0 0 ≤ *A*;
*G* ¢ *E* undefined; *G* ¢ *D* ≤ *A*; *G* ¢ *B*, *G* ¢ *A*, *H* ¢ *G* undefined;
*H* ¢ *F* = *B*; *H* ¢ *E*, *H* ¢ *D*, *H* ¢ *B*, *H* ¢ *A*, undefined; *J* ¢ *H* = *E*;
*J* ¢ *G*, *J* ¢ *E*, *J* ¢ *B*, *J* ¢ *A* undefined; *J* ¢ *F* ≤ *D*, *J* ¢ *D* ≤ *A*

Thus, all terms but term *C* are prime implicants. Although there are eight prime implicants, only three are used in any minimum solution (as we will see in the next section).

### 3.2.2 Prime Implicant Tables for One Output

Once we have a complete list of prime implicants, a table is constructed with one row for each prime implicant and one column for each minterm included in the function (not don't cares). An X is entered in the column of a minterm that is covered by that prime implicant. Thus, for the prime implicants found in Table 3.1, the prime implicant (PI) table is shown in Table 3.3.

**Table 3.3** A prime implicant (PI) table.

| PI | Numeric | $ | | 0 | 4 | 5 | 7 | 8 | 11 | 12 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *w′xy′* | 0 1 0 – | 4 | *B* | | X | X | | | | | |
| *xyz* | – 1 1 1 | 4 | *D* | | | | X | | | | X |
| *wyz* | 1 – 1 1 | 4 | *E* | | | | | | X | | X |
| *w′xz* | 0 1 – 1 | 4 | *H* | | | X | X | | | | |
| *y′z′* | – – 0 0 | 3 | *J* | X | X | | | X | | X | |

The first column is the list of prime implicants in algebraic form; the second is in numeric form. The latter makes it easy to find a list of minterms that are covered by this term, since each – can represent either a 0 or a 1. The third column is the number of gate inputs when that term is used in a two-level circuit, that is, just one for each literal plus one for the input to the output gate (OR). The fourth column is just the label (to save writing the whole term later).

Our job is to find a minimum set of rows such that using only these rows, every column has at least one X, that is, all of the minterms are included in the expression. If there is more than one set, the total number of gate inputs ($ column) is minimized. The first step in the process is to find essential prime implicants. They correspond to rows where the X is the only one in at least one column. Those squares are shaded; the minterms covered by each of the essential prime implicants are checked off; and an asterisk is placed next to the prime implicant as shown in Table 3.4.

**Table 3.4** Finding essential prime implicants.

| PI | Numeric | $ | | 0 √ | 4 √ | 5 | 7 | 8 √ | 11 √ | 12 √ | 15 √ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w'xy'$ | 0 1 0 – | 4 | *B* | | X | X | | | | | |
| $xyz$ | – 1 1 1 | 4 | *D* | | | | X | | | | X |
| $wyz$* | 1 – 1 1 | 4 | *E* | | | | | | X | | X |
| $w'xz$ | 0 1 – 1 | 4 | *H* | | | X | X | | | | |
| $y'z'$* | – – 0 0 | 3 | *J* | X | X | | | X | | X | |

Note that all of the minterms covered by the essential prime implicants are checked, not just those columns with shaded X's. The table is now reduced to that of Table 3.5 by eliminating the essential prime implicant rows and the covered minterms.

In this simple example, the answer is apparent. Prime implicant *H* covers the remaining 1's; any other solution would require at least two more terms, for a total of four. Thus, the solution is

$$E + J + H = wyz + y'z' + w'xz$$

Before looking at some more complex examples that will require us to develop additional techniques, we will complete Example 3.37 (with don't cares), for which we have already developed a list of prime implicants. The only thing that is different from the first example is that we only have columns for minterms included in the function—not for don't cares. That is really what happened in the reduced table above; the columns that were eliminated correspond to minterms that became don't cares after having chosen the essential prime implicants (as in Map Method 3).

**Table 3.5** The reduced table.

| $ | | 5 | 7 |
|---|---|---|---|
| 4 | *B* | X | |
| 4 | *D* | | X |
| 4 | *H* | X | X |

**EXAMPLE 3.37**

**(Continued)**

| PI | | $ | | 1 | 3 | 4 √ | 6 √ | 11 |
|---|---|---|---|---|---|---|---|---|
| $y'z'$ | – – 0 0 | 3 | *A* | | | X | | |
| $w'x'z$ | 0 0 – 1 | 4 | *B* | X | X | | | |
| $wxy'$ | 1 1 0 – | 4 | *D* | | | | | |
| $wx'y$ | 1 0 1 – | 4 | *E* | | | | | X |
| $w'x'y'$ | 0 0 0 – | 4 | *F* | X | | | | |
| $w'xz'$* | 0 1 – 0 | 4 | *G* | | | X | X | |
| $x'yz$ | – 0 1 1 | 4 | *H* | | X | | | X |
| $wx'z'$ | 1 0 – 0 | 4 | *J* | | | | | |

The first thing to note about this table is that rows *D* and *J* have no X's in them; they correspond to prime implicants that cover only don't cares. *G* is essential, as indicated by the shading. We can now eliminate rows, *D*, *J*, and *G* and columns 4 and 6, producing the reduced table:

| $ | | 1 | 3 | 11 |
|---|---|---|---|---|
| 3 | *A* | | | |
| 4 | *B* | X | X | |
| 4 | *E* | | | X |
| 4 | *F* | X | | |
| 4 | *H* | | X | X |

Note that row *A* has no X's; the minterm that it covered was already covered by the essential prime implicant. There are several ways to proceed from here. By looking at the table, we can see that we need at least one prime implicant that covers two minterms (either *B* or *H*). In either case, one minterm is left. There are three solutions:

$$G + B + E = w'xz' + w'x'z + wx'y$$
$$G + B + H = w'xz' + w'x'z + x'yz$$
$$G + H + F = w'xz' + x'yz + w'x'y'$$

All of these are equal cost, since each of the prime implicants used have the same number of literals. (We will see in other examples that some of the covers that use the same number of terms may have a different number of literals.)

If we are looking for only one of the minimum solutions, instead of all of them, we can often reduce a prime implicant table by removing *dominated* or equal rows. A row dominates another if the term it represents costs no more than the other and has X's in every column that the dominated row does (and possibly more).

**EXAMPLE 3.38**

In Example 3.37, row *F* is dominated by *B*, and row *E* is dominated by *H*. Removing them, the table reduces to

| $ | | 1 | 3 | 11 |
|---|---|---|---|---|
| 4 | *B* | X | X | |
| 4 | *H* | | X | X |

and the only solution produced is

$$G + B + H = w'xz' + w'x'z + x'yz$$

Finally, a third approach, utilizes the table we have obtained after removing the essential prime implicants, but before removing dominated and equal rows. Create a product of sums expression by producing one term for each column. For the last example, the expression is

$$(B + F)(B + H)(E + H)$$

Minterm 1 must be covered by $B$ or $F$, minterm 3 by $B$ or $H$, and minterm 11 by $E$ or $H$. Expanding that expression to sum of products form, we get

$$(B + FH)(E + H) = BE + BH + EFH + FH$$

Each product term corresponds to a set of prime implicants that could be used to cover the function. Obviously, $EFH$ involves more terms; the other three are, of course, the solutions that we found.

We are now ready to look at some more complex examples.

**EXAMPLE 3.39**

$f(a, b, c, d) = \Sigma m(1, 3, 4, 6, 7, 9, 11, 12, 13, 15)$

Either from the map, or using iterated consensus[2], we found all of the prime implicants and constructed the following table:

| | | | | √ | √ | | | | √ | √ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | **$** | | **1** | **3** | **4** | **6** | **7** | **9** | **11** | **12** | **13** | **15** |
| $b'd$* | – 0 – 1 | 3 | $A$ | X | X | | | | X | X | | | |
| $cd$ | – – 1 1 | 3 | $B$ | | X | | | X | | X | | | X |
| $ad$ | 1 – – 1 | 3 | $C$ | | | | | | X | X | | X | X |
| $abc'$ | 1 1 0 – | 4 | $D$ | | | | | | | | X | X | |
| $bc'd'$ | – 1 0 0 | 4 | $E$ | | | X | | | | | X | | |
| $a'bd'$ | 0 1 – 0 | 4 | $F$ | | | X | X | | | | | | |
| $a'bc$ | 0 1 1 – | 4 | $G$ | | | | X | X | | | | | |

There is one essential prime implicant, $b'd$, as shown in the table above. The table is then reduced, by eliminating that row and the terms that have been covered.

| **$** | | **4** | **6** | **7** | **12** | **13** | **15** |
|---|---|---|---|---|---|---|---|
| 3 | $B$ | | | X | | | X |
| 3 | $C$ | | | | | X | X |
| 4 | $D$ | | | | X | X | |
| 4 | $E$ | X | | | X | | |
| 4 | $F$ | X | X | | | | |
| 4 | $G$ | | X | X | | | |

[2]An effective approach is to map the function and find as many prime implicants as possible. Then, use iterated consensus to check that none have been left out.

The reduced table has two X's in each column and two X's in each row. Since there are six minterms to be covered, we need at least three prime implicants. Also, since *B* and *C* cost less than the other terms, we should try to use them. A careful study of the table will show that there are two covers that use three terms, each of which uses one of the less costly terms, namely,

$A + B + D + F$
$A + C + E + G.$

(We cannot complete the cover with three terms in addition to *A* by using both of the less costly rows, since they only cover three 1's between them.) The more systematic approach is to choose one of the minterms that can be covered in the fewest number of ways, for example, 4. We then recognize that we must choose either *E* or *F* in order to cover minterm 4. We will next derive a minimum solution using each of those and compare them. After we choose *E*, the table reduces to

| **$** | | **6** | **7** | **13** | **15** |
|---|---|---|---|---|---|
| 3 | *B* | | X | | X |
| 3 | *C* | | | X | X |
| 4 | *D* | | | X | |
| 4 | *F* | X | | | |
| 4 | *G* | X | X | | |

Note that row *D* is dominated by *C* and costs more than *C*. It can be removed. (This row is shaded in the table above.) If that is eliminated, *C* is needed to cover minterm 13. (It also covers minterm 15.) Now, only minterms 6 and 7 need to be covered; the only way to do that with one term is with *G*. That produces the solution

$A + C + E + G.$

Row *F* is also dominated (by *G*); but those two terms cost the same. In general (although not in this example), we risk losing other equally good solutions if we delete dominated rows that are not more expensive.

If, instead, we chose prime implicant *F* to cover minterm 4, we would have

| **$** | | **7** | **12** | **13** | **15** |
|---|---|---|---|---|---|
| 3 | *B* | X | | | X |
| 3 | *C* | | | X | X |
| 4 | *D* | | X | X | |
| 4 | *E* | | X | | |
| 4 | *G* | X | | | |

Row *G* is dominated by row *B* and costs more. Thus, prime implicant *B* is needed to cover the function. With only minterms 12 and 13 left, we must choose term *D*, giving the other solution

$$A + F + B + D.$$

Finally, we could go back to the second table (with six minterms) and consider the prime implicants needed to cover each minterm. That produces the following expression

$$\begin{aligned}&(E + F)(F + G)(B + G)(D + E)(C + D)(B + C)\\ &\quad = (F + EG)(B + CG)(D + CE)\\ &\quad = (BF + BEG + CFG + CEG)(D + CE)\\ &\quad = \underline{BDF} + BDEG + CDFG + CDEG + BCEF\\ &\qquad + BCEG + CEFG + \underline{CEG}\end{aligned}$$

Any of these eight combinations could be used; but only the two underlined correspond to three terms (in addition to *A*). This approach produces the same two minimum solutions.

**EXAMPLE 3.40**

$$g(a, b, c, d) = \Sigma m(0, 1, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15)$$

Using the map of Example 3.14, we came up with the list of nine prime implicants shown in the table below. (We can check that this list is complete and that all of these are prime implicants by using them as the starting point for iterated consensus. If we do that, no new terms are produced in this example.) We do not need a cost column since all terms consist of two literals.

| | | **0** | **1** | **3** | **4** | **6** | **7** | **8** | **9** | **11** | **12** | **13** | **14** | **15** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| – – 0 0 | *A* | X | | | X | | | X | | | X | | | |
| – 0 0 – | *B* | X | X | | | | | X | X | | | | | |
| – 0 – 1 | *C* | | X | X | | | | | X | X | | | | |
| – 1 – 0 | *D* | | | | X | X | | | | | X | | X | |
| – 1 1 – | *E* | | | | | X | X | | | | | | X | X |
| – – 1 1 | *F* | | | X | | | X | | | X | | | | X |
| 1 1 – – | *G* | | | | | | | | | | X | X | X | X |
| 1 – 0 – | *H* | | | | | | | X | X | | X | X | | |
| 1 – – 1 | *J* | | | | | | | | X | X | | X | | X |

All of the minterms are covered by at least two prime implicants (some by as many as four). We will choose one of the columns that has only two X's and try to minimize the function first using one of them, and then using the other. For this example, we will use either term *A* or term *B* to cover $m_0$; first we will use *A* and reduce the table by removing the minterms covered by *A*.

| | | √ | √ | √ | √ | √ | √ | | √ | √ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | **1** | **3** | **6** | **7** | **9** | **11** | **13** | **14** | **15** |
| – 0 0 – | *B* | X | | | | X | | | | |
| – 0 – 1* | *C* | X | X | | | X | X | | | |
| – 1 – 0 | *D* | | | X | | | | | X | |
| – 1 1 –* | *E* | | | X | X | | | | X | X |
| – – 1 1 | *F* | | X | | X | | X | | | X |
| 1 1 – – | *G* | | | | | | | X | X | X |
| 1 – 0 – | *H* | | | | | X | | X | | |
| 1 – – 1 | *J* | | | | | X | X | X | | X |

Row *B* is dominated by *C*; and row *D* is dominated by *E*. Although row *H* is dominated by *J*, we will leave that for now. Thus, we will choose terms *C* and *E*. Reducing the table once more, we get

| | | **13** |
|---|---|---|
| – – 1 1 | *F* | |
| 1 1 – – | *G* | X |
| 1 – 0 – | *H* | X |
| 1 – – 1 | *J* | X |

Obviously, any of *G*, *H*, or *J* could be used to cover minterm 13. Notice that row *H*, even though it was dominated, is used in one of the minimum solutions. We must now ask if that might be true of row *B* or row *D*. To be sure, we must go back to the previous table and see what happens if we don't eliminate them. We will choose *B* (rather than *C* to cover $m_1$ and $m_4$) and *E* and leave it to the reader to do it for *D* (rather than *E*) and *C*. The reduced table now becomes

| | | **3** | **11** | **13** |
|---|---|---|---|---|
| – 0 – 1 | *C* | X | X | |
| – – 1 1 | *F* | X | X | |
| 1 1 – – | *G* | | | X |
| 1 – 0 – | *H* | | | X |
| 1 – – 1 | *J* | | X | X |

Now, however, we need two more prime implicants to complete the cover, a total of five. Those solutions cannot be minimum since we found three (so far) with only four terms. Thus, the three minimum solutions using term *A* are

$$f = c'd' + b'd + bc + ab$$
$$f = c'd' + b'd + bc + ac'$$
$$f = c'd' + b'd + bc + ad$$

We will now go back and repeat the process, starting with term *B*. We can eliminate row *A*, since we already found all minimum solutions using row *A*.

| | | 3 | √ 4 | √ 6 | 7 | 11 | √ 12 | 13 | √ 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|
| – 0 – 1 | *C* | X | | | | X | | | | |
| – 1 – 0* | *D* | | X | X | | | X | | X | |
| – 1 1 – | *E* | | | X | X | | | | X | X |
| – – 1 1 | *F* | X | | | X | X | | | | X |
| 1 1 – – | *G* | | | | | | X | X | X | X |
| 1 – 0 – | *H* | | | | | | X | X | | |
| 1 – – 1 | *J* | | | | | X | | X | | X |

Row *D* is now required. We will reduce the table one more time.

| | | 3 | 7 | 11 | 13 | 15 |
|---|---|---|---|---|---|---|
| – 0 – 1 | *C* | X | | X | | |
| – 1 1 – | *E* | | X | | | X |
| – – 1 1 | *F* | X | X | X | | X |
| 1 1 – – | *G* | | | | X | X |
| 1 – 0 – | *H* | | | | X | |
| 1 – – 1 | *J* | | | X | X | X |

It is clear now that *F* is necessary, covering all the remaining minterms except $m_{13}$. (Otherwise, we would need both *C* and *E*, and would still leave $m_{13}$ uncovered.) As before, prime implicants *G*, *H*, and *J* could be used to complete the function. The three solutions using term *B* are thus

$$f = b'c' + bd' + cd + ab$$
$$f = b'c' + bd' + cd + ac'$$
$$f = b'c' + bd' + cd + ad$$

giving a total of six solutions. *[SP 11; EX 13]*

### 3.2.3 Iterated Consensus for Multiple Output Problems

The iterated consensus algorithm needs only minor modifications to produce all of the terms that may be used for sum of product expressions for multiple output problems. Candidates are terms that are prime implicants of any one function or prime implicants of the product of functions. (Although we did not make use of this property when we considered the map approach, if we look back, we will find that all terms that were shared between two functions were indeed prime implicants of the product of those two functions and terms that were shared among three functions were prime implicants of the product of the three functions.)

To begin the iterated consensus procedure, we must either start with minterms or include not only a cover of each function, but also a cover of all possible products of functions. We will follow the first approach in this example and use the second later. To each product term on our list for iterated consensus, we add a tag section with a dummy variable for each output. That tag contains a 0 (complemented output variable) if the term is not an implicant of that function and a blank if it is. We will illustrate the process with the functions of Example 3.29:

$$f(a, b, c) = \Sigma m(2, 3, 7)$$
$$g(a, b, c) = \Sigma m(4, 5, 7)$$

The initial list then becomes

| | | | | | |
|---|---|---|---|---|---|
| $a'$ | $b$ | $c'$ | $g'$ | 0 1 0 | – 0 |
| $a'$ | $b$ | $c$ | $g'$ | 0 1 1 | – 0 |
| $a$ | $b'$ | $c'$ | $f'$ | 1 0 0 | 0 – |
| $a$ | $b'$ | $c$ | $f'$ | 1 0 1 | 0 – |
| $a$ | $b$ | $c$ | | 1 1 1 | – – |

We now proceed as before, taking the consensus of each pair of terms (including the tag), adding new terms and deleting terms included in others. The only new rule is that terms that have an all 0 tag section are also deleted. (They correspond to a grouping made of a 1 from one function with a 1 from the other function; they are not implicants of either function.) Note that the tag never affects whether or not a consensus exists, since there are no 1's in the tag section.

We now proceed, as in Table 3.6.

**Table 3.6** Iterated consensus for multiple output functions.

| | | | | |
|---|---|---|---|---|
| ~~*A*~~ | ~~0 1 0~~ | ~~– 0~~ | | |
| ~~*B*~~ | ~~0 1 1~~ | ~~– 0~~ | | |
| ~~*C*~~ | ~~1 0 0~~ | ~~0 –~~ | | |
| ~~*D*~~ | ~~1 0 1~~ | ~~0 –~~ | | |
| *E* | 1 1 1 | – – | | |
| *F* | 0 1 – | – 0 | *B* ¢ *A* ≥ *B*, *A* | |
| *G* | 1 0 – | 0 – | *D* ¢ *C* ≥ *D*, *C* | |
| *H* | – 1 1 | – 0 | *F* ¢ *E* | |
| *J* | 1 – 1 | 0 – | *G* ¢ *E* | (*G* ¢ *F* undefined) |
| | | *H* ¢ *G* zero tag; *H* ¢ *F*, *H* ¢ *E* undefined | | |
| | | *J* ¢ *H*, *J* ¢ *F* zero tag; *J* ¢ *G*, *J* ¢ *E* undefined | | |

In Table 3.7, the prime implicant table is constructed in two sections, one for each function. An X is only placed in the column of a function for which the term is an implicant. (For example, there is no X in column 7 of *G* or for term *H*.) Essential prime implicants are found as before ($a'b$ for *f* and $ab'$ for *g*).

**Table 3.7** A multiple output prime implicant table.

| | | | f | | | g | | |
|---|---|---|---|---|---|---|---|---|
| | | | √ | √ | | √ | √ | |
| | $ | | 2 | 3 | 7 | 4 | 5 | 7 |
| 1 1 1 | 4 | *E* | | | X | | | X |
| 0 1 –* | 3 | *F* | X | X | | | | |
| 1 0 –* | 3 | *G* | | | | X | X | |
| – 1 1 | 3 | *H* | | X | X | | | |
| 1 – 1 | 3 | *J* | | | | | X | X |

The table is then reduced as in Table 3.8.

**Table 3.8** A reduced prime implicant table.

| | | | f | g |
|---|---|---|---|---|
| | $ | | 7 | 7 |
| 1 1 1 | 4 | *E* | X | X |
| – 1 1 | 3 | *H* | X | |
| 1 – 1 | 3 | *J* | | X |

Now, it is clear that we can use term *E* to cover both functions, rather than two separate terms, even though *E* costs 4 and the others cost 3. Indeed, the cost to use a term in each function after the first is only 1, the input to another OR gate. (We only build one AND gate for that term.) The solution using *E* thus costs 5, compared to 6 for a solution that uses both *H* and *J*. The solution is thus,

$$f = a'b + abc$$
$$g = ab' + abc$$

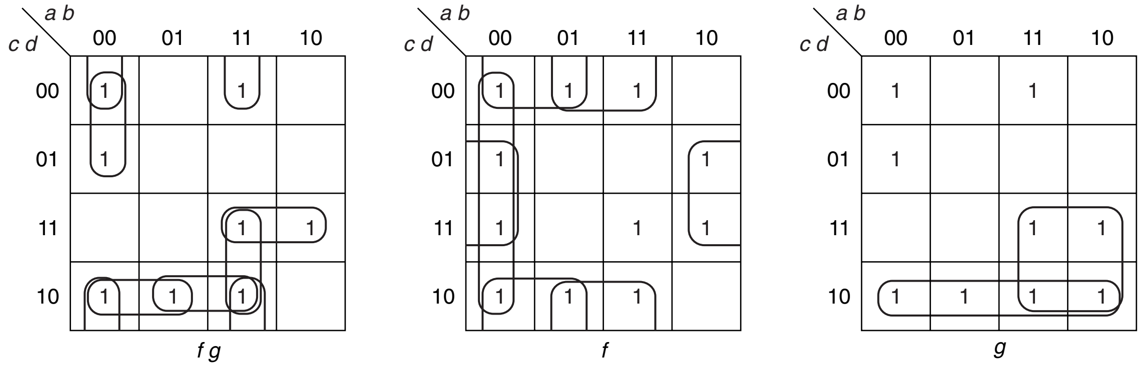
**EXAMPLE 3.41**

We will consider functions from Example 3.36, a two-output problem with don't cares.

$$f(a, b, c, d) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$$
$$g(a, b, c, d) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$$

To obtain the list of prime implicants to include in the prime implicant table, we can start with minterms, treating all don't cares as 1's and work the iterated consensus algorithm. It is very time-consuming and prone to error (although it would be fairly straightforward to write a computer routine to process it).[3] The other approach is to map *fg* (the product of the two functions), find all of the prime implicants of that plus those terms that are only prime implicants of one of the functions. The following maps show the prime implicants of *fg* and those of *f* and *g* that are not prime implicants of both functions, where all don't cares have been made 1 on the maps, since we must include all prime implicants that cover don't cares, as well.

[3] Another example of this approach is given in Solved Problem 12d.




This produces the following prime implicant table

| | | | f | | | | | | | g | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | √ | | | √ | √ | | | | | | √ |
| | | | 2 | 3 | 4 | 6 | 9 | 11 | 12 | 2 | 6 | 10 | 11 | 12 |
| 0 0 0 – | *A* | 4 | | | | | | | | | | | | |
| 0 0 – 0 | *B* | 4 | X | | | | | | | X | | | | |
| 0 – 1 0 | *C* | 4 | X | | | X | | | | X | X | | | |
| – 1 1 0 | *D* | 4 | | | | X | | | | | X | | | |
| 1 – 1 1 | *E* | 4 | | | | | | X | | | | | X | |
| 1 1 1 – | *F* | 4 | | | | | | | | | | | | |
| 1 1 – 0* | *G* | 4 | | | | | | | X | | | | | X |
| – 1 – 0 | *H* | 3 | | | X | X | | | X | | | | | |
| 0 – – 0 | *J* | 3 | X | | X | X | | | | | | | | |
| 0 0 – – | *K* | 3 | X | X | | | | | | | | | | |
| – 0 – 1* | *L* | 3 | | X | | | X | X | | | | | | |
| – – 1 0 | *M* | 3 | | | | | | | | X | X | X | | |
| 1 – 1 – | *N* | 3 | | | | | | | | | | X | X | |

Note that the table is divided into three sections of rows. The first (*A* to *G*) includes the prime implicants of *fg,* that is terms that are eligible for sharing. The second section contains the prime implicants of *f* that are not also prime implicants of *g*, and the last section contains those of *g* that are not prime implicants of *f*. Notice that rows *A* and *F* have no *X*'s; they are prime implicants made up of only don't cares. (Of course, there are no columns corresponding to the don't cares.)

Row *L*, *b*′*d*, is an essential prime implicant of *f* and row *G*, *abd*′ is an essential prime implicant of *g*. Although the latter is also useful for *f*, it is not essential and we may or may not want to use it. The reduced table is shown next.

| | | | f | | | | g | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | √ | √ |
| | | | 2 | 4 | 6 | 12 | 2 | 6 | 10 | 11 |
| 0 0 – 0 | *B* | 4 | X | | | | X | | | |
| 0 – 1 0 | *C* | 4 | X | | X | | X | X | | |
| – 1 1 0 | *D* | 4 | | | X | | | X | | |
| 1 – 1 1 | *E* | 4 | | | | | | | | X |
| 1 1 – 0* | *G* | 1 | | | | X | | | | |
| – 1 – 0 | *H* | 3 | | X | X | X | | | | |
| 0 – – 0 | *J* | 3 | X | X | X | | | | | |
| 0 0 – – | *K* | 3 | X | | | | | | | |
| – – 1 0 | *M* | 3 | | | | | X | X | X | |
| 1 – 1 – | *N* | 3 | | | | | | | X | X |

Note that the cost for term *G* has been reduced to 1, since the AND gate has already been built; we only need an input to the OR gate. Term *E* is dominated by and costs more than term *N*, and can be eliminated. (It will never be part of a minimum solution, since it is less expensive to use term *N*.) That makes term *N, ac*, necessary for *g*. With these two terms and the minterms they cover removed, the table reduces to:

| | | | f | | | | g | |
|---|---|---|---|---|---|---|---|---|
| | | | 2 | 4 | 6 | 12 | 2 | 6 |
| 0 0 – 0 | *B* | 4 | X | | | | X | |
| 0 – 1 0 | *C* | 4 | X | | X | | X | X |
| – 1 1 0 | *D* | 4 | | | X | | | X |
| 1 1 – 0 | *G* | 1 | | | | X | | |
| – 1 – 0 | *H* | 3 | | X | X | X | | |
| 0 – – 0 | *J* | 3 | X | X | X | | | |
| 0 0 – – | *K* | 3 | X | | | | | |
| – – 1 0 | *M* | 3 | | | | | X | X |

Neither *B* nor *D* would be used for *g*, unless we used both of them. However, then we could use *C*, which would cover both minterms in *g* and also be shared with *f* (covering the same minterms in *f* that were covered by *B* and *D*). At this point, we are left with two choices. Either we choose term *G* for *f* (at a cost of 1), which would then allow us to finish covering *f* with term *J* (a total of one new gate, four inputs). Then, term *M* would be used for *g* (one new gate, three inputs). The other choice is to use term *C* to cover the 1's in both *f* and *g* and then use *H* to cover the remaining 1's in *f*. That would also require two new gates and a total of eight inputs—three for *H*, four for *C* in *f*, and one more for the OR gate in *g*. (That solution requires one extra gate input.) Notice that the cost column only refers to the number of inputs for one function; we need to add 1 for each additional function in which it is used.

Thus, the minimum solution is, as we found in Example 3.36,

$$f = b'd + abd' + a'd'$$
$$g = ac + abd' + cd'$$

**EXAMPLE 3.42**

$$f(x, y, z) = \Sigma m(0, 2, 5, 6, 7)$$
$$g(x, y, z) = \Sigma m(2, 3, 5, 6, 7)$$
$$h(x, y, z) = \Sigma m(0, 2, 3, 4, 5)$$

We start by listing all of the minterms used for any of the functions, including the tag, and then perform the iterated consensus algorithm to find all of the prime implicants.

| | | | | | | |
|---|---|---|---|---|---|---|
| ~~*A*~~ | ~~0 0 0~~ | ~~– 0 –~~ | *H* | 0 – 0 | – 0 – | *B* ¢ *A* ≥ *A* |
| *B* | 0 1 0 | – – – | *J* | 0 1 – | 0 – – | *C* ¢ *B* ≥ *C* |
| ~~*C*~~ | ~~0 1 1~~ | ~~0 – –~~ | *K* | 1 0 – | 0 0 – | *E* ¢ *D* ≥ *D* |
| ~~*D*~~ | ~~1 0 0~~ | ~~0 0 –~~ | *L* | – 1 0 | – – 0 | *F* ¢ *B* ≥ *F* |
| *E* | 1 0 1 | – – – | *M* | 1 – 1 | – – 0 | *G* ¢ *E* ≥ *G* |
| ~~*F*~~ | ~~1 1 0~~ | ~~– – 0~~ | *N* | – 0 0 | 0 0 – | *K* ¢ *H* |
| ~~*G*~~ | ~~1 1 1~~ | ~~– – 0~~ | *P* | 1 1 – | – – 0 | *M* ¢ *L* |
| | | | ~~*Q*~~ | ~~– 1 1~~ | ~~0 – 0~~ | *M* ¢ *J* |
| | | | *R* | – 1 – | 0 – 0 | *Q* ¢ *L* ≥ *Q* |

We did not show any term that produced an all 0 tag section and we did not list the consensus operations that led to undefined terms or to terms included in other terms already on the list. This leaves a total of 10 prime implicants (of one of the functions or the product of functions). Note that two of the minterms remain, since they can be used for all three functions and are not part of any one larger group in all three. The prime implicant table becomes

| | | | ***f*** | | | | | ***g*** | | | | | ***h*** | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | √ | √ | | | | √ | √ | | √ | √ | | √ | √ | | |
| | | | 0 | 2 | 5 | 6 | 7 | 2 | 3 | 5 | 6 | 7 | 0 | 2 | 3 | 4 | 5 |
| 0 1 0 | 4 | *B* | | X | | | | X | | | | | | X | | | |
| 1 0 1 | 4 | *E* | | | X | | | | | X | | | | | | | X |
| 0 – 0* | 3 | *H* | X | X | | | | | | | | | X | X | | | |
| 0 1 –* | 3 | *J* | | | | | | X | X | | | | | | X | X | |
| 1 0 – | 3 | *K* | | | | | | | | | | | | | | X | X |
| – 1 0 | 3 | *L* | | X | | X | | X | | | X | | | | | | |
| 1 – 1 | 3 | *M* | | | X | | X | | | X | | X | | | | | |
| – 0 0 | 3 | *N* | | | | | | | | | | | X | | | X | |
| 1 1 – | 3 | *P* | | | | X | X | | | | X | X | | | | | |
| – 1 –* | 1 | *R* | | | | | | X | X | | X | X | | | | | |

We see that term $H$ is an essential prime implicant of $f$, but not of $h$. (We will thus check off the terms in $f$ and leave those in $h$, but reduce the cost of this term to 1 in the reduced table, since the AND gate is already accounted for; only the input to the $h$ OR gate needs to be charged.) Similarly, term $J$ is an essential prime implicant of $h$, but not of $g$. Finally, term $R$ will be used for $g$, since it only costs 1 (the OR gate input). Even if we could cover that with two shared terms, that would cost two inputs to the OR gate. The table thus reduces to

| | | | ***f*** | | | ***g*** | ***h*** | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 5 | 6 | 7 | 5 | 0 | 4 | 5 |
| 0 1 0 | 4 | *B* | | | | | | | |
| 1 0 1 | 4 | *E* | X | | | X | | | X |
| 0 – 0 | 1 | *H* | | | | | X | | |
| 0 1 – | 1 | *J* | | | | | | | |
| 1 0 – | 3 | *K* | | | | | | X | X |
| – 1 0 | 3 | *L* | | X | | | | | |
| 1 – 1 | 3 | *M* | X | | X | X | | | |
| – 0 0 | 3 | *N* | | | | | X | X | |
| 1 1 – | 3 | *P* | | X | X | | | | |

We can see that terms $B$ and $J$ no longer cover any terms; those rows can be eliminated. We seem to have two choices now. First, we can use $E$ for all three functions, at a cost of 6. We would then use $P$ for $f$ and $N$ for $h$, for a cost of 12 (on this table). This solution requires eight gates and 19 inputs.

$$f = x'z' + xy'z + xy$$
$$g = y + xy'z$$
$$h = x'y + xy'z + y'z'$$

The other choice is to use $M$ for $f$ and $g$ (at a cost of 4). Then $L$ or $P$ can be used for $f$; and $H$ (since it costs only 1) and $K$ for $h$. The total cost is 11 inputs and three gates ($M$, $L$ or $P$, and $K$), and thus this second solution is best. (Note that the gate to create term $H$ is not included in the gate count here, since it was already built.) The equations are

$$f = x'z' + xz + (yz' \quad \text{or} \quad xy)$$
$$g = y + xz$$
$$h = x'y + x'z' + xy'$$

It also uses eight gates, but has only 18 inputs. *[SP 12; EX 14]*

## 3.3 Solved Problems

**1.** For each of the following, find all minimum sum of product expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

**a.** $G(X, Y, Z) = \Sigma m(1, 2, 3, 4, 6, 7)$

**b.** $f(w, x, y, z) = \Sigma m(2, 5, 7, 8, 10, 12, 13, 15)$

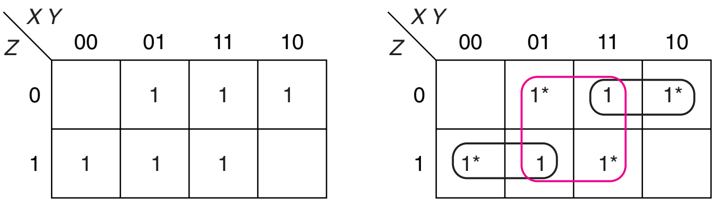
**c.** $g(a, b, c, d) = \Sigma m(0, 6, 8, 9, 10, 11, 13, 14, 15)$ (2 solutions)

**d.** $f(a, b, c, d) = \Sigma m(0, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15)$ (2 solutions)

**e.** $f(a, b, c, d) = \Sigma m(0, 1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 15)$

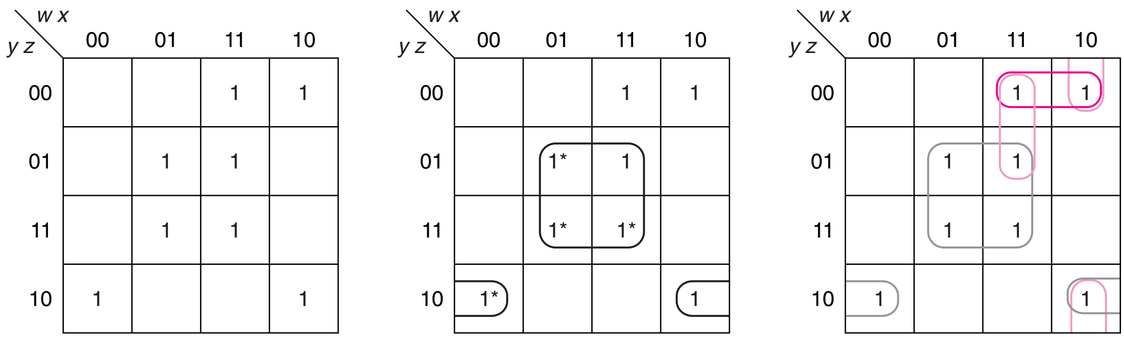
**f.** $g(a, b, c, d) = \Sigma m(0, 2, 3, 5, 7, 8, 10, 11, 12, 13, 14, 15)$ (4 solutions)

**a.** All of the prime implicants are essential, as shown on the map to the right.




$G = Y + XZ' + X'Z$

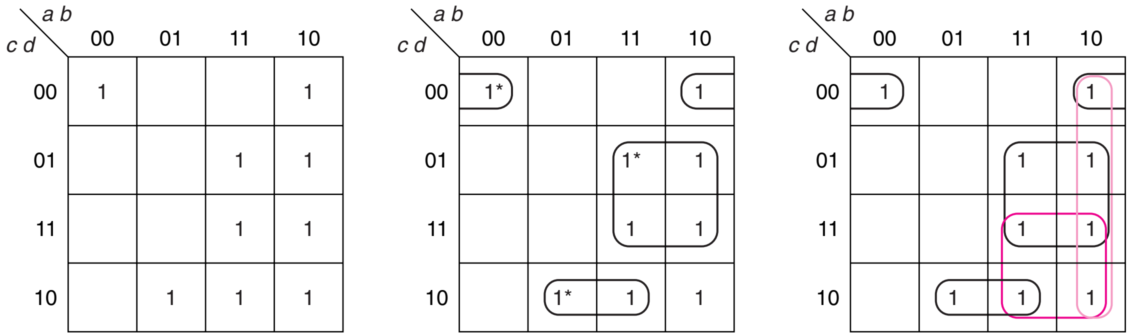
**b.**




The essential prime implicants are shown on the second map, leaving two 1's to be covered. The third map shows that each can be covered by two different prime implicants, but the red group shown is the only one that covers both with one term. We would require both pink terms. The minimum is

$f = xz + x'yz' + wy'z'$

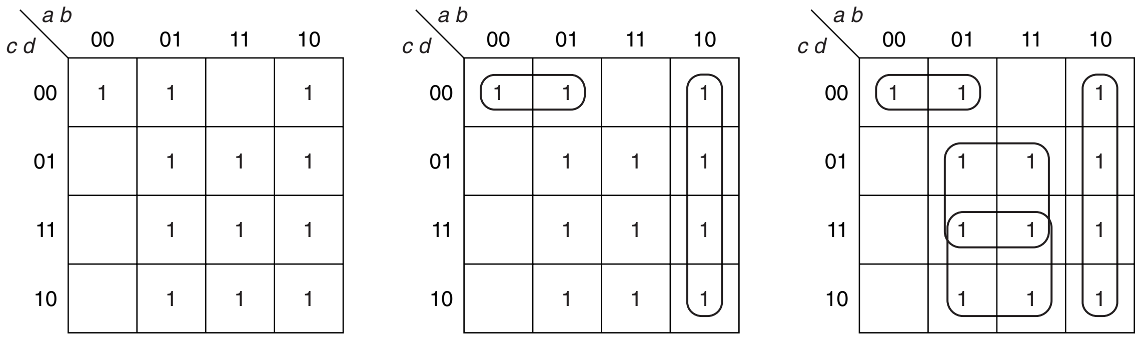
**c.**




The three essential prime implicants are shown on the center map. The only 1 left to be covered can be covered by either of two groups of four, as shown circled in red on the third map, producing

$$g = b'c'd' + bcd' + ad + ab'$$
$$g = b'c'd' + bcd' + ad + ac$$

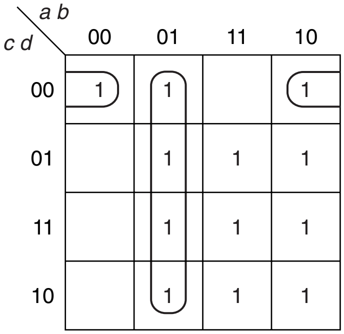
**d.**

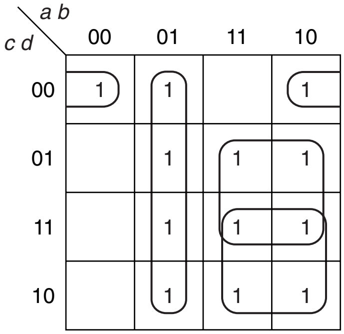



There are no essential prime implicants. We need one group of two to cover $m_0$; all other 1's can be covered by groups of four. Once we have chosen $a'c'd'$ to cover $m_0$ (center map), we would choose $ab'$ to cover $m_8$. (Otherwise, we must use $b'c'd'$, a group of two to cover that 1. Not only is that more literals, it covers nothing else new; whereas $ab'$ covered three additional uncovered 1's.) Once that has been done, the other two prime implicants become obvious, giving

$$f = a'c'd' + ab' + bc + bd$$

In a similar fashion (on the next map), once we choose $b'c'd'$ (the other prime implicant that covers $m_0$), $a'b$ is the appropriate choice to cover $m_4$:

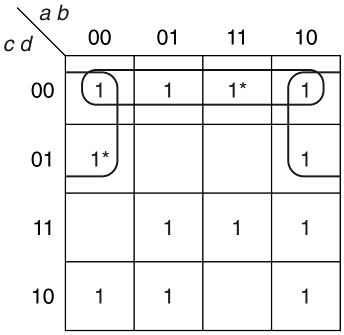


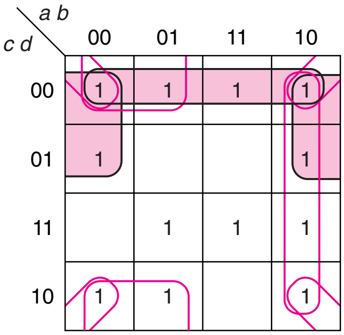



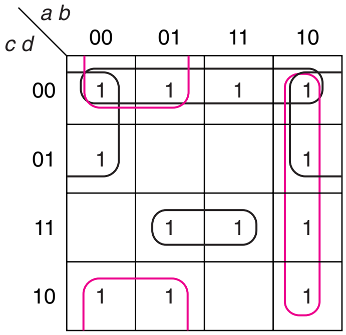

The only way to cover the remaining 1's in two terms is with $ac$ and $ad$, as shown on the second map, leaving

$$f = b'c'd' + a'b + ac + ad$$

**e.** There are two essential prime implicants, as indicated on the first map, leaving six 1's to be covered. The essential implicants are shaded on the second map.




No prime implicant covers more than two of the remaining 1's; thus three more terms are needed. The three groups of four (two literal terms) are circled in red on the second map. We can cover four new 1's only using $a'd'$ and $ab'$. Note that $m_7$ and $m_{15}$ are uncovered; they require a group of two, $bcd$. The only minimum solution, requiring five terms and 11 literals,

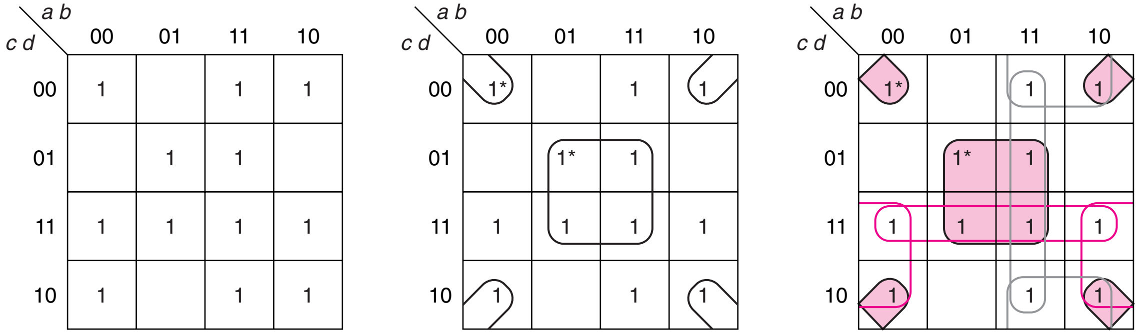
$$f = c'd' + b'c' + a'd' + ab' + bcd$$

is shown on the third map. There is another solution that uses five terms, but it requires 12 literals, namely,

$$f = c'd' + b'c' + b'd' + a'bc + acd$$

Obviously, it is not minimum (since it has an extra literal); it only used one of the groups of four instead of two.

**f.** On the second map, the two essential prime implicants have been highlighted ($b'd' + bd$), leaving four 1's uncovered. On the third map, we have shown the 1's covered by these prime implicants shaded.




We can cover $m_3$ and $m_{11}$ by either $cd$ or $b'c$ (shown with red lines), and we can cover $m_{12}$ and $m_{14}$ by either $ab$ or $ad'$ (shown in gray lines). Thus, there are four solutions:

$$f = b'd' + bd + cd + ab$$
$$f = b'd' + bd + cd + ad'$$
$$f = b'd' + bd + b'c + ab$$
$$f = b'd' + bd + b'c + ad'$$

The term $ac$ is also a prime implicant. However, it is not useful in a minimum solution since it leaves two isolated 1's to be covered, resulting in a five-term solution.

**2.** For the following functions,

i. List all prime implicants, indicating which are essential.

ii. Show the minimum sum of products expression(s).

**a.** $G(A, B, C, D) = \Sigma m(0, 1, 4, 5, 7, 8, 10, 13, 14, 15)$ (3 solutions)

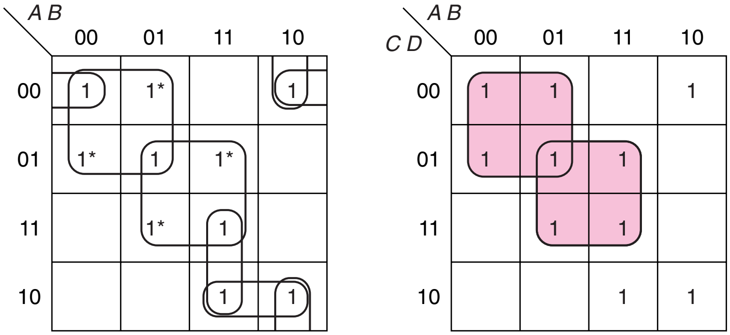
**b.** $f(w, x, y, z) = \Sigma m(2, 3, 4, 5, 6, 7, 9, 10, 11, 13)$

**c.** $h(a, b, c, d) = \Sigma m(1, 2, 3, 4, 8, 9, 10, 12, 13, 14, 15)$ (2 solutions)

**a.** The first map shows all of the prime implicants circled; the 1's that have been covered only once are indicated with an asterisk:

Essential prime implicants: $A'C'$, $BD$

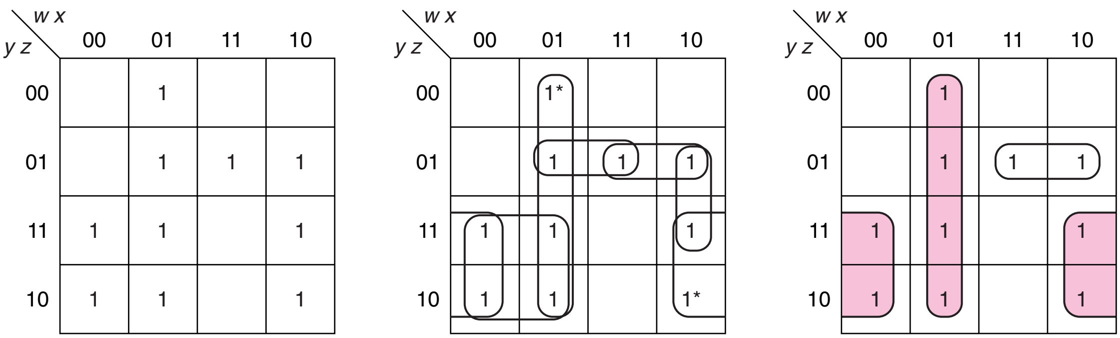
Other prime implicants: $B'C'D'$, $AB'D'$, $ACD'$, $ABC$




On the second map, the essential prime implicants have been shaded, highlighting the three 1's remaining to be covered. We need two terms to cover them, at least one of which must cover two of these remaining 1's. The three solutions are thus

$$F = A'C' + BD + ACD' + B'C'D'$$
$$F = A'C' + BD + AB'D' + ACD'$$
$$F = A'C' + BD + AB'D' + ABC$$

**b.**



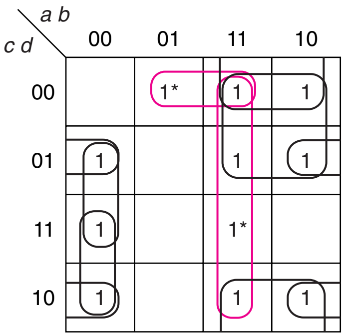

The second map shows all of the prime implicants circled and the 1's that have been covered only once are indicated with an asterisk:

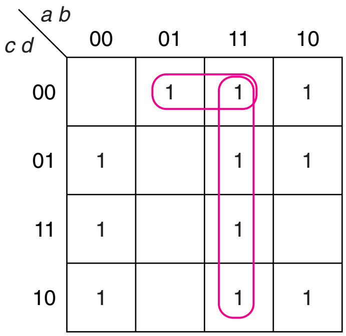
Essential prime implicants: $w'x$, $x'y$
Other prime implicants: $w'y$, $xy'z$, $wy'z$, $wx'z$

With the essential prime implicants shaded on the third map, it is clear that the only minimum solution is

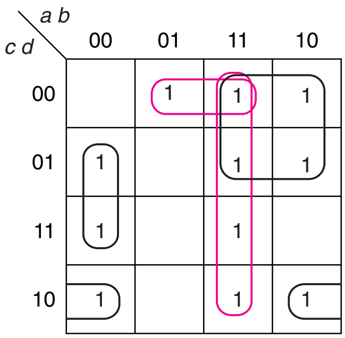
$$f = w'x + x'y + wy'z$$

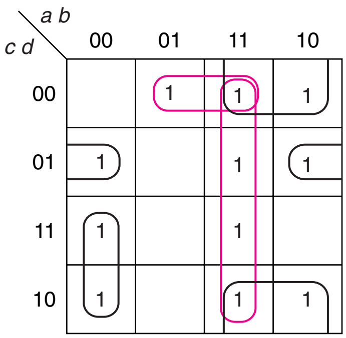
**c.** All of the prime implicants are circled on the first map, with the essential prime implicants shown in red.




Once we chose the essential prime implicants, there are six 1's left to be covered. We can only cover two at a time. There are two groups of four 1's, either of which can be used. (We cannot use both, since that would only cover three 1's.) The two solutions are shown on the maps below.




$$h = ab + bc'd' + ac' + a'b'd + b'cd'$$
$$h = ab + bc'd' + ad' + b'c'd + a'b'c$$

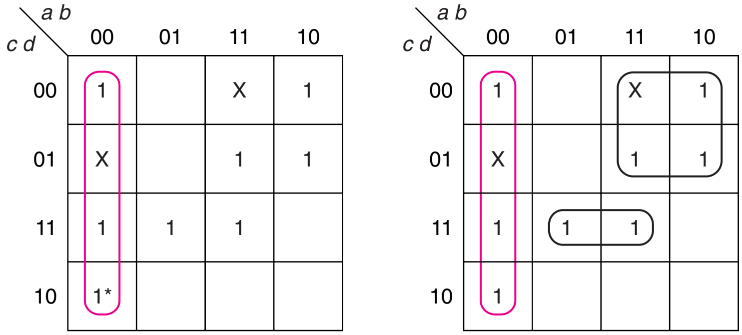
**3.** For each of the following, find all minimum sum of product expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

**a.** $f(a, b, c, d) = \Sigma m(0, 2, 3, 7, 8, 9, 13, 15) + \Sigma d(1, 12)$

**b.** $F(W, X, Y, Z) = \Sigma m(1, 3, 5, 6, 7, 13, 14) + \Sigma d(8, 10, 12)$ (2 solutions)

**c.** $f(a, b, c, d) = \Sigma m(3, 8, 10, 13, 15) + \Sigma d(0, 2, 5, 7, 11, 12, 14)$ (8 solutions)

**a.**

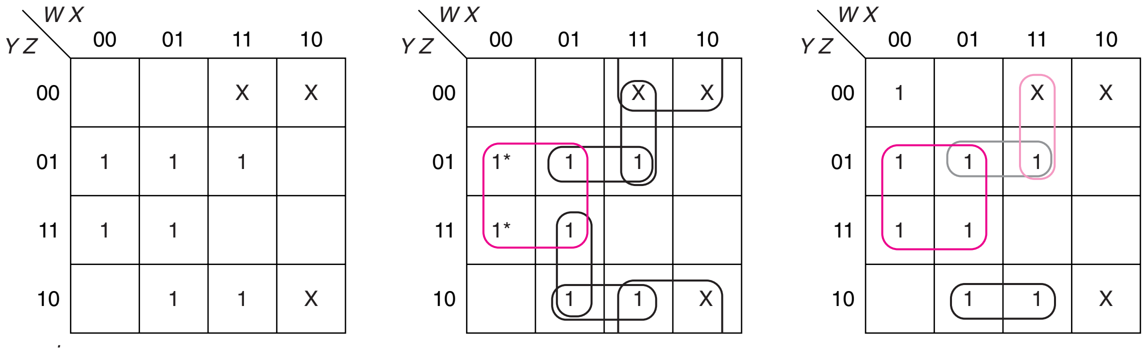


The first map shows the one essential prime implicant, $a'b'$. The remaining 1's can be covered by two additional terms, as shown on the second map. In this example, all don't cares are treated as 1's. The resulting solution is

$$f = a'b' + ac' + bcd$$

Although there are other prime implicants, such as $b'c'$, $abd$, and $a'cd$, three prime implicants would be needed in addition to $a'b'$ if any of them were chosen.

**b.**

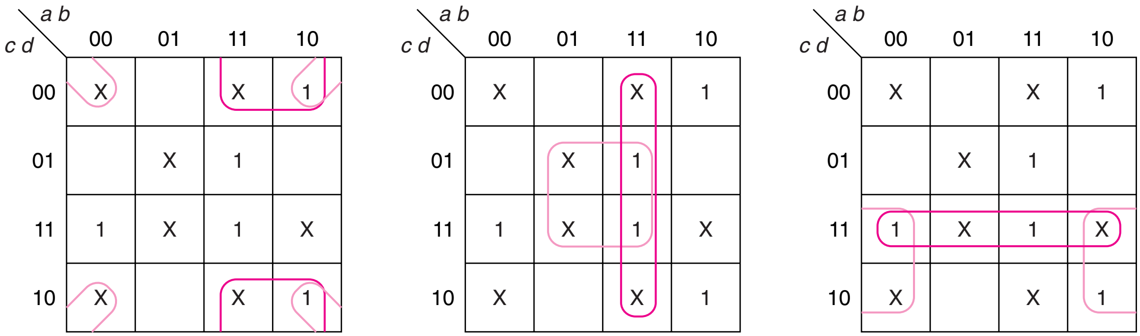


The second map shows all of the prime implicants circled. It is clear that only $W'Z$ is essential, after which three 1's remain uncovered. The prime implicant $XYZ'$ is the only one that can cover two of these and thus appears in both minimum solutions. That leaves a choice of two terms to cover the remaining one—either $WXY'$ (pink) or $XY'Z$ (gray). Note that they treat the don't care at $m_{12}$ differently and thus, although the two solutions shown below both satisfy the requirements of the problem, they are not equal:

$$F = W'Z + XYZ' + WXY'$$
$$F = W'Z + XYZ' + XY'Z$$

Also, the group of four ($WZ'$) is not used; that would require a four term solution.

**c.** There are no essential prime implicants in this problem. The left map shows the only two prime implicants that cover $m_8$; they also cover $m_{10}$. We must choose one of these. The next map shows the only prime implicants that cover $m_{13}$; both also cover $m_{15}$. We must choose one of these also. Finally, the last map shows the only two prime implicants that cover $m_3$.




So, our final solution takes one from each group, giving us a total of eight solutions:

$$f = \begin{Bmatrix} ad' \\ b'd' \end{Bmatrix} + \begin{Bmatrix} ab \\ bd \end{Bmatrix} + \begin{Bmatrix} cd \\ b'c \end{Bmatrix}$$

or, written out

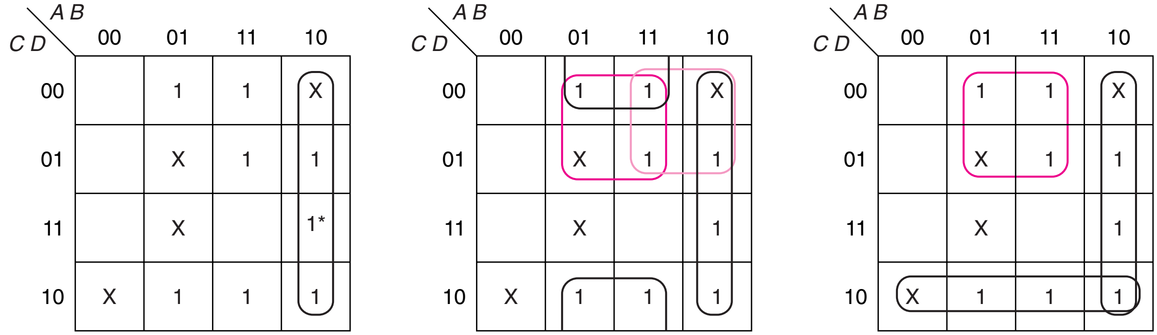
$$f = ad' + ab + cd$$
$$f = ad' + ab + b'c$$
$$f = ad' + bd + cd$$
$$f = ad' + bd + b'c$$
$$f = b'd' + ab + cd$$
$$f = b'd' + ab + b'c$$
$$f = b'd' + bd + cd$$
$$f = b'd' + bd + b'c$$

**4.** For each of the following, find all minimum sum of product expressions. Label the solutions $f_1, f_2, \ldots$ and indicate which solutions are equal.

**a.** $F(A, B, C, D) = \Sigma m(4, 6, 9, 10, 11, 12, 13, 14) + \Sigma d(2, 5, 7, 8)$ (3 solutions)

**b.** $f(a, b, c, d) = \Sigma m(0, 1, 4, 6, 10, 14) + \Sigma d(5, 7, 8, 9, 11, 12, 15)$ (13 solutions)

**a.**




On the first map, we have shown the one essential prime implicant, $AB'$. Neither $A'B$ nor $CD'$ are essential, since the 1's covered by them can each be covered by some other prime implicant. (That there is a don't care that can only be covered by one of these terms does not make that term essential.) With five 1's left to be covered, we need two additional terms. The first that stands out is $BD'$, circled on the middle map, since it covers four of the remaining 1's. If that is chosen, it leaves only $m_{13}$, which can be covered by $BC'$ or $AC'$. However, the third map shows still another cover, utilizing $BC'$ and $CD'$. Thus, the three solutions are

$$F_1 = AB' + BD' + BC'$$
$$F_2 = AB' + BD' + AC'$$
$$F_3 = AB' + BC' + CD'$$

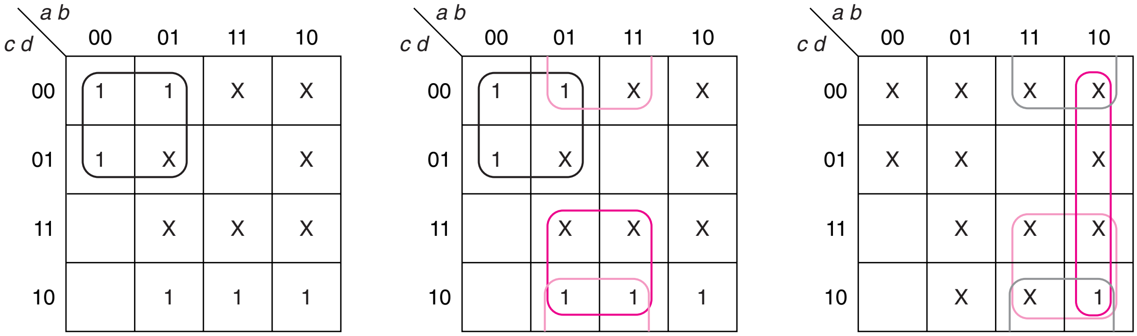
Notice that none of the solutions utilize the remaining prime implicant, $A'B$.

Next is the question of whether or not these three solutions are equal. The answer can be determined by examining how the don't cares are treated by each of the functions. The following table shows that:

| | **2** | **5** | **7** | **8** |
|---|---|---|---|---|
| $F_1$ | 0 | 1 | 0 | 1 |
| $F_2$ | 0 | 0 | 0 | 1 |
| $F_3$ | 1 | 1 | 0 | 1 |

In all functions, $m_7$ is treated as 0 (that is, it is not included in any prime implicant used) and $m_8$ as 1 (since it is included in the essential prime implicant, $AB'$); but the first two columns show that no two functions treat $m_2$ and $m_5$ the same. Thus, none of these is equal to any other.

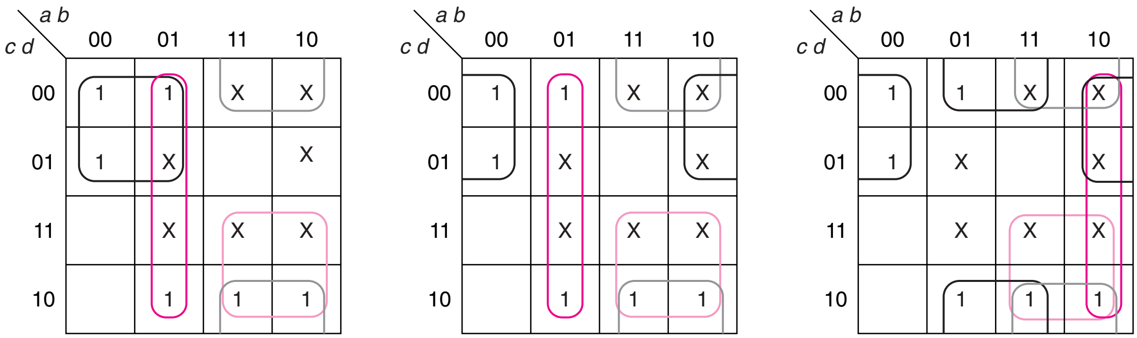
**b.** There are no essential prime implicants. The best place to start is with a 1 that can only be covered in two ways; in this problem there is only one, $m_1$. Any solution must contain either the term $a'c'$ (as shown on the first four maps) or the term $b'c'$ (as shown on the remaining two maps). There is no reason to use both, since $b'c'$ does not cover any 1's that are not already covered by $a'c'$. The first map shows $a'c'$. Note that there are three 1's left, requiring two more terms. At least one of these terms must cover two of the remaining 1's.




The second map shows two ways of covering $m_6$ and $m_{14}$, $bc$ and $bd'$. In either case, only one 1 is left to be covered. The third map shows the previously covered 1's as don't cares and three ways of covering the last 1, $m_{10}$. Thus, we have as the first six solutions

$$f_1 = a'c' + bc + ab'$$
$$f_2 = a'c' + bc + ac$$
$$f_3 = a'c' + bc + ad'$$
$$f_4 = a'c' + bd' + ab'$$
$$f_5 = a'c' + bd' + ac$$
$$f_6 = a'c' + bd' + ad'$$

Next, we consider how we may cover both $m_{10}$ and $m_{14}$ with one term (in addition to those already found). That provides two more solutions shown on the left map below. (Other solutions that use these terms have already been listed.)



$$f_7 = a'c' + a'b + ad'$$
$$f_8 = a'c' + a'b + ac$$

We next consider the solutions that use $b'c'$. The middle map shows two of these, utilizing $a'b$. The last map shows the final three, utilizing $bd'$, instead; it has the same three last terms as in the first series. Thus, we have

$$f_9 = b'c' + a'b + ad'$$
$$f_{10} = b'c' + a'b + ac$$
$$f_{11} = b'c' + bd' + ab'$$
$$f_{12} = b'c' + bd' + ac$$
$$f_{13} = b'c' + bd' + ad'$$

Finally, the table below shows how each of the functions treats the don't cares:

| | 5 | 7 | 8 | 9 | 11 | 12 | 15 |
|---|---|---|---|---|---|---|---|
| $f_1$ | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| $f_2$ | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| $f_3$ | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $f_4$ | 1 | 0 | 1 | 1 | 1 | 1 | 0 |
| $f_5$ | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| $f_6$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| $f_7$ | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| $f_8$ | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| $f_9$ | 1 | 1 | 1 | 1 | 0 | 1 | 0 |
| $f_{10}$ | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| $f_{11}$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| $f_{12}$ | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| $f_{13}$ | 0 | 0 | 1 | 1 | 0 | 1 | 0 |

Comparing the rows, the only two pairs that are equal are

$$f_1 = f_{10} \text{ and } f_2 = f_8.$$

**5.** For each of the following functions, find all of the minimum sum of product expressions and all of the minimum product of sums expressions:

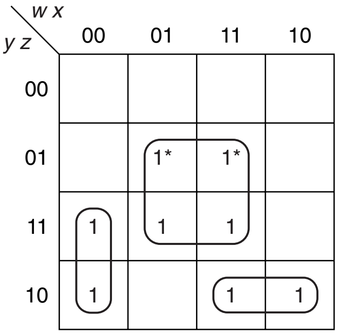
**a.** $f(w, x, y, z) = \Sigma m(2, 3, 5, 7, 10, 13, 14, 15)$
(1 SOP, 1 POS solution)

**b.** $f(a, b, c, d) = \Sigma m(3, 4, 9, 13, 14, 15) + \Sigma d(2, 5, 10, 12)$
(1 SOP, 2 POS solutions)

**c.** $f(a, b, c, d) = \Sigma m(4, 6, 11, 12, 13) + \Sigma d(3, 5, 7, 9, 10, 15)$
(2 SOP and 8 POS solutions)

**a.** The map of *f* is shown below.

| yz \ wx | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | | | | |
| 01 | | 1 | 1 | |
| 11 | 1 | 1 | 1 | |
| 10 | 1 | | 1 | 1 |

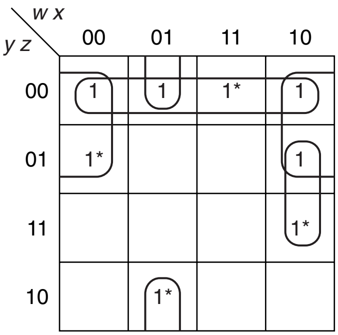



Although there is only one essential prime implicant, there is only one way to complete the cover with two more terms, namely,

$$f = xz + w'x'y + wyz'$$

By replacing all the 1's with 0's and 0's with 1's, or by plotting all the minterms not in *f*, we get the map for $f'$

| yz \ wx | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 1 | 1 | 1 |
| 01 | 1 | | | 1 |
| 11 | | | | 1 |
| 10 | | 1 | | |




There are four essential prime implicants, covering all of $f'$, giving

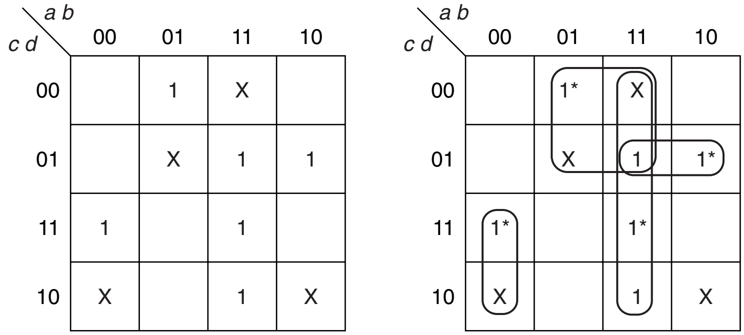
$$f' = x'y' + y'z' + w'xz' + wx'z$$

Using DeMorgan's theorem, we get

$$f = (x + y)(y + z)(w + x' + z)(w' + x + z')$$

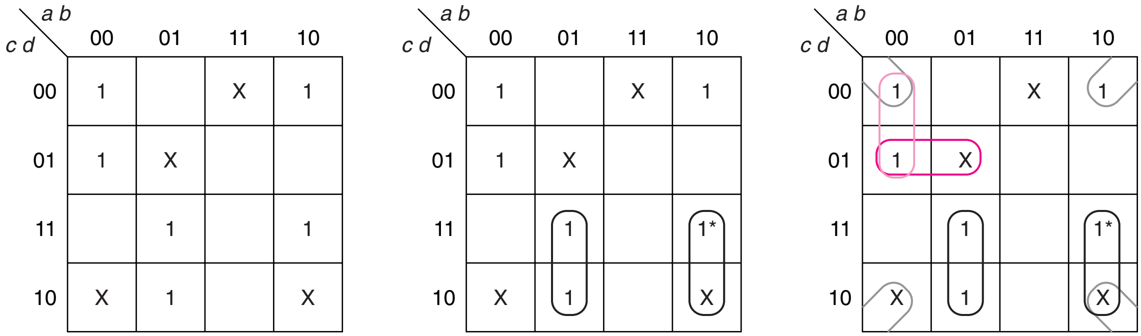
In this case, the sum of products solution requires fewer terms.

**b.** As indicated on the map below, all of the 1's are covered by essential prime implicants, producing the minimum sum of product expression




$$f_1 = bc' + ab + a'b'c + ac'd$$

Now, replacing all of the 1's by 0's and 0's by 1's and leaving the X's unchanged, we get the map for $f'$




There is one essential prime implicant, $ab'c$. Although $m_6$ and $m_7$ can each be covered in two ways, only $a'bc$ covers them both (and neither of the other terms cover additional 1's). The middle map shows each of these terms circled, leaving three 1's to be covered. There is a group of four, covering two of the 1's (as shown on the third map), $b'd'$. That leaves just $m_1$, which can be covered in two ways, as shown on the third map in red and pink lines. Thus, the two minimum sum of product expressions for $f'$ are

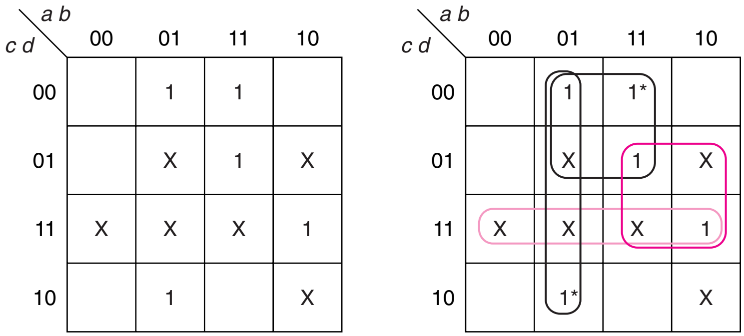
$$f_2' = ab'c + a'bc + b'd' + a'c'd$$
$$f_3' = ab'c + a'bc + b'd' + a'b'c'$$

producing the two minimum product of sums solutions

$$f_2 = (a' + b + c')(a + b' + c')(b + d)(a + c + d')$$
$$f_3 = (a' + b + c')(a + b' + c')(b + d)(a + b + c)$$

**c.** The map for $f$ is shown next (on the left). There are two essential prime implicants, leaving only $m_{11}$ to be covered.

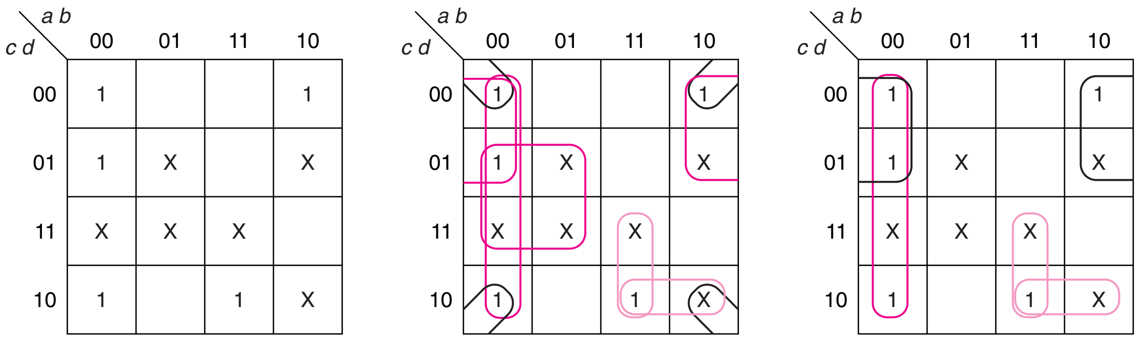
There are two groups of four that can be used, as indicated on the right map.




Thus the two sum of products solutions are

$$f_1 = a'b + bc' + ad$$
$$f_2 = a'b + bc' + cd$$

We then mapped $f'$ and found no essential prime implicants.




We chose as a starting point $m_8$. It can be covered either by the four corners, $b'd'$ (as shown on the second map) or by $b'c'$, as shown on the third map. Whichever solution we choose, we need a group of two to cover $m_{14}$ (as shown in pink); neither covers any other 1. After choosing one of these (and $b'd'$), all that remains to be covered is $m_1$. The three red lines show the covers. (Notice that one of those is $b'c'$.) If we don't choose $b'd'$, then we must choose $b'c'$ to cover $m_0$ and $a'b'$ to cover $m_2$ (since the only other prime implicant that covers $m_2$ is $b'd'$ and we have already found all of the solutions using that term). Thus, the eight solutions for $f'$ are

$$f'_3 = b'd' + abc + a'b'$$
$$f'_4 = b'd' + abc + a'd$$
$$f'_5 = b'd' + abc + b'c'$$

$$f_6' = b'd' + acd' + a'b'$$
$$f_7' = b'd' + acd' + a'd$$
$$f_8' = b'd' + acd' + b'c'$$
$$f_9' = b'c' + abc + a'b'$$
$$f_{10}' = b'c' + acd' + a'b'$$

The product of sums solutions for $f$ are thus

$$f_3 = (b + d)(a' + b' + c')(a + b)$$
$$f_4 = (b + d)(a' + b' + c')(a + d')$$
$$f_5 = (b + d)(a' + b' + c')(b + c)$$
$$f_6 = (b + d)(a' + c' + d)(a + b)$$
$$f_7 = (b + d)(a' + c' + d)(a + d')$$
$$f_8 = (b + d)(a' + c' + d)(b + c)$$
$$f_9 = (b + c)(a' + b' + c')(a + b)$$
$$f_{10} = (b + c)(a' + c' + d)(a + b)$$

**6.** Label the solutions of each part of problem 5 as $f_1, f_2, \ldots$, and indicate which solutions are equal.

**a.** Since this problem does not involve don't cares, all solutions are equal.

**b.**

| | 2 | 5 | 10 | 12 |
|---|---|---|---|---|
| $f_1$ | 1 | 1 | 0 | 1 |
| $f_2'$ | 1 | 1 | 1 | 0 |
| $f_2$ | 0 | 0 | 0 | 1 |
| $f_3'$ | 1 | 0 | 1 | 0 |
| $f_3$ | 0 | 1 | 0 | 1 |

All of the solutions are unique. The sum of products solution treats $m_2$ as a 1; the product of sums treats it as a 0. The two product of sums solutions treat $m_5$ differently.

**c.**

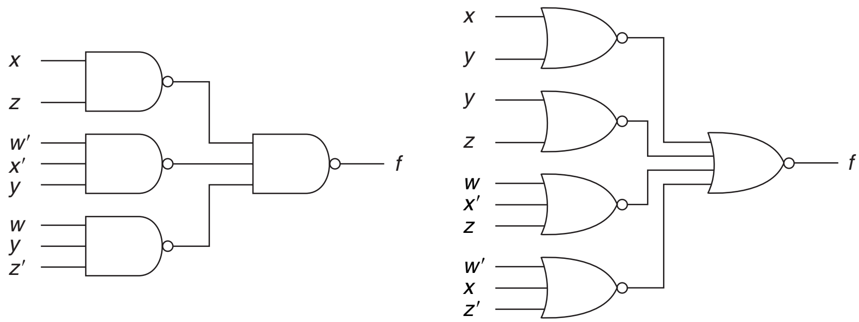
| | 3 | 5 | 7 | 9 | 10 | 15 |
|---|---|---|---|---|---|---|
| $f_1$ | 0 | 1 | 1 | 1 | 0 | 1 |
| $f_2$ | 1 | 1 | 1 | 0 | 0 | 1 |
| $f_3'$ | 1 | 0 | 0 | 0 | 1 | 1 |
| $f_4'$ | 1 | 1 | 1 | 0 | 1 | 1 |
| $f_5'$ | 0 | 0 | 0 | 1 | 1 | 1 |
| $f_6'$ | 1 | 0 | 0 | 0 | 1 | 0 |
| $f_7'$ | 1 | 1 | 1 | 0 | 1 | 0 |
| $f_8'$ | 0 | 0 | 0 | 1 | 1 | 0 |
| $f_9'$ | 1 | 0 | 0 | 1 | 0 | 1 |
| $f_{10}'$ | 1 | 0 | 0 | 1 | 1 | 0 |

For one of the sum of product expressions to be equal to one of the product of sum expressions, the pattern must be opposite (since we are showing the values of the don't cares for $f'$ for the POS forms). Thus, $f_1 = f_6$, and $f_2 = f_8$, that is

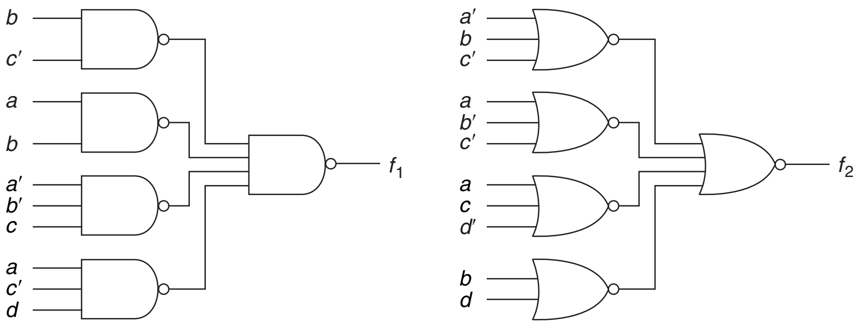
$$a'b + bc' + ad = (b + d)(a' + c' + d)(a + b)$$
$$a'b + bc' + cd = (b + d)(a' + c' + d)(b + c)$$

**7.** For each part of problem 5, draw the block diagram of a two-level NAND gate circuit and a two-level NOR gate circuit. (For those parts with multiple solutions, you need only draw one NAND and one NOR solution.)

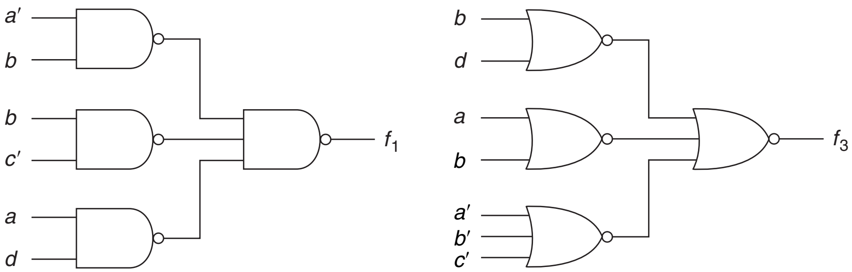
**a.**




**b.**




**c.**



**8.** Find the minimum sum of products solution(s) for each of the following:

**a.** $F(A, B, C, D, E) = \Sigma m(0, 5, 7, 9, 11, 13, 15, 18, 19, 22, 23, 25, 27, 28, 29, 31)$

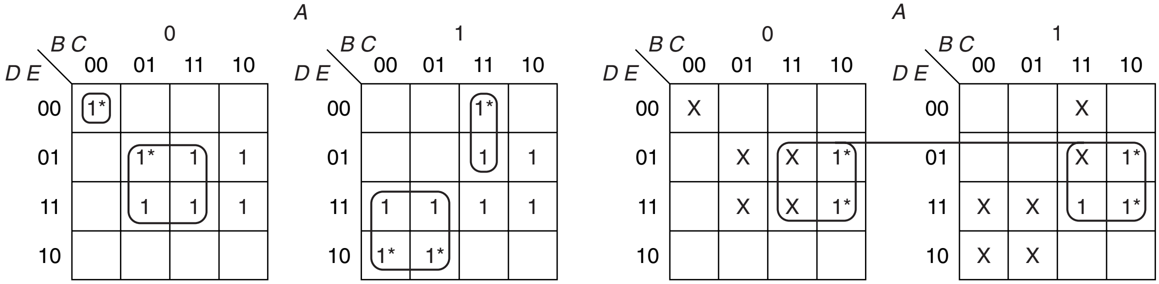
**b.** $F(A, B, C, D, E) = \Sigma m(0, 2, 4, 7, 8, 10, 15, 17, 20, 21, 23, 25, 26, 27, 29, 31)$

**c.** $G(V, W, X, Y, Z) = \Sigma m(0, 1, 4, 5, 6, 7, 10, 11, 14, 15, 21, 24, 25, 26, 27)$ (3 solutions)

**d.** $G(V, W, X, Y, Z) = \Sigma m(0, 1, 5, 6, 7, 8, 9, 14, 17, 20, 21, 22, 23, 25, 28, 29, 30)$ (3 solutions)

**e.** $H(A, B, C, D, E) = \Sigma m(1, 3, 10, 14, 21, 26, 28, 30) + \Sigma d(5, 12, 17, 29)$

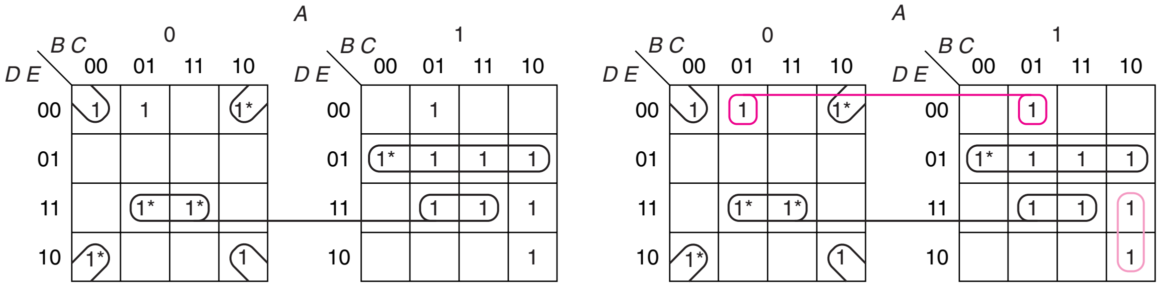
**a.** We begin by looking at 1's for which the corresponding position on the other layer is 0. On the first map, all of the essential prime implicants that are totally contained on one layer of the map, $A'B'C'D'E'$, $A'CE$, $AB'D$, and $ABCD'$, are circled.



The 1's covered by these essential prime implicants are shown as don't cares on the second map. The remaining 1's are all part of the group of eight, $BE$, shown on the second map. Thus, the minimum solution is

$$F = A'B'C'D'E' + A'CE + AB'D + ABCD' + BE$$

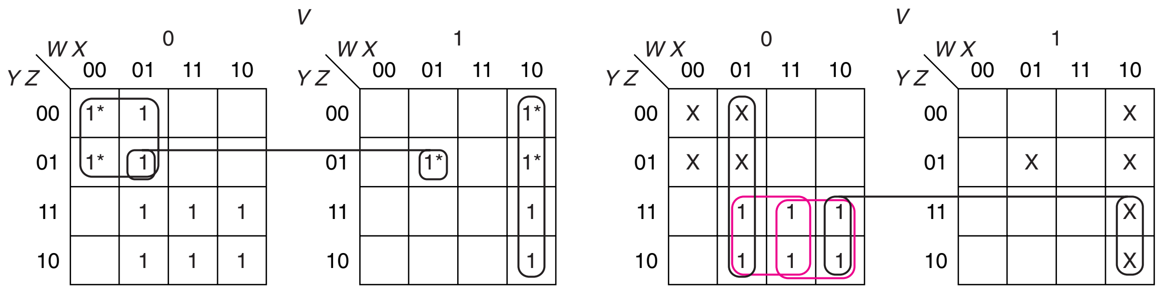
**b.** On the left map below, the essential prime implicants are circled. Note that $A'C'E'$ is on the top layer, $AD'E$ is on the lower layer and $CDE$ is split between the layers.



That leave four 1's to be covered, using two groups of two as shown on the right map. The minimum is thus

$$F = A'C'E' + AD'E + CDE + B'CD'E' + ABC'D$$

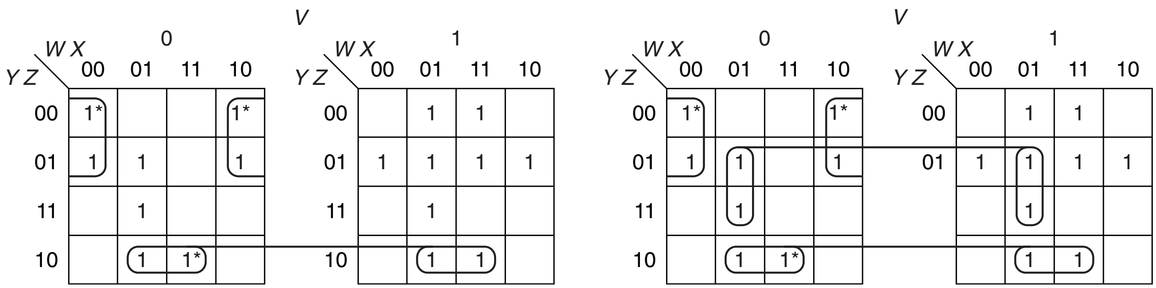
**c.** The map, with essential prime implicants circled, is shown on the left. After choosing $V'W'Y' + VWX' + W'XY'Z$, there are still six 1's uncovered. On the right map, the minterms covered by essential prime implicants are shown as don't cares. Each of the 1's can be covered by two different groups of four, which are shown on the map on the right.



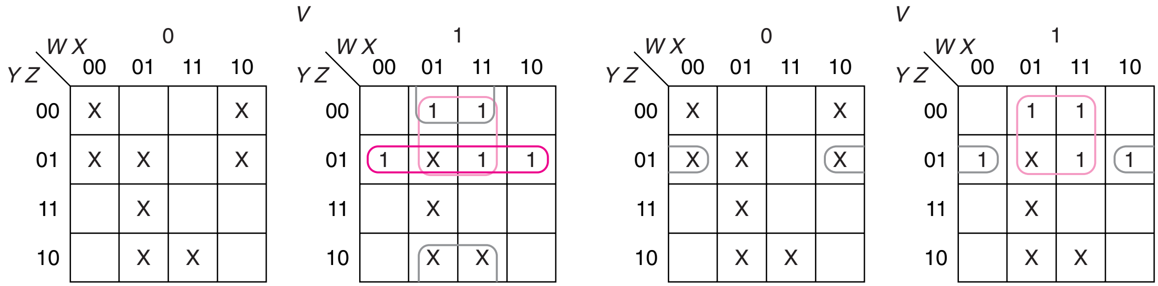

One group that covers four new 1's must be used (or both of them may be used), giving the following solutions:

$$G = V'W'Y' + VWX' + W'XY'Z + V'XY + V'WY$$
$$G = V'W'Y' + VWX' + W'XY'Z + V'XY + WX'Y$$
$$G = V'W'Y' + VWX' + W'XY'Z + V'WY + V'W'X$$

**d.** On the first map, the two essential prime, $V'X'Y'$ and $XYZ'$, implicants are circled. The term $W'XZ$ is circled on the second map; if it is not used, $W'XY$ would be needed to cover $m_7$ and $m_{23}$. But then, three more terms would be needed to cover the function.



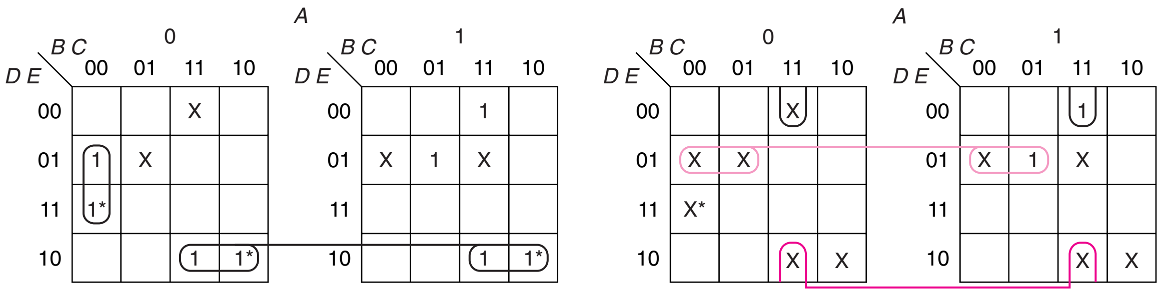

The following maps show the covered terms as don't cares and three ways of covering the remaining 1's. On the left map, the red term, $VY'Z$, is used with either of the other terms, $VXY'$ or $VXZ'$. On the right map, $VXY'$ and $X'Y'Z$ are used.




The three minimum solutions are thus

$$G = V'X'Y' + XYZ' + W'XZ + VY'Z + VXY'$$
$$G = V'X'Y' + XYZ' + W'XZ + VY'Z + VXZ'$$
$$G = V'X'Y' + XYZ' + W'XZ + VXY' + X'Y'Z$$

**e.** The two essential prime implicants, $A'B'C'E$ and $BDE'$, are circled on the first map. Each of the remaining 1's can be covered in two ways, by a group of two contained completely on one layer or by the group of four shown.



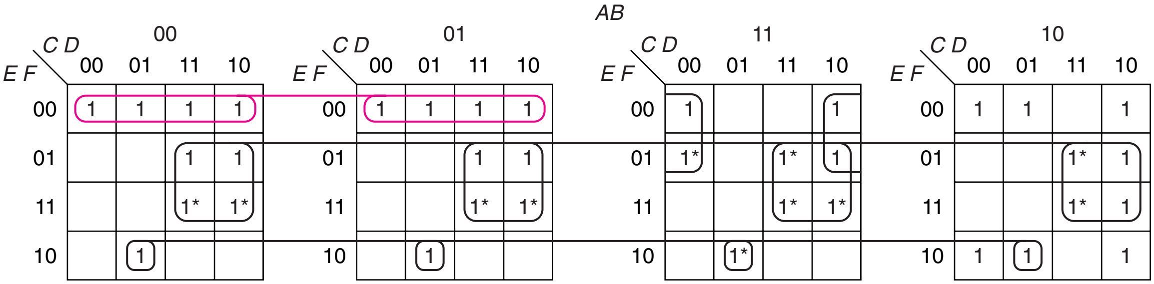

Thus, the minimum solution is

$$H = A'B'C'E + BDE' + BCE' + B'D'E$$

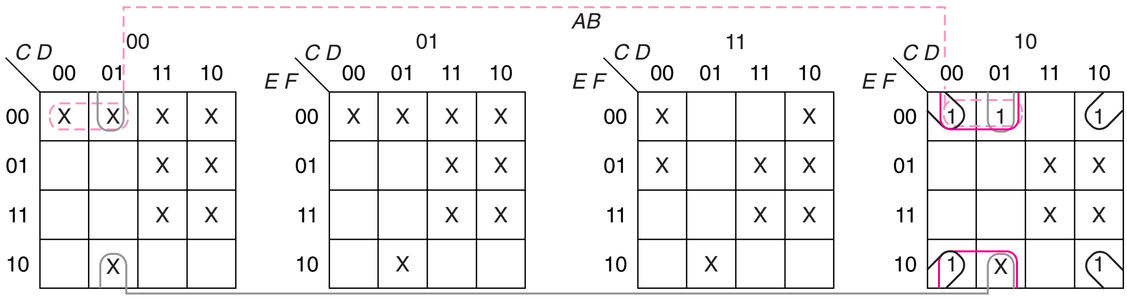
**9.** Find the four minimum sum of product expressions for the following six-variable function

$$G(A, B, C, D, E, F) = \Sigma m(0, 4, 6, 8, 9, 11, 12, 13, 15, 16, 20, 22, 24, 25, 27, 28, 29, 31, 32, 34, 36, 38, 40, 41, 42, 43, 45, 47, 48, 49, 54, 56, 57, 59, 61, 63)$$

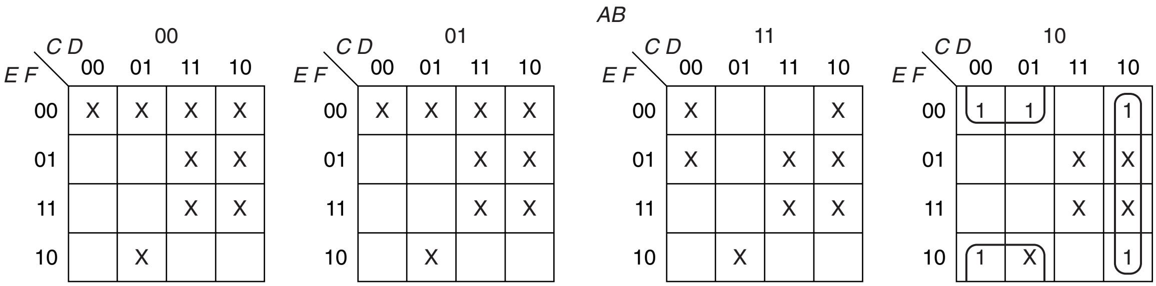
On the first map, the three essential prime implicants, $ABD'E'$, $CF$, and $C'DEF'$, are circled in black. The first is on just the third layer. The other two include 1's on all four layers (and thus do not involve the variable $A$ and $B$). Also circled (in red) is a group of eight, $A'E'F'$, that is not essential (since each of the 1's is part of some other prime implicant). If that is not used, however, at least two terms would be needed to cover those 1's.




On the next map, the 1's that have been covered are shown as don't cares. The remaining 1's are all on the bottom (10) layer. The four corners, $AB'D'F'$, covers four of the five remaining 1's. Then, either $AB'C'F'$ (on the bottom layer) or $B'C'E'F'$ or $B'C'DF'$ (both half on the top layer and half on the bottom) can be used to cover the remaining 1's. These terms are circled below.




Also, as shown on the map below, $AB'C'F'$ could be used with $AB'CD'$.




Thus, we have the following four solutions

$$H = ABD'E' + CF + C'DEF' + A'E'F' + AB'D'F' + AB'C'F'$$

$$H = ABD'E' + CF + C'DEF' + A'E'F' + AB'D'F' + B'C'E'F'$$

$$H = ABD'E' + CF + C'DEF' + A'E'F' + AB'D'F' + B'C'DF'$$

$$H = ABD'E' + CF + C'DEF' + A'E'F' + AB'C'F' + AB'CD'$$

**10.** Find a minimum two-level circuit (corresponding to sum of products expressions) using AND gates and one OR gate per function for each of the following sets of functions:

**a.** $f(a, b, c, d) = \Sigma m(0, 1, 2, 3, 5, 7, 8, 10, 11, 13)$
$g(a, b, c, d) = \Sigma m(0, 2, 5, 8, 10, 11, 13, 15)$
(7 gates, 19 inputs)

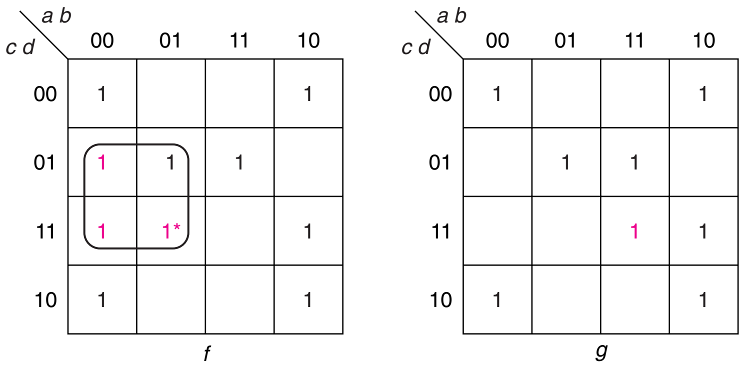
**b.** $f(a, b, c, d) = \Sigma m(1, 2, 4, 5, 6, 9, 11, 13, 15)$
$g(a, b, c, d) = \Sigma m(0, 2, 4, 8, 9, 11, 12, 13, 14, 15)$
(8 gates, 23 inputs)

**c.** $F(W, X, Y, Z) = \Sigma m(2, 3, 6, 7, 8, 9, 13)$
$G(W, X, Y, Z) = \Sigma m(2, 3, 6, 7, 9, 10, 13, 14)$
$H(W, X, Y, Z) = \Sigma m(0, 1, 4, 5, 9, 10, 13, 14)$
(8 gates, 22 inputs)

**d.** $f(a, b, c, d) = \Sigma m(0, 2, 3, 8, 9, 10, 11, 12, 13, 15)$
$g(a, b, c, d) = \Sigma m(3, 5, 7, 12, 13, 15)$
$h(a, b, c, d) = \Sigma m(0, 2, 3, 4, 6, 8, 10, 14)$
(10 gates, 28 inputs)

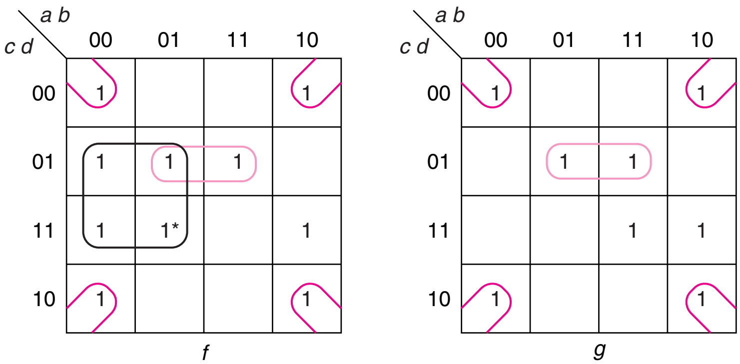
**e.** $f(a, b, c, d) = \Sigma m(0, 3, 5, 7) + \Sigma d(10, 11, 12, 13, 14, 15)$
$g(a, b, c, d) = \Sigma m(0, 5, 6, 7, 8) + \Sigma d(10, 11, 12, 13, 14, 15)$
(6 gates, 19 inputs)

**a.** The maps below show the only prime implicant, $a'd$ in $f$, that covers a 1 not part of the other function.

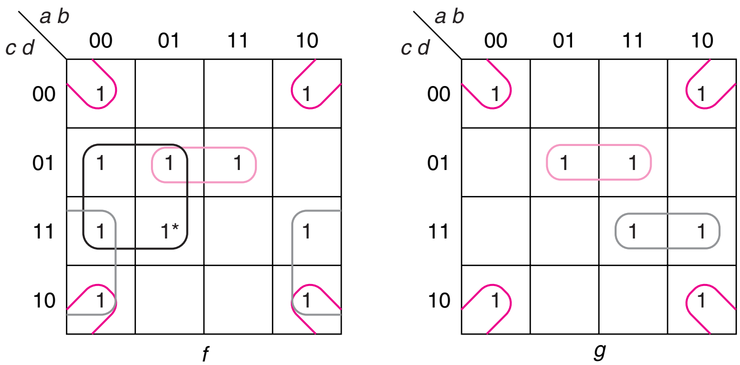



No other 1 (of either $f$ or $g$) that is not shared makes a prime implicant essential ($m_1$ or $m_3$ in $f$ or $m_{15}$ in $g$). Two other terms,

$b'd'$ and $bc'd$, are essential prime implicants of both $f$ and $g$ and have been thus chosen in the maps below.




Although the term $ab'c$ could be shared, another term would be needed for $g$ (either $abd$ or $acd$). This would require seven gates and 20 gate inputs (one input too many). But, if $acd$ is used for $g$, we could then complete covering both functions using $b'c$ for $f$ as shown on the maps below.



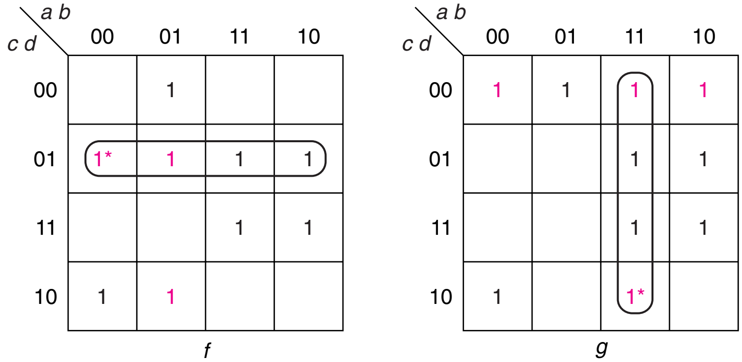

Thus,

$$f = a'd + b'd' + bc'd + b'c$$
$$g = b'd' + bc'd + acd$$

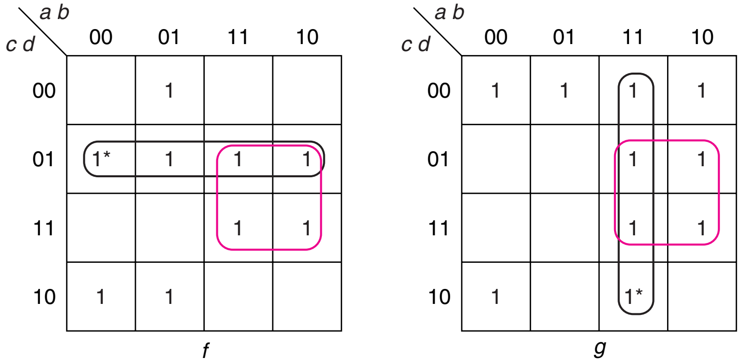
requiring seven gates and 19 inputs.

**b.** Scanning each function for 1's that are not part of the other function, we find $m_1$, $m_5$, and $m_6$ in $f$ and $m_0$, $m_8$, $m_{12}$, and $m_{14}$

in $g$. The only ones that make a prime implicant essential are indicated on the map below.



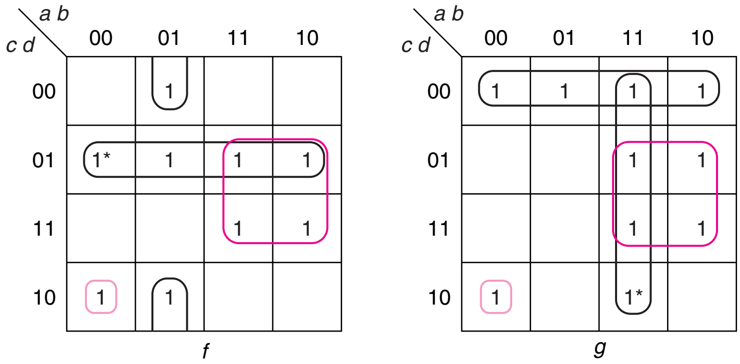

Next, we note that *ad* is an essential prime implicant of both functions, producing the following maps:



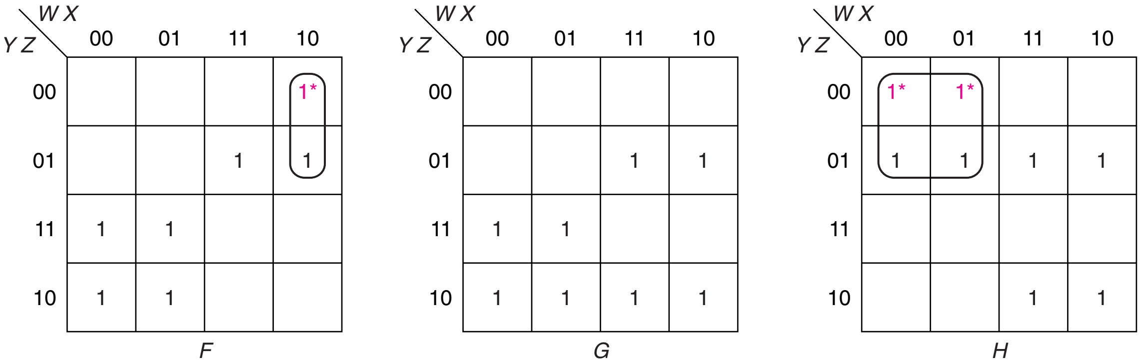

Unless we choose $c'd'$ to cover the remaining three 1's in the first row of $g$, we will need an extra term. Once we have done that, we see that the last 1 ($m_2$) of $g$ can be covered by the minterm and shared with $f$. That leaves just two 1's of $f$ that can be covered with the term $a'bd'$. The functions and the maps are shown next:

$$f = c'd + ad + a'b'cd' + a'bd'$$
$$g = ab + ad + c'd' + a'b'cd'$$

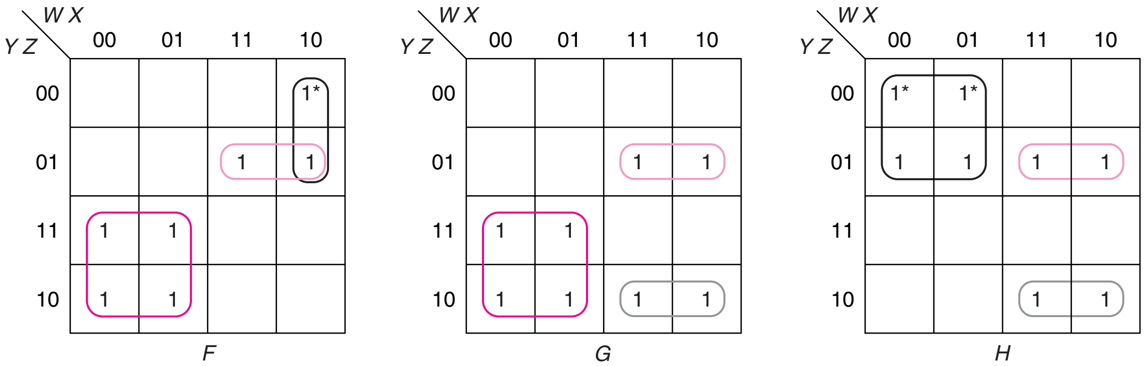
for a total of eight gates and 23 inputs.




**c.** When minimizing three functions, we still look for 1's that are only included in one of the functions and that make a prime implicant essential. In this problem, the only ones that satisfy these conditions are $m_8$ in $F$ and $m_0$ and $m_4$ in $H$, as shown on the map below.

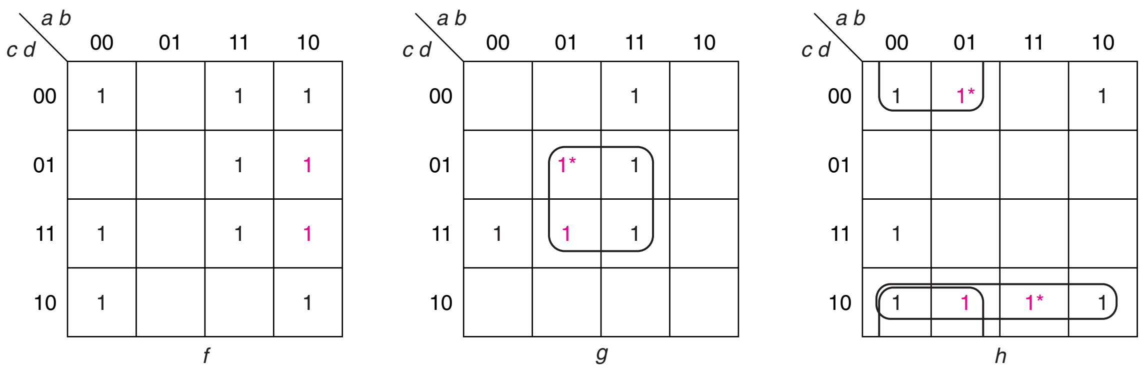



Next, notice that $W'Y$ is an essential prime implicant of both $F$ and $G$. Once that is chosen, the term $WY'Z$ covers the remaining 1 of $F$ and two 1's in $G$ and $H$. (That term would be used for both $F$ and $G$ in any case since it is an essential prime implicant of both and is shareable. It is used for $H$ since the remaining 1's in the prime implicant $Y'Z$ are already covered.) Finally, $WYZ'$, an essential prime implicant of $H$, finishes the cover of $G$ and $H$. The maps and functions below show the final solution, utilizing eight gates and 22 inputs.

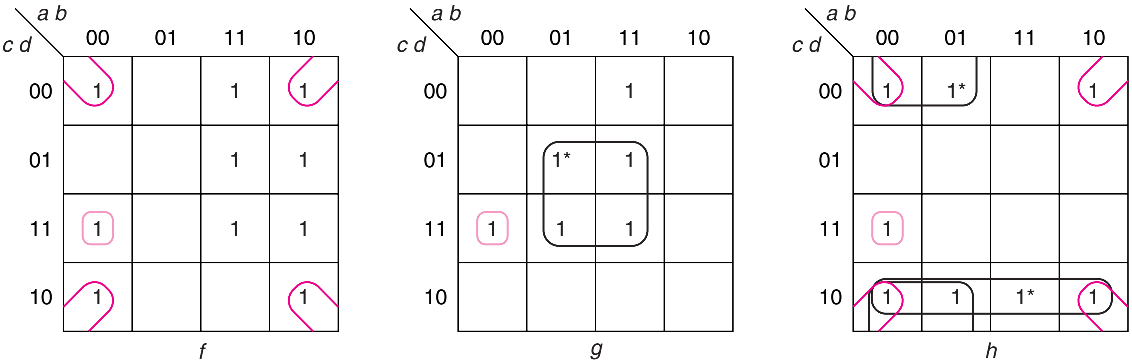



$$F = WX'Y' + W'Y + WY'Z$$
$$G = W'Y + WY'Z + WYZ'$$
$$H = W'Y' + WY'Z + WYZ'$$

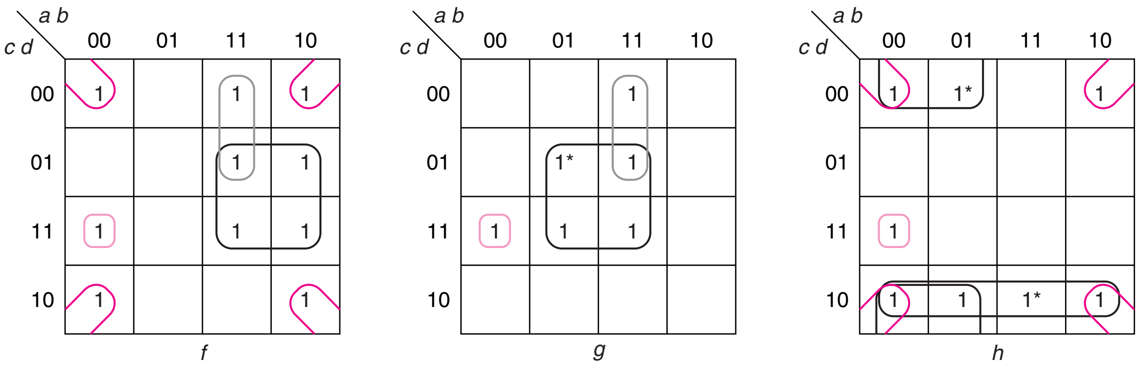
**d.** On the maps below, the essential prime implicants that cover 1's not part of any other function are circled. In *f*, $m_9$ and $m_{11}$ can be covered with any of three prime implicants.




Next, we note that $m_8$ can only be covered by $b'd'$ in *h* and that $b'd'$ is also an essential prime implicant of *f*. That leaves only $m_3$ uncovered in *h*; by using the minterm for that, it can be shared with both *f* and *g*. (Otherwise, a new term would be required in each of those functions.) The resulting maps are shown below.

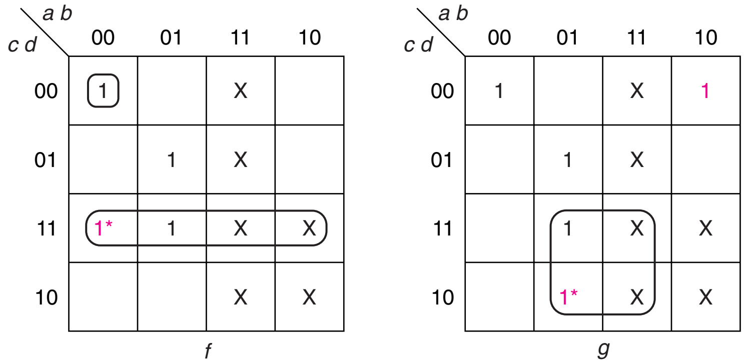



The only uncovered 1 in $g$ is $m_{12}$. By using $abc'$ for both that and for $f$, we save a term. Finally, the three remaining 1's in $f$ are covered by $ad$, yielding the maps and equations below.

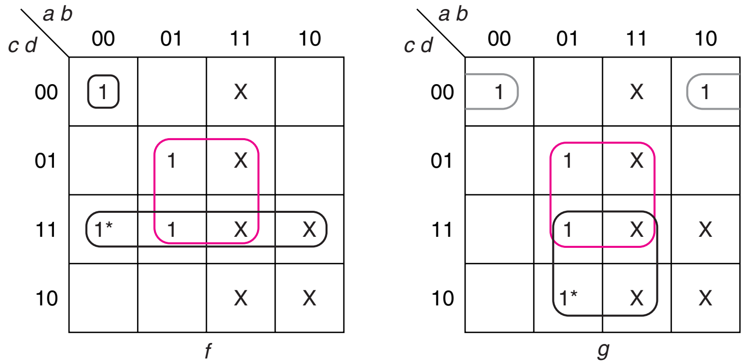



$$f = b'd' + a'b'cd + abc' + ad$$
$$g = bd + a'b'cd + abc'$$
$$h = a'd' + cd' + b'd' + a'b'cd$$

**e.** This example includes a number of don't cares, but that does not change the process significantly. There are two essential prime implicants, $cd$ in $f$ and $bc$ in $g$, that cover 1's that cannot be shared. In addition, $a'b'c'd'$ must be used in $f$ since it is the only prime implicant that covers $m_0$. (If a minterm is a prime implicant, we have no choice but to use it.) The maps below show these terms circled.




Next, we use $bd$ to cover $m_5$ in both functions, and complete the cover of $f$. The obvious choice is to use $b'c'd'$ for the remaining 1's of $g$, producing the following maps and equations:

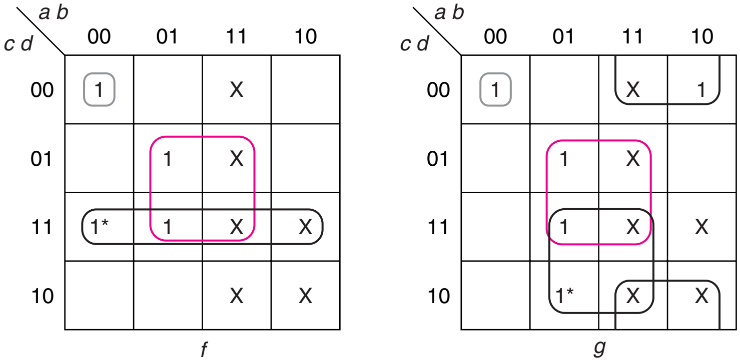



$$f = cd + a'b'c'd' + bd$$
$$g = bc + bd + b'c'd'$$

But, there is another solution, as illustrated below. By using $a'b'c'd'$ to cover $m_0$ in $g$ (we already needed that term for $f$), we can cover the remaining 1 in $g$ with a group of four, $ad'$, producing the solution

$$f = cd + a'b'c'd' + bd$$
$$g = bc + bd + a'b'c'd' + ad'$$

as shown on the following maps. Both solutions require seven gates and 19 inputs.




**11.** For each of the following functions, use the iterated consensus method to first find all of the prime implicants and then all of the minimum solutions.

**a.** $f(w, x, y, z) = \Sigma m(2, 5, 7, 8, 10, 12, 13, 15)$

**b.** $f(a, b, c, d) = \Sigma m(0, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15)$ (2 solutions)

**c.** $F(W, X, Y, Z) = \Sigma m(1, 3, 5, 6, 7, 13, 14) + \Sigma d(8, 10, 12)$ (2 solutions)

**d.** $f(a, b, c, d) = \Sigma m(0, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14)$ (32 solutions)

**e.** $G(V, W, X, Y, Z) = \Sigma m(0, 1, 5, 6, 7, 8, 9, 14, 17, 20, 21, 22, 23, 25, 28, 29, 30)$ (3 solutions)

Note that we have already solved each of these problems with the Karnaugh map (Solved Problems 1b, 1d, 3b, Example 3.15, Solved Problem 8d).

**a.** We will start with the minterms for this solution, listing only those consensus terms that are to be added to the list.

| | | | | |
|---|---|---|---|---|
| ~~*A*~~ | ~~0 0 1 0~~ | ~~*J*~~ | ~~0 1 – 1~~ | *C* ¢ *B* ≥ *C*, *B* |
| ~~*B*~~ | ~~0 1 0 1~~ | *K* | 1 0 – 0 | *E* ¢ *D* ≥ *D*, *E* |
| ~~*C*~~ | ~~0 1 1 1~~ | *L* | 1 1 0 – | *G* ¢ *F* ≥ *F*, *G* |
| ~~*D*~~ | ~~1 0 0 0~~ | ~~*M*~~ | ~~– 1 1 1~~ | *J* ¢ *H* ≥ *H* |
| ~~*E*~~ | ~~1 0 1 0~~ | *N* | – 0 1 0 | *K* ¢ *A* ≥ *A* |
| ~~*F*~~ | ~~1 1 0 0~~ | *P* | 1 – 0 0 | *L* ¢ *K* |
| ~~*G*~~ | ~~1 1 0 1~~ | ~~*Q*~~ | ~~– 1 0 1~~ | *L* ¢ *J* |
| ~~*H*~~ | ~~1 1 1 1~~ | ~~*R*~~ | ~~1 1 – 1~~ | *M* ¢ *L* |
| | | *S* | – 1 – 1 | *Q* ¢ *M* ≥ *J*, *M*, *Q*, *R* |

All other consensus operations are either undefined or produce a term that is already on the list. The terms remaining on the list are all the prime implicants—$wx'z'$, $wxy'$, $x'yz'$, $wy'z'$, and $xz$.

The prime implicant table is

| | | | $ | √ 2 | √ 5 | √ 7 | 8 | √ 10 | 12 | √ 13 | √ 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $wx'z'$ | 1 0 – 0 | K | 4 | | | | X | X | | | |
| $wxy'$ | 1 1 0 – | L | 4 | | | | | | X | X | |
| $x'yz'$* | – 0 1 0 | N | 4 | X | | | | X | | | |
| $wy'z'$ | 1 – 0 0 | P | 4 | | | | X | | X | | |
| $xz$* | – 1 – 1 | S | 3 | | X | X | | | | X | X |

The 1's that make two prime implicants essential, $x'yz'$ and $xz$, are shaded, and the minterms covered by them are checked. The table then reduces to

| | | $ | 8 | 12 |
|---|---|---|---|---|
| $wx'z'$ | K | 4 | X | |
| $wxy'$ | L | 4 | | X |
| $wy'z'$ | P | 4 | X | X |

Clearly, term *P* must be used; otherwise, two more terms would be necessary. The solution becomes

$$f = x'yz' + xz + wy'z'$$

**b.** We first map the function (as in Solved Problem 1d) and find four prime implicants that cover the function. We then use iterated consensus to generate the rest.

| | | | | |
|---|---|---|---|---|
| A | 0 – 0 0 | ~~E~~ | ~~0 1 0 –~~ | B ¢ A |
| B | – 1 – 1 | ~~F~~ | ~~0 1 – 0~~ | C ¢ A |
| C | – 1 1 – | G | 1 – 1 – | D ¢ C |
| D | 1 0 – – | H | 1 – – 1 | D ¢ B |
| | | J | – 0 0 0 | D ¢ A |
| | | K | 0 1 – – | E ¢ C ≥ E, F |

No other consensus terms are formed. The prime implicant table is

| | | $ | 0 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *A* | 0 – 0 0 | 4 | X | X | | | | | | | | | | |
| *B* | – 1 – 1 | 3 | | | X | | X | | | | | X | | X |
| *C* | – 1 1 – | 3 | | | | X | X | | | | | | X | X |
| *D* | 1 0 – – | 3 | | | | | | X | X | X | X | | | |
| *G* | 1 – 1 – | 3 | | | | | | | | X | X | | X | X |
| *H* | 1 – – 1 | 3 | | | | | | | X | | X | X | | X |
| *J* | – 0 0 0 | 4 | X | | | | | X | | | | | | |
| *K* | 0 1 – – | 3 | | X | X | X | X | | | | | | | |

There are no essential prime implicants. The starting point should be one of the columns in which there are only two X's. We will choose minterm 5, since both terms will cover four 1's (but we could have used minterm 0, 4, 5, 6, 8, 9, 10, 13, or 14). We will first try prime implicant *B*; then we will try prime implicant *K*. If we choose *B*, the table reduces to

| | | | | | | √ | √ | √ | √ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $ | 0 | 4 | 6 | 8 | 9 | 10 | 11 | 14 |
| *A* | 0 – 0 0 | 4 | X | X | | | | | | |
| *C* | – 1 1 – | 3 | | | X | | | | | X |
| *D* | 1 0 – –* | 3 | | | | X | X | X | X | |
| *G* | 1 – 1 – | 3 | | | | | | X | X | X |
| *H* | 1 – – 1 | 3 | | | | | X | | X | |
| *J* | – 0 0 0 | 4 | X | | | X | | | | |
| *K* | 0 1 – – | 3 | | X | X | | | | | |

Row *H* is dominated by row *D*. If row *D* is chosen, the table reduces to

| | | $ | 0 | 4 | 6 | 14 |
|---|---|---|---|---|---|---|
| *A* | 0 – 0 0 | 4 | X | X | | |
| *C* | – 1 1 – | 3 | | | X | X |
| *G* | 1 – 1 – | 3 | | | | X |
| *J* | – 0 0 0 | 4 | X | | | |
| *K* | 0 1 – – | 3 | | X | X | |

At this point, the only way to cover the function with two terms is to choose *A* and *C*, giving a solution of

$$f = bd + ab' + a'c'd' + bc$$

Notice that if the dominated term *H* had been chosen instead of *D*, three additional terms would be required to cover the function, since minterms 8 and 10 are not covered by *H*.

Now, we must consider what happens if we chose term *K* instead of *B*. The resulting table is

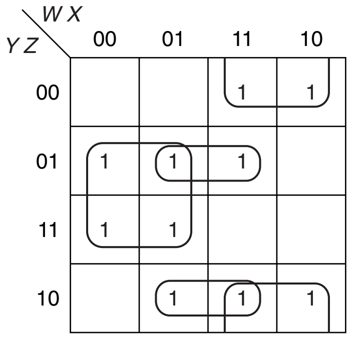
| | | | √ | √ | | √ | √ | | √ | √ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | **$** | **0** | **8** | **9** | **10** | **11** | **13** | **14** | **15** |
| *A* | 0 – 0 0 | 4 | X | | | | | | | |
| *B* | – 1 – 1 | 3 | | | | | | X | | X |
| *C* | – 1 1 – | 3 | | | | | | | X | X |
| *D* | 1 0 – – | 3 | | X | X | X | X | | | |
| *G* | 1 – 1 –* | 3 | | | | X | X | | X | X |
| *H* | 1 – – 1 | 3 | | | X | | X | X | | X |
| *J* | – 0 0 0* | 4 | X | X | | | | | | |

Prime implicant *A* is dominated by *J*, and *C* is dominated by *G*. Eliminating them, we must choose *J* and *G*, leaving only minterms 9 and 13 uncovered. They can both be covered by term *H*. No other solution (that used *K*) requires as few as four terms (even those using one of the dominated terms, *A* or *C*). The resulting function, the second equally good solution, is

$$f = a'b + ac + b'c'd' + ad$$

**c.** First, we took the map of the function and converted all of the don't cares to 1's. We then found a set of prime implicants that covered the function. (We could have used any set of product terms that covered the function, but starting with prime implicants usually reduces the amount of work.)

| YZ \ WX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | | | X | X |
| 01 | 1 | 1 | 1 | |
| 11 | 1 | 1 | | |
| 10 | | 1 | 1 | X |



Iterated consensus then proceeds very smoothly

| | | | | | |
|---|---|---|---|---|---|
| *A* | 0 | – | – | 1 | |
| *B* | 1 | – | – | 0 | |
| *C* | – | 1 | 0 | 1 | |
| *D* | – | 1 | 1 | 0 | |
| *E* | 1 | 1 | 0 | – | $C \text{¢} B$ |
| *F* | 0 | 1 | 1 | – | $D \text{¢} A$ |

No other new terms are formed. The only other consensus terms formed are

$$E \text{¢} D = 1\ 1\ -\ 0 \le B$$
$$E \text{¢} A = C$$
$$F \text{¢} C \le A$$
$$F \text{¢} B = D$$

The prime implicant table becomes

| | | $ | √ 1 | √ 3 | √ 5 | 6 | √ 7 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|
| 0 – – 1* | *A* | 3 | X | X | X | | X | | |
| 1 – – 0 | *B* | 3 | | | | | | | X |
| – 1 0 1 | *C* | 4 | | | X | | | X | |
| – 1 1 0 | *D* | 4 | | | | X | | | X |
| 1 1 0 – | *E* | 4 | | | | | | X | |
| 0 1 1 – | *F* | 4 | | | | X | X | | |

Note that there are no columns for the don't cares; they do not need to be covered. There is one essential prime implicant, *A* ($W'Z$), and the table can then be reduced to

| | | $ | 6 | 13 | 14 |
|---|---|---|---|---|---|
| 1 – – 0 | *B* | 3 | | | X |
| – 1 0 1 | *C* | 4 | | X | |
| – 1 1 0 | *D* | 4 | X | | X |
| 1 1 0 – | *E* | 4 | | X | |
| 0 1 1 – | *F* | 4 | X | | |

A study of the reduced table reveals that row *D* must be chosen; otherwise, it would take both terms *B* and *F* to cover minterms 6 and 14. That leaves us two choices to conclude the cover, *C* or *E*. Thus, the two solutions to the problem are

$$F = W'Z + XYZ' + XY'Z$$
$$F = W'Z + XYZ' + WXY'$$

**d.** Next, we will look at the solution for the complex function of Example 3.15, the map of which is shown below:

| cd \ ab | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | | 1 | |
| 01 | | 1 | 1 | 1 |
| 11 | 1 | 1 | | 1 |
| 10 | | 1 | 1 | 1 |

A set of prime implicants that covers this function is shown in the following list (above the dashed line); the remaining prime implicants are found applying iterated consensus:

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | *A* |
| 0 | – | 1 | 1 | *B* |
| – | 1 | 0 | 1 | *C* |
| – | 1 | 1 | 0 | *D* |
| 1 | 1 | 0 | – | *E* |
| 1 | 0 | – | 1 | *F* |
| 1 | 0 | 1 | – | *G* |

| | | | | |
|---|---|---|---|---|
| 0 | 1 | – | 1 | *H* |
| 0 | 1 | 1 | – | *J* |
| 1 | 1 | – | 0 | *K* |
| 1 | – | 0 | 1 | *L* |
| – | 0 | 1 | 1 | *M* |
| 1 | – | 1 | 0 | *N* |

The prime implicant table is shown below.

| PI | | 0 | 3 | 5 | 6 | 7 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 0 0 0* | *A* | X | | | | | | | | | | |
| 0 – 1 1 | *B* | | X | | | X | | | | | | |
| – 1 0 1 | *C* | | | X | | | | | | | X | |
| – 1 1 0 | *D* | | | | X | | | | | | | X |
| 1 1 0 – | *E* | | | | | | | | | X | X | |
| 1 0 – 1 | *F* | | | | | | X | | X | | | |
| 1 0 1 – | *G* | | | | | | | X | X | | | |
| 0 1 – 1 | *H* | | | X | | X | | | | | | |
| 0 1 1 – | *J* | | | | X | X | | | | | | |
| 1 1 – 0 | *K* | | | | | | | | | X | | X |
| 1 – 0 1 | *L* | | | | | | X | | | | X | |
| – 0 1 1 | *M* | | X | | | | | | X | | | |
| 1 – 1 0 | *N* | | | | | | | X | | | | X |

The cost column has been omitted since all terms, other than the one essential prime implicant, have three literals. There is one essential prime implicant, *A*, which covers minterm 0. Otherwise, each minterm is covered by two or three prime implicants. We will start by choosing one of the two prime implicants that covers minterm 3, *B*. The reduced table becomes:

| | | √ | √ | | | | | √ | √ |
|---|---|---|---|---|---|---|---|---|---|
| **PI** | | **5** | **6** | **9** | **10** | **11** | **12** | **13** | **14** |
| – 1 0 1* | *C* | X | | | | | | X | |
| – 1 1 0* | *D* | | X | | | | | | X |
| 1 1 0 – | *E* | | | | | | X | X | |
| 1 0 – 1 | *F* | | | X | | X | | | |
| 1 0 1 – | *G* | | | | X | X | | | |
| 0 1 – 1 | *H* | X | | | | | | | |
| 0 1 1 – | *J* | | X | | | | | | |
| 1 1 – 0 | *K* | | | | | | X | | X |
| 1 – 0 1 | *L* | | | X | | | | X | |
| – 0 1 1 | *M* | | | | | X | | | |
| 1 – 1 0 | *N* | | | | X | | | | X |

As can be seen from this table, *H*, *J*, and *M* are dominated. If they are eliminated, then *C* and *D* must be used. That would result in the table shown below. Understand that we may have eliminated some of the equally good solutions (that is, they may have used one of the dominated rows). However, no solution that has been eliminated will be any better than one of these that we will now find

| PI | | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|
| 1 1 0 – | *E* | | | | X |
| 1 0 – 1 | *F* | X | | X | |
| 1 0 1 – | *G* | | X | X | |
| 1 1 – 0 | *K* | | | | X |
| 1 – 0 1 | *L* | X | | | |
| 1 – 1 0 | *N* | | X | | |

As can be seen from the table, we need three more prime implicants to cover the remaining four minterms. We can use *EFG*, *EFN*, *EGL*, *FGK*, *FKN*, and *GKL* (in addition to $a'b'c'd'$). Thus, we need seven prime implicants.

Returning to the original table, we could have chosen *M* instead of *B*. That would produce

| | | | | | √ | √ | | √ | √ |
|---|---|---|---|---|---|---|---|---|---|
| **PI** | | **5** | **6** | **7** | **9** | **10** | **12** | **13** | **14** |
| 0 – 1 1 | *B* | | | X | | | | | |
| – 1 0 1 | *C* | X | | | | | | X | |
| – 1 1 0 | *D* | | X | | | | | | X |
| 1 1 0 – | *E* | | | | | | X | X | |
| 1 0 – 1 | *F* | | | | X | | | | |
| 1 0 1 – | *G* | | | | | X | | | |
| 0 1 – 1 | *H* | X | | X | | | | | |
| 0 1 1 – | *J* | | X | X | | | | | |
| 1 1 – 0 | *K* | | | | | | X | | X |
| 1 – 0 1* | *L* | | | | X | | | X | |
| 1 – 1 0* | *N* | | | | | X | | | X |

*B*, *F*, and *G* are dominated; if they are removed, then *L* and *N* must be used, producing the following table:

| PI | | 5 | 6 | 7 | 12 |
|---|---|---|---|---|---|
| – 1 0 1 | *C* | X | | | |
| – 1 1 0 | *D* | | X | | |
| 1 1 0 – | *E* | | | | X |
| 0 1 – 1 | *H* | X | | X | |
| 0 1 1 – | *J* | | X | X | |
| 1 1 – 0 | *K* | | | | X |

As before, we need three more prime implicants to cover the remaining four minterms. We can use *CEJ*, *CJK*, *DEH*, *DHK*, *EHJ*, and *HJK* (in addition to *ALMN*). This method has produced 12 of the 32 minimum covers.

A complete list could be obtained by considering the use of some of the dominated rows or by the algebraic method, but that would require a great deal of manipulation.

**e.** We will start the process by finding a cover using only terms on one layer of the map. Obviously, all of them are not prime implicants; iterated consensus will find all of the prime implicants

| | | | | | | |
|---|---|---|---|---|---|---|
| *A* | 0 | – | 0 | 0 | – | |
| ~~*B*~~ | ~~0~~ | ~~0~~ | ~~1~~ | ~~–~~ | ~~1~~ | |
| ~~*C*~~ | ~~0~~ | ~~–~~ | ~~1~~ | ~~1~~ | ~~0~~ | |
| *D* | 1 | – | 1 | – | 0 | |
| *E* | 1 | – | – | 0 | 1 | |
| *F* | 1 | 0 | 1 | – | – | |
| ~~*G*~~ | ~~0~~ | ~~0~~ | ~~–~~ | ~~0~~ | ~~1~~ | *B* ¢ *A* |
| ~~*H*~~ | ~~0~~ | ~~0~~ | ~~1~~ | ~~1~~ | ~~–~~ | *C* ¢ *B* |
| *J* | – | – | 1 | 1 | 0 | *D* ¢ *C* ≥ *C* |
| *K* | 1 | – | 1 | 0 | – | *E* ¢ *D* |
| ~~*L*~~ | ~~–~~ | ~~0~~ | ~~1~~ | ~~0~~ | ~~1~~ | *E* ¢ *B* |
| *M* | – | – | 0 | 0 | 1 | *E* ¢ *A* |
| *N* | – | 0 | 1 | – | 1 | *F* ¢ *B* ≥ *L*, *B* |
| *P* | – | 0 | – | 0 | 1 | *G* ¢ *E* ≥ *G* |
| *Q* | – | 0 | 1 | 1 | – | *H* ¢ *F* ≥ *H* |

The balance of the consensus operations produce no new terms. There are 10 prime implicants, all groups of four. The prime implicant table is thus

| | 0 | 1 | 5 | 6 | 7 | 8 | 9 | 14 | 17 | 20 | 21 | 22 | 23 | 25 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | √ | √ | | √ | | √ | √ | √ | | | | √ | | | | | √ |
| *A** | X | X | | | | X | X | | | | | | | | | | |
| *D* | | | | | | | | | | X | | X | | | X | | X |
| *E* | | | | | | | | | X | | X | | | X | | X | |
| *F* | | | | | | | | | | X | X | X | X | | | | |
| *J** | | | | X | | | | X | | | | X | | | | | X |
| *K* | | | | | | | | | | X | X | | | | X | X | |
| *M* | | X | | | | | X | | X | | | | | X | | | |
| *N* | | | X | | X | | | | | | X | | X | | | | |
| *P* | | X | X | | | | | | X | | X | | | | | | |
| *Q* | | | | X | X | | | | | | | X | X | | | | |

There are two essential prime implicants $A(V'X'Y')$ and $J(XYZ')$. We can then reduce the table to

| | 5 | 7 | 17 | 20 | 21 | 23 | 25 | 28 | 29 |
|---|---|---|---|---|---|---|---|---|---|
| *D* | | | | X | | | | X | |
| *E* | | | X | | X | | X | | X |
| *F* | | | | X | X | X | | | |
| *K* | | | | X | X | | | X | X |
| *M* | | | X | | | | X | | |
| *N* | X | X | | | X | X | | | |
| *P* | X | | X | | X | | | | |
| *Q* | | X | | | | X | | | |

There are three dominated rows, *D* by *K*, *M* by *E*, and *Q* by *N*. If all of the dominated rows are eliminated, we have the following table:

| | 5 | 7 | 17 | 20 | 21 | 23 | 25 | 28 | 29 |
|---|---|---|---|---|---|---|---|---|---|
| | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| *E* | | | X | | X | | X | | X |
| *F* | | | | X | X | X | | | |
| *K* | | | | X | X | | | X | X |
| *N* | X | X | | | X | X | | | |
| *P* | X | | X | | X | | | | |

With these five prime implicants available, $E$ must be chosen to cover $m_{25}$, $N$ must be chosen for $m_7$, and $K$ must be chosen for $m_{28}$. Thus, one of the minimum solutions is

$$A + J + E + N + K$$

If we want all of the minimum solutions, we must see if any of the dominated terms could be used. We cannot use more than one of these, since each covers only two 1's. Using two of them would leave five 1's uncovered, requiring two new terms. We could use $D$ in place of $K$; $K$ covers terms 21 and 29, in addition to terms 20 and 28 covered by both $D$ and $K$. But, 21 and 29 are also covered by $E$. Thus, a second solution is

$$A + J + E + N + D$$

Next, we will consider what would happen if we used $M$ in place of $E$ (the row that dominated it). If we add $M$ back into the table, we get

| | **5** | **7** | **17** | **20** | **21** | **23** | **25** | **28** | **29** |
|---|---|---|---|---|---|---|---|---|---|
| | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| *E* | | | X | | X | | X | | X |
| *F* | | | | X | X | X | | | |
| *K** | | | | X | X | | | X | X |
| *M** | | | X | | | | X | | |
| *N** | X | X | | | X | X | | | |
| *P* | X | | X | | X | | | | |

Prime implicants $N$ and $K$ can be used to complete the function, producing another solution,

$$A + J + M + N + K$$

Finally, if $Q$ were used instead of $N$, but we continued to use $E$ and $K$, we would get the following table

| | **5** | **7** | **17** | **20** | **21** | **23** | **25** | **28** | **29** |
|---|---|---|---|---|---|---|---|---|---|
| | | √ | √ | √ | √ | √ | √ | √ | √ |
| *E** | | | X | | X | | X | | X |
| *F* | | | | X | X | X | | | |
| *K** | | | | X | X | | | X | X |
| *N* | X | X | | | X | X | | | |
| *P* | X | | X | | X | | | | |
| *Q** | | X | | | | X | | | |

Terms $E$ and $K$ would still be required (in addition to $A$, $J$, and $Q$), but $m_5$ is left uncovered, requiring a sixth term. Obviously, this is not minimum. Thus, the three minimum solutions are

$$G = V'X'Y' + XYZ' + VY'Z + W'XZ + VXY'$$
$$G = V'X'Y' + XYZ' + VY'Z + W'XZ + VXZ'$$
$$G = V'X'Y' + XYZ' + X'Y'Z + W'XZ + VXY'$$

**12.** For each of the following sets of functions, find all of the minimum sum of product expressions using the iterated consensus method. (The first four are the same problems as SP10a, c, d, e.)

**a.** $f(a, b, c, d) = \Sigma m(0, 1, 2, 3, 5, 7, 8, 10, 11, 13)$
$g(a, b, c, d) = \Sigma m(0, 2, 5, 8, 10, 11, 13, 15)$

**b.** $F(W, X, Y, Z) = \Sigma m(2, 3, 6, 7, 8, 9, 13)$
$G(W, X, Y, Z) = \Sigma m(2, 3, 6, 7, 9, 10, 13, 14)$
$H(W, X, Y, Z) = \Sigma m(0, 1, 4, 5, 9, 10, 13, 14)$

**c.** $f(a, b, c, d) = \Sigma m(0, 2, 3, 8, 9, 10, 11, 12, 13, 15)$
$g(a, b, c, d) = \Sigma m(3, 5, 7, 12, 13, 15)$
$h(a, b, c, d) = \Sigma m(0, 2, 3, 4, 6, 8, 10, 14)$

**d.** $f(a, b, c, d) = \Sigma m(0, 3, 5, 7) + \Sigma d(10, 11, 12, 13, 14, 15)$
$g(a, b, c, d) = \Sigma m(0, 5, 6, 7, 8) + \Sigma d(10, 11, 12, 13, 14, 15)$

**e.** $f(w, x, y, z) = \Sigma m(5, 7, 9, 11, 13, 15)$
$g(w, x, y, z) = \Sigma m(1, 5, 7, 9, 10, 11, 14)$

**a.** The maps of $f$, $g$, and $fg$ are shown below

| cd \ ab | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | | | 1 |
| 01 | 1 | 1 | 1 | |
| 11 | 1 | 1 | | 1 |
| 10 | 1 | | | 1 |

*f*

| cd \ ab | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | | | 1 |
| 01 | | 1 | 1 | |
| 11 | | | 1 | 1 |
| 10 | 1 | | | 1 |

*g*

| cd \ ab | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | | | 1 |
| 01 | | 1 | 1 | |
| 11 | | | | 1 |
| 10 | 1 | | | 1 |

*f g*

We will start by finding all of the prime implicants of the product, $fg$, and then those that are prime implicants of the individual functions but not of the product. (Of course, we will add the appropriate tag section, – if it is included in the function, 0 if it is not.)

| | | |
|---|---|---|
| $A$ | – 0 – 0 | – – |
| $B$ | 1 0 1 – | – – |
| $C$ | – 1 0 1 | – – |
| $D$ | 0 0 – – | – 0 |
| $E$ | 0 – – 1 | – 0 |
| $F$ | 1 1 – 1 | 0 – |
| $G$ | 1 – 1 1 | 0 – |

After "completing" the list, it is a good idea to try the consensus of all pairs of terms, in case we missed one. In this case, we did.

$H$ – 0 1 – – 0 $D$ ¢ $B$

Note that in trying the consensus, there is no need to take the consensus of $F$ or $G$ with $D$ or $E$ (or $H$) since the tag would be 0 0, indicating that the term is in neither function.

We can now construct a prime implicant table

| | | | f | | | | | | | | | | g | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | √ | √ | √ | √ | √ | √ | √ | √ | | √ | √ | √ | √ | √ | √ | | √ | |
| | | | **0** | **1** | **2** | **3** | **5** | **7** | **8** | **10** | **11** | **13** | **0** | **2** | **5** | **8** | **10** | **11** | **13** | **15** |
| – 0 – 0* | 3 | *A* | X | | X | | | | X | X | | | X | X | | X | X | | | |
| 1 0 1 – | 4 | *B* | | | | | | | | X | X | | | | | | X | X | | |
| – 1 0 1* | 4 | *C* | | | | | X | | | | | X | | | X | | | | X | |
| 0 0 – – | 3 | *D* | X | X | X | X | | | | | | | | | | | | | | |
| 0 – – 1* | 3 | *E* | | X | | X | X | X | | | | | | | | | | | | |
| 1 1 – 1 | 4 | *F* | | | | | | | | | | | | | | | | | X | X |
| 1 – 1 1 | 4 | *G* | | | | | | | | | | | | | | | | X | | X |
| – 0 1 – | 3 | *H* | | | X | X | | | | X | X | | | | | | | | | |

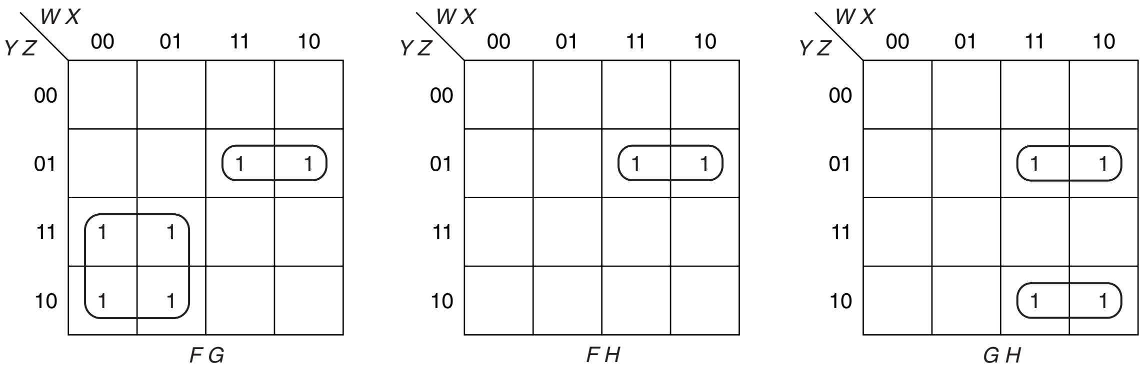
Terms $A$ and $C$ are essential prime implicants of both $f$ and $g$, and $E$ is an essential prime implicant of $f$. The reduced table thus becomes

| | | | f | g | |
|---|---|---|---|---|---|
| | | | **11** | **11** | **15** |
| 1 0 1 – | 4 | *B* | X | X | |
| 0 0 – – | 3 | *D* | | | |
| 1 1 – 1 | 4 | *F* | | | X |
| 1 – 1 1 | 4 | *G* | | X | X |
| – 0 1 – | 3 | *H* | X | | |

Term $G$ completes the cover of $g$ and term $H$ would then be used for $f$ since it is less expensive than $B$. The other option, using $B$ for both $f$ and $g$, and then using either $F$ or $G$ to cover $m_{15}$ in $g$, would cost an extra input. The solution thus becomes

$$f = b'd' + bc'd + a'd + b'c$$
$$g = b'd' + bc'd + acd$$

**b.** We first map the prime implicants of the products, $FG$, $FH$, and $GH$. (We also need the product $FGH$, but that is equal to $FH$.)




From these maps, and the maps for the solution of SP11c, we get the following terms:

| | | |
|---|---|---|
| $A$ | 0 – 1 – | – – 0 |
| $B$ | 1 – 0 1 | – – – |
| $C$ | 1 – 1 0 | 0 – – |
| $D$ | 1 0 0 – | – 0 0 |
| $E$ | – – 1 0 | 0 – 0 |
| $F$ | – – 0 1 | 0 0 – |
| $G$ | 0 – 0 – | 0 0 – |

This produces the prime implicant table shown next. Notice that essential prime implicants cover all of $F$, and all but two minterms of $G$ and of $H$.

| | | | F | | | | | | | G | | | | | | | | H | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | | √ | | √ | √ | √ | √ | | √ | | √ |
| | | | 2 | 3 | 6 | 7 | 8 | 9 | 13 | 2 | 3 | 6 | 7 | 9 | 10 | 13 | 14 | 0 | 1 | 4 | 5 | 9 | 10 | 13 | 14 |
| 0 – 1 –* | *A* | 3 | X | X | X | X | | | | X | X | X | X | | | | | | | | | | | | |
| 1 – 0 1* | *B* | 4 | | | | | | X | X | | | | | X | | X | | | | | | X | | X | |
| 1 – 1 0* | *C* | 4 | | | | | | | | | | | | | X | | X | | | | | | X | | X |
| 1 0 0 –* | *D* | 4 | | | | | X | X | | | | | | | | | | | | | | | | | |
| – – 1 0 | *E* | 3 | | | | | | | | X | | X | | | X | | X | | | | | | | | |
| – – 0 1 | *F* | 3 | | | | | | | | | | | | | | | | | X | | X | X | | X | |
| 0 – 0 –* | *G* | 3 | | | | | | | | | | | | | | | | X | X | X | X | | | | |

The table reduces to only four rows. Clearly terms *B* and *C* are used since they cost only 1; the AND gates were implemented for other functions.

| | | | G | | H | |
|---|---|---|---|---|---|---|
| | | | 10 | 14 | 9 | 13 |
| 1 – 0 1* | *B* | 1 | | | X | X |
| 1 – 1 0* | *C* | 1 | X | X | | |
| – – 1 0 | *E* | 3 | X | X | | |
| – – 0 1 | *F* | 3 | | | X | X |

The resulting equations are

$$F = W'Y + WY'Z + WX'Y'$$
$$G = W'Y + WY'Z + WYZ'$$
$$H = WYZ' + W'Y' + WY'Z$$

**c.** When we map the various products and find all of the prime implicants, we come up with the following prime implicant table. Note that because of its size, we have broken it into two parts. We show all of the prime implicants in each part of the table, although some of the rows are empty in one part of the table. After finding essential prime implicants, we will be able to combine the tables and complete the problem.

| | | | *f* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | √ | √ | | √ | | √ | | | | |
| | | | **0** | **2** | **3** | **8** | **9** | **10** | **11** | **12** | **13** | **15** |
| 0 0 1 1 | *A* | 5 | | | X | | | | | | | |
| 1 1 0 – | *B* | 4 | | | | | | | | X | X | |
| 1 1 – 1 | *C* | 4 | | | | | | | | | X | X |
| – 0 – 0* | *D* | 3 | X | X | | X | | X | | | | |
| 0 0 1 – | *E* | 4 | | X | X | | | | | | | |
| 1 0 – – | *F* | 3 | | | | X | X | X | X | | | |
| 1 – 0 – | *G* | 3 | | | | X | X | | | X | X | |
| 1 – – 1 | *H* | 3 | | | | | X | | X | | X | X |
| – 0 1 – | *J* | 3 | | X | X | | | X | X | | | |
| – 1 – 1 | *K* | 3 | | | | | | | | | | |
| 0 – 1 1 | *L* | 4 | | | | | | | | | | |
| 0 – – 0 | *M* | 3 | | | | | | | | | | |
| – – 1 0 | *N* | 3 | | | | | | | | | | |

| | | | *g* | | | | | | *h* | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | √ | √ | √ | √ | √ | √ | √ | | √ | √ | √ | √ | √ |
| | | | **3** | **5** | **7** | **12** | **13** | **15** | **0** | **2** | **3** | **4** | **6** | **8** | **10** | **14** |
| 0 0 1 1 | *A* | 5 | X | | | | | | | | X | | | | | |
| 1 1 0 –* | *B* | 4 | | | | X | X | | | | | | | | | |
| 1 1 – 1 | *C* | 4 | | | | | X | X | | | | | | | | |
| – 0 – 0* | *D* | 3 | | | | | | | X | X | | | | X | X | |
| 0 0 1 – | *E* | 4 | | | | | | | | X | X | | | | | |
| 1 0 – – | *F* | 3 | | | | | | | | | | | | | | |
| 1 – 0 – | *G* | 3 | | | | | | | | | | | | | | |
| 1 – – 1 | *H* | 3 | | | | | | | | | | | | | | |
| – 0 1 – | *J* | 3 | | | | | | | | | | | | | | |
| – 1 – 1* | *K* | 3 | | X | X | | X | X | | | | | | | | |
| 0 – 1 1 | *L* | 4 | X | | X | | | | | | | | | | | |
| 0 – – 0* | *M* | 3 | | | | | | | X | X | | X | X | | | |
| – – 1 0* | *N* | 3 | | | | | | | | X | | | X | | X | X |

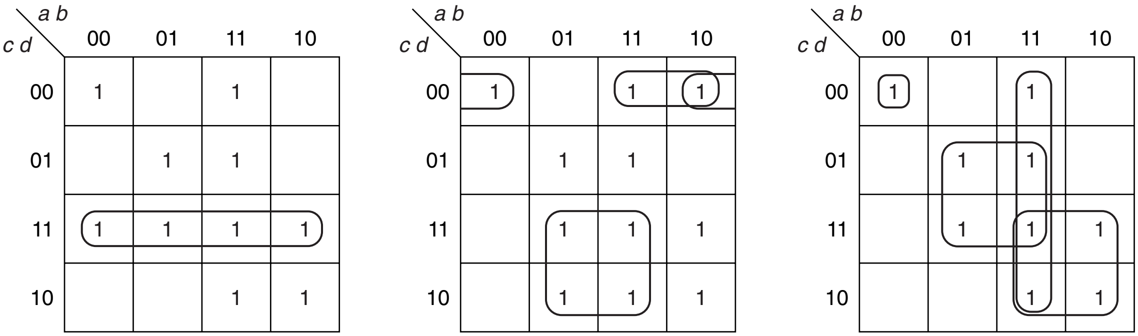
The table can be reduced and the two halves combined as shown below. Note that all of $g$ and $h$ other than minterm 3 have already been covered and that the cost of prime implicant $B$ has been reduced to 1, since it is an essential prime implicant of $g$.

| | | | f: 3 | f: 9 √ | f: 11 √ | f: 12 | f: 13 √ | f: 15 √ | g: 3 | h: 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 0 1 1 | *A* | 5 | X | | | | | | X | X |
| 1 1 0 – | *B* | 1 | | | | X | X | | | |
| 1 1 – 1 | *C* | 4 | | | | | X | X | | |
| 0 0 1 – | *E* | 4 | X | | | | | | | X |
| 1 0 – – | *F* | 3 | | X | X | | | | | |
| 1 – 0 – | *G* | 3 | | X | | X | X | | | |
| 1 – – 1 | *H* | 3 | | X | X | | X | X | | |
| – 0 1 – | *J* | 3 | X | | X | | | | | |
| 0 – 1 1 | *L* | 4 | | | | | | | X | |

Clearly, prime implicant *A* should be used to cover $m_3$ in both *g* and *h* (at a cost of $5 + 1 = 6$), since otherwise we would need both *E* and *L* at a cost of 8. For *f*, we can eliminate prime implicant *C*, since that row is dominated by row *H* and costs more. That requires us to choose *H* to cover $m_{15}$. Once *H* is chosen, all that remains to be covered are $m_3$ and $m_{12}$, which can be covered by *A* and *B* (respectively), each at a cost of 1. (*J* or *G* could have been used, but they would cost 3 each.) The final functions are

$$f = b'd' + ad + a'b'cd + abc'$$
$$g = abc' + bd + a'b'cd$$
$$h = b'd' + a'd' + cd' + a'b'cd$$

**d.** In finding the prime implicants, we must treat all don't cares as 1's. When we construct the prime implicant table, we do not have columns for the don't cares, since they need not be covered. We first map *f*, *g*, and *fg*, converting all X's to 1's to find the prime implicants. (Once again, it is a good idea to check that none have been missed by using the iterated consensus algorithm on the result.)




*f* *g* *fg*

This produces the following prime implicant table.

| | | | f | | | | g | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | √ 0 | √ 3 | √ 5 | √ 7 | 0 | √ 5 | √ 6 | √ 7 | 8 |
| 1 1 – – | *A* | 3 | | | | | | | | | |
| – 1 – 1* | *B* | 3 | | | X | X | | X | | X | |
| 1 – 1 – | *C* | 3 | | | | | | | | | |
| 0 0 0 0* | *D* | 5 | X | | | | X | | | | |
| – – 1 1* | *E* | 3 | | X | | X | | | | | |
| – 0 0 0 | *F* | 4 | | | | | X | | | | X |
| – 1 1 –* | *G* | 3 | | | | | | | X | X | |
| 1 – 0 0 | *H* | 4 | | | | | | | | | X |

Notice that prime implicants *A* and *C* cover no minterms; they are both groups of four don't cares. The function *f* is covered by essential prime implicants; only minterms 0 and 8 of *g* are left. The reduced prime implicant table (for *g*) becomes

| | | | g 0 | 8 |
|---|---|---|---|---|
| 0 0 0 0 | *D* | 1 | X | |
| – 0 0 0 | *F* | 4 | X | X |
| 1 – 0 0 | *H* | 4 | | X |

There are two equally good solutions. Prime implicant *F* covers both minterms, but requires one AND gate and five inputs. Prime implicant *D* was an essential prime implicant of *f* and thus does not require a new AND gate and only one gate input. Thus, *F* and *H* also produce a solution that requires one new gate and five inputs. The two solutions are

$$f = bd + a'b'c'd' + cd$$

and

$$g = bd + bc + b'c'd'$$

or

$$g = bd + bc + a'b'c'd' + ad'$$

**e.** We will solve this problem by starting with minterms and finding all of the prime implicants.

| | | |
|---|---|---|
| ~~1 0 0 0 1 0 –~~ | *A* 0 – 0 1 0 – | 5 ¢ 1 ≥ 1 |
| ~~5 0 1 0 1 – –~~ | *B* 0 1 – 1 – – | 7 ¢ 5 ≥ 7, 5 |
| ~~7 0 1 1 1 – –~~ | *C* 1 0 1 – 0 – | 11 ¢ 10 ≥ 10 |
| ~~9 1 0 0 1 – –~~ | *D* 1 0 – 1 – – | 11 ¢ 9 ≥ 11, 9 |
| ~~10 1 0 1 0 0 –~~ | ~~*E* 1 1 – 1 – 0~~ | 15 ¢ 13 ≥ 15, 13 |
| ~~11 1 0 1 1 – –~~ | *F* 1 – 1 0 0 – | *C* ¢ 14 ≥ 14 |
| ~~13 1 1 0 1 – 0~~ | *G* – 0 0 1 0 – | *D* ¢ *A* |
| ~~14 1 1 1 0 0 –~~ | *H* 1 – – 1 – 0 | *E* ¢ *D* ≥ *E* |
| ~~15 1 1 1 1 – 0~~ | *J* – 1 – 1 – 0 | *H* ¢ *B* |

Each of the new terms that is created by consensus is shown; all of the original terms and one of the groups of 2 are included in a larger prime implicant. The resulting table is

| | | | f | | | | | | g | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | √ | √ | | √ | | √ |
| | | | **5** | **7** | **9** | **11** | **13** | **15** | **1** | **5** | **7** | **9** | **10** | **11** | **14** |
| 0 – 0 1 | *A* | 4 | | | | | | | X | X | | | | | |
| 0 1 – 1* | *B* | 4 | X | X | | | | | | X | X | | | | |
| 1 0 1 – | *C* | 4 | | | | | | | | | | | X | X | |
| 1 0 – 1 | *D* | 4 | | | X | X | | | | | | X | | X | |
| 1 – 1 0* | *F* | 4 | | | | | | | | | | | X | | X |
| – 0 0 1 | *G* | 4 | | | | | | | X | | | X | | | |
| 1 – – 1 | *H* | 3 | | | X | X | X | X | | | | | | | |
| – 1 – 1 | *J* | 3 | X | X | | | X | X | | | | | | | |

The two essential prime implicants of *g*, $w'xz$ and $wyz'$, are shown. The table is then reduced (and the cost of *B* is made 1, since the AND gate was already built).

| | | | f | | | | | | g | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | **5** | **7** | **9** | **11** | **13** | **15** | **1** | **9** | **11** |
| 0 – 0 1 | *A* | 4 | | | | | | | X | | |
| 0 1 – 1 | *B* | 1 | X | X | | | | | | | |
| 1 0 1 – | *C* | 4 | | | | | | | | | X |
| 1 0 – 1 | *D* | 4 | | | X | X | | | | X | X |
| – 0 0 1 | *G* | 4 | | | | | | | X | X | |
| 1 – – 1 | *H* | 3 | | | X | X | X | X | | | |
| – 1 – 1 | *J* | 3 | X | X | | | X | X | | | |

If we are looking for just one of the minimum solutions, we can eliminate rows *A* and *C*, because they are dominated by *D* and *G*, respectively, and cost the same to implement. If we do that, we would choose *D* and *G* to cover function *g*, leaving the following reduced table:

| | | | **f** 5 | 7 | 9 | 11 | 13 | 15 |
|---|---|---|---|---|---|---|---|---|
| 0 1 – 1 | *B* | 1 | X | X | | | | |
| 1 0 1 – | *C* | 4 | | | | | | |
| 1 0 – 1 | *D* | 1 | | | X | X | | |
| 1 – – 1 | *H* | 3 | | | X | X | X | X |
| – 1 – 1 | *J* | 3 | X | X | | | X | X |

We can cover function *f* with either *B* and *H* (at a cost of 4) or *D* and *J* (at a cost of 4). (Note that *H* and *J* would cover the function also, but the cost would be 6.) Thus, two of the minimum solutions (at a cost of seven gates and 20 inputs) are

$$f = w'xz + wz$$
$$g = w'xz + wyz' + wx'z + x'y'z$$

$$f = wx'z + xz$$
$$g = w'xz + wyz' + wx'z + x'y'z$$

However, there are other equally good solutions, using the two dominated rows we eliminated for the last table. In order to achieve the minimum cost, we must share either *B* or *D* between the two functions. If we share *B*, then we must use *H* for *f* (as in the first solution above). But, we can now use one of three solutions for the remainder of *g* (in addition to the essential prime implicants):

$$A + D$$
$$C + G$$
$$D + G$$

The third is the solution already found. Thus, the three solutions that share $w'xz$ are

$$f = w'xz + wz$$
$$g_1 = w'xz + wyz' + w'y'z + wx'z$$
$$g_2 = w'xz + wyz' + wx'y + x'y'z$$
$$g_3 = w'xz + wyz' + wx'z + x'y'z$$

If we share $wx'z$ (term $D$), then we can use either $A$ or $G$ to complete the cover of $g$, giving the two solutions below (one of which we found before), for a total of five equally good solutions.

$$f = wx'z + xz$$
$$g_1 = w'xz + wyz' + wx'z + w'y'z$$
$$g_2 = w'xz + wyz' + wx'z + x'y'z$$

## 3.4 Exercises

**1.** For each of the following, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

a. $f(a, b, c) = \Sigma m(1, 2, 3, 6, 7)$

b. $g(w, x, y) = \Sigma m(0, 1, 5, 6, 7)$ (2 solutions)

c. $h(a, b, c) = \Sigma m(0, 1, 2, 5, 6, 7)$ (2 solutions)

d. $f(a, b, c, d) = \Sigma m(1, 2, 3, 5, 6, 7, 8, 11, 13, 15)$

e. $G(W, X, Y, Z) = \Sigma m(0, 2, 5, 7, 8, 10, 12, 13)$

f. $h(a, b, c, d) = \Sigma m(2, 4, 5, 6, 7, 8, 10, 12, 13, 15)$ (2 solutions)

g. $f(a, b, c, d) = \Sigma m(1, 3, 4, 5, 6, 11, 12, 13, 14, 15)$ (2 solutions)

h. $g(w, x, y, z) = \Sigma m(2, 3, 6, 7, 8, 10, 11, 12, 13, 15)$ (2 solutions)

i. $h(p, q, r, s) = \Sigma m(0, 2, 3, 4, 5, 8, 11, 12, 13, 14, 15)$ (3 solutions)

j. $F(W, X, Y, Z) = \Sigma m(0, 2, 3, 4, 5, 8, 10, 11, 12, 13, 14, 15)$ (4 solutions)

k. $f(w, x, y, z) = \Sigma m(0, 1, 2, 4, 5, 6, 9, 10, 11, 13, 14, 15)$ (2 solutions)

l. $g(a, b, c, d) = \Sigma m(0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15)$

m. $H(W, X, Y, Z) = \Sigma m(0, 2, 3, 5, 7, 8, 10, 12, 13)$ (4 solutions)

n. $f(a, b, c, d) = \Sigma m(0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15)$ (6 solutions)

o. $g(w, x, y, z) = \Sigma m(0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 13, 14, 15)$ (6 solutions)

p. $f(a, b, c, d) = \Sigma m(0, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14)$ (32 solutions)

**2.** For the following functions,

i. List all prime implicants, indicating which are essential.

ii. Show the minimum sum of products expression(s).

a. $f(a, b, c, d) = \Sigma m(0, 3, 4, 5, 8, 11, 12, 13, 14, 15)$
b. $g(w, x, y, z) = \Sigma m(0, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15)$

**3.** Map each of the following functions and find the minimum sum of products expression:
a. $F = AD + AB + A'CD' + B'CD + A'BC'D'$
b. $g = w'yz + xy'z + wy + wxy'z' + wz + xyz'$

**4.** For each of the following, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.) Label the solutions $f_1, f_2, \ldots$.
a. $f(w, x, y, z) = \Sigma m(1, 3, 6, 8, 11, 14) + \Sigma d(2, 4, 5, 13, 15)$ (3 solutions)
b. $f(a, b, c, d) = \Sigma m(0, 3, 6, 9, 11, 13, 14) + \Sigma d(5, 7, 10, 12)$
c. $f(a, b, c, d) = \Sigma m(0, 2, 3, 5, 7, 8, 9, 10, 11) + \Sigma d(4, 15)$ (3 solutions)
d. $f(w, x, y, z) = \Sigma m(0, 2, 4, 5, 10, 12, 15) + \Sigma d(8, 14)$ (2 solutions)
e. $f(a, b, c, d) = \Sigma m(5, 7, 9, 11, 13, 14) + \Sigma d(2, 6, 10, 12, 15)$ (4 solutions)
f. $f(a, b, c, d) = \Sigma m(0, 2, 4, 5, 6, 7, 8, 9, 10, 14) + \Sigma d(3, 13)$ (3 solutions)
g. $f(w, x, y, z) = \Sigma m(1, 2, 5, 10, 12) + \Sigma d(0, 3, 4, 8, 13, 14, 15)$ (7 solutions)

**5.** For each of the functions of problem 4, indicate which solutions are equal.

**6.** For each of the following functions, find all of the minimum sum of products expressions and all of the minimum product of sums expressions:
a. $f(A, B, C, D) = \Sigma m(1, 4, 5, 6, 7, 9, 11, 13, 15)$
b. $f(W, X, Y, Z) = \Sigma m(2, 4, 5, 6, 7, 10, 11, 15)$
c. $f(A, B, C, D) = \Sigma m(1, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15)$ (1 SOP and 2 POS solutions)
d. $f(a, b, c, d) = \Sigma m(0, 2, 4, 6, 7, 9, 11, 12, 13, 14, 15)$ (2 SOP and 1 POS solutions)
e. $f(w, x, y, z) = \Sigma m(0, 4, 6, 9, 10, 11, 14) + \Sigma d(1, 3, 5, 7)$
f. $f(a, b, c, d) = \Sigma m(0, 1, 2, 5, 7, 9) + \Sigma d(6, 8, 11, 13, 14, 15)$ (4 SOP and 2 POS solutions)
g. $f(w, x, y, z) = \Sigma m(4, 6, 9, 10, 11, 13) + \Sigma d(2, 12, 15)$ (2 SOP and 2 POS solutions)
h. $f(a, b, c, d) = \Sigma m(0, 1, 4, 6, 10, 14) + \Sigma d(5, 7, 8, 9, 11, 12, 15)$ (13 SOP and 3 POS solutions)
i. $f(w, x, y, z) = \Sigma m(1, 3, 7, 11, 13, 14) + \Sigma d(0, 2, 5, 8, 10, 12, 15)$ (6 SOP and 1 POS solutions)

j. $f(a, b, c, d) = \Sigma m(0, 1, 6, 15) + \Sigma d(3, 5, 7, 11, 14)$
(1 SOP and 2 POS solutions)

**7.** Label the solutions of each part of problem 6 as $f_1, f_2, \ldots$ and indicate which solutions are equal.

**8.** For each part of problem 6, draw the block diagram of a two-level NAND gate circuit and a two-level NOR gate circuit. (For those parts with multiple solutions, you need only draw one NAND and one NOR solution.)

**9.** For each of the following five variable functions, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

a. $F(A, B, C, D, E) = \Sigma m(0, 1, 5, 7, 8, 9, 10, 11, 13, 15, 18, 20, 21, 23, 26, 28, 29, 31)$

b. $G(A, B, C, D, E) = \Sigma m(0, 1, 2, 4, 5, 6, 10, 13, 14, 18, 21, 22, 24, 26, 29, 30)$

c. $H(A, B, C, D, E) = \Sigma m(5, 8, 12, 13, 15, 17, 19, 21, 23, 24, 28, 31)$

d. $F(V, W, X, Y, Z) = \Sigma m(2, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 24, 25, 29, 30, 31)$

e. $G(V, W, X, Y, Z) = \Sigma m(0, 1, 4, 5, 8, 9, 10, 15, 16, 18, 19, 20, 24, 26, 28, 31)$

f. $H(V, W, X, Y, Z) = \Sigma m(0, 1, 2, 3, 5, 7, 10, 11, 14, 15, 16, 18, 24, 25, 28, 29, 31)$ (2 solutions)

g. $F(A, B, C, D, E) = \Sigma m(0, 4, 6, 8, 12, 13, 14, 15, 16, 17, 18, 21, 24, 25, 26, 28, 29, 31)$ (6 solutions)

h. $G(A, B, C, D, E) = \Sigma m(0, 3, 5, 7\ 12, 13, 14, 15, 19, 20, 21, 22, 23, 25, 26, 29, 30)$ (3 solutions)

i. $H(A, B, C, D, E) = \Sigma m(0, 1, 5, 6, 7, 8, 9, 14, 17, 20, 21, 22, 23, 25, 28, 29, 30)$ (3 solutions)

j. $F(V, W, X, Y, Z) = \Sigma m(0, 4, 5, 7, 10, 11, 14, 15, 16, 18, 20, 21, 23, 24, 25, 26, 29, 31)$ (4 solutions)

k. $G(V, W, X, Y, Z) = \Sigma m(0, 2, 5, 6, 8, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 29, 31)$
(3 solutions)

l. $H(V, W, X, Y, Z) = \Sigma m(0, 1, 2, 3, 5, 8, 9, 10, 13, 17, 18, 19, 20, 21, 26, 28, 29)$ (3 solutions)

m. $F(A, B, C, D, E) = \Sigma m(1, 2, 5, 8, 9, 10, 12, 13, 14, 15, 16, 18, 21, 22, 23, 24, 26, 29, 30, 31)$
(18 solutions)

n. $G(V, W, X, Y, Z) = \Sigma m(0, 1, 5, 7, 8, 13, 24, 25, 29, 31) + \Sigma d(9, 15, 16, 17, 23, 26, 27, 30)$
(2 solutions)

o. $H(A, B, C, D, E) = \Sigma m(0, 4, 12, 15, 27, 29, 30) + \Sigma d(1, 5, 9, 10, 14, 16, 20, 28, 31)$ (4 solutions)

p. $F(A, B, C, D, E) = \Sigma m(8, 9, 11, 14, 28, 30) + d(0, 3, 4, 6, 7, 12, 13, 15, 20, 22, 27, 29, 31)$
(8 solutions)

**10.** For each of the following six-variable functions, find all minimum sum of products expressions. (The number of terms and literals and, if there is more than one solution, the number of solutions is given in parentheses.)

a. $G(A, B, C, D, E, F) = \Sigma m(4, 5, 6, 7, 8, 10, 13, 15, 18, 20, 21, 22, 23, 26, 29, 30, 31, 33, 36, 37, 38, 39, 40, 42, 49, 52, 53, 54, 55, 60, 61)$
(6 terms, 21 literals)

b. $G(A, B, C, D, E, F) = \Sigma m(2, 3, 6, 7, 8, 12, 14, 17, 19, 21, 23, 25, 27, 28, 29, 30, 32, 33, 34, 35, 40, 44, 46, 49, 51, 53, 55, 57, 59, 61, 62, 63)$
(8 terms, 30 literals)

c. $G(A, B, C, D, E, F) = \Sigma m(0, 1, 2, 4, 5, 6, 7, 9, 13, 15, 17, 19, 21, 23, 26, 27, 29, 30, 31, 33, 37, 39, 40, 42, 44, 45, 46, 47, 49, 53, 55, 57, 59, 60, 61, 62, 63)$
(8 terms, 28 literals, 2 solutions)

**11.** Find a minimum two-level circuit (corresponding to sum of products expressions) using AND and one OR gate per function for each of the following sets of functions.

a. $f(a, b, c, d) = \Sigma m(1, 3, 5, 8, 9, 10, 13, 14)$
$g(a, b, c, d) = \Sigma m(4, 5, 6, 7, 10, 13, 14)$ (7 gates, 21 inputs)

b. $f(a, b, c, d) = \Sigma m(0, 1, 2, 3, 4, 5, 8, 10, 13)$
$g(a, b, c, d) = \Sigma m(0, 1, 2, 3, 8, 9, 10, 11, 13)$
(6 gates, 16 inputs)

c. $f(a, b, c, d) = \Sigma m(5, 8, 9, 12, 13, 14)$
$g(a, b, c, d) = \Sigma m(1, 3, 5, 8, 9, 10)$
(3 solutions, 8 gates, 25 inputs)

d. $f(a, b, c, d) = \Sigma m(1, 3, 4, 5, 10, 11, 12, 14, 15)$
$g(a, b, c, d) = \Sigma m(0, 1, 2, 8, 10, 11, 12, 15)$
(9 gates, 28 inputs)

e. $F(W, X, Y, Z) = \Sigma m(1, 5, 7, 8, 10, 11, 12, 14, 15)$
$G(W, X, Y, Z) = \Sigma m(0, 1, 4, 6, 7, 8, 12)$ (8 gates, 23 inputs)

f. $F(W, X, Y, Z) = \Sigma m(0, 2, 3, 7, 8, 9, 13, 15)$
$G(W, X, Y, Z) = \Sigma m(0, 2, 8, 9, 10, 12, 13, 14)$
(2 solutions, 8 gates, 23 inputs)

g. $f(a, b, c, d) = \Sigma m(1, 3, 5, 7, 8, 9, 10)$
$g(a, b, c, d) = \Sigma m(0, 2, 4, 5, 6, 8, 10, 11, 12)$
$h(a, b, c, d) = \Sigma m(1, 2, 3, 5, 7, 10, 12, 13, 14, 15)$
(2 solutions, 12 gates, 33 inputs)

h. $f(a, b, c, d) = \Sigma m(0, 3, 4, 5, 7, 8, 12, 13, 15)$
$g(a, b, c, d) = \Sigma m(1, 5, 7, 8, 9, 10, 11, 13, 14, 15)$
$h(a, b, c, d) = \Sigma m(1, 2, 4, 5, 7, 10, 13, 14, 15)$
(2 solutions, 11 gates, 33 inputs)

i. $f(a, b, c, d) = \Sigma m(0, 2, 3, 4, 6, 7, 9, 11, 13)$
$g(a, b, c, d) = \Sigma m(2, 3, 5, 6, 7, 8, 9, 10, 13)$
$h(a, b, c, d) = \Sigma m(0, 4, 8, 9, 10, 13, 15)$
(2 solutions for $f$ and $g$, 11 gates, 35 inputs)

j. $f(a, c, b, d) = \Sigma m(0, 1, 2, 3, 4, 9) + \Sigma d(10, 11, 12, 13, 14, 15)$
$g(a, c, b, d) = \Sigma m(1, 2, 6, 9) + \Sigma d(10, 11, 12, 13, 14, 15)$
(2 solutions for $f$, 6 gates, 15 inputs)

k. $f(a, c, b, d) = \Sigma m(5, 6, 11) + \Sigma d(0, 1, 2, 4, 8)$
$g(a, c, b, d) = \Sigma m(6, 9, 11, 12, 14) + \Sigma d(0, 1, 2, 4, 8)$
(7 gates, 18 inputs)

**12.** In each of the following sets, the functions have been minimized individually. Find a minimum two-level circuit (corresponding to sum of products expressions) using AND and one OR gate per function for each.

a. $F = B'D' + CD' + AB'C$
$G = BC + ACD$ (6 gates, 15 inputs)

b. $F = A'B'C'D + BC + ACD + AC'D'$
$G = A'B'C'D' + A'BC + BCD'$
$H = B'C'D' + BCD + AC' + AD$
(2 solutions for $H$, 12 gates, 40 inputs)

c. $f = a'b' + a'd + b'c'd'$
$g = b'c'd' + bd + acd + abc$
$h = a'd' + a'b + bc'd + b'c'd'$ (11 gates, 32 inputs)

**13.** Use the iterated consensus method to find the minimum sum of products expression(s) for the functions of Exercises 1, 4, and 9.

**14.** Use the iterated consensus method to find the minimum sum of products expression(s) for the sets of functions of Exercise 11.