Section 7.2
Graph Terminology

Undirected Graphs

Definition: Two vertices $u$, $v$ in $V$ are adjacent or neighbors if there is an edge $e$ between $u$ and $v$.

The edge $e$ connects $u$ and $v$.

The vertices $u$ and $v$ are endpoints of $e$.

Definition: The degree of a vertex $v$, denoted $\text{deg}(v)$, is the number of edges for which it is an endpoint.

A loop contributes twice in an undirected graph.

Example:

- If $\text{deg}(v) = 0$, $v$ is called isolated.
• If \( \text{deg}(v) = 1 \), \( v \) is called pendant.

The Handshaking Theorem:

Let \( G = (V, E) \). Then

\[
2|E| = \sum_{v \in V} \text{deg}(v)
\]

Proof:

Each edge represents contributes twice to the degree count of all vertices.

Q. E. D.

Example:

If a graph has 5 vertices, can each vertex have degree 3? 4?

• The sum is \( 3 \cdot 5 = 15 \) which is an odd number. Not possible.

• The sum is \( 20 = 2 \cdot |E| \) and \( 20/2 = 10 \). May be possible.
Theorem: A graph has an even number of vertices of odd degree.

Proof:

Let \( V_1 \) = vertices of odd degree

\( V_2 = \) vertices of even degree

The sum must be even. But

- odd times odd = odd
- odd times even = even
- even times even = even
- even plus odd = odd

It doesn't matter whether \( V_2 \) has odd or even cardinality.

\( V_1 \) cannot have odd cardinality.

Q. E. D.

Example:

It is not possible to have a graph with 3 vertices each of which has degree 1.
Directed Graphs

**Definition:** Let \( <u, v> \) be an edge in \( G \). Then \( u \) is an *initial vertex* and is *adjacent to* \( v \) and \( v \) is a *terminal vertex* and is *adjacent from* \( u \).

**Definition:** The *in degree* of a vertex \( v \), denoted \( \text{deg}^-(v) \) is the number of edges which terminate at \( v \).

Similarly, the *out degree* of \( v \), denoted \( \text{deg}^+(v) \), is the number of edges which initiate at \( v \).

**Theorem:** \( |E| = \sum_{v \in V} \text{deg}^{-}(v) = \sum_{v \in V} \text{deg}^{+}(v) \)

**Special Simple Graphs**

- Complete graphs - \( K_n \): the simple graph with

  - \( n \) vertices
  - exactly one edge between every pair of distinct vertices.

Maximum redundancy in local area networks and processor connection in parallel machines.
Examples:
Note: K5 is important because it is the simplest nonplanar graph: It cannot be drawn in a plane with nonintersecting edges.

- Cycles:

C_n is an n vertex graph which is a cycle. Local area networks are sometimes configured this way called Ring networks.
• Wheels:

Add one additional vertex to the cycle $C_n$ and add an edge from each vertex to the new vertex to produce $W_n$.

Provides redundancy in local area networks.
• n-Cubes:

$Q_n$ is the graph with $2^n$ vertices representing bit strings of length $n$.

An edge exists between two vertices that differ by one bit position.

A common way to connect processors in parallel machines.

Intel Hypercube.
Bipartite Graphs

**Definition:** A simple graph \( G \) is *bipartite* if \( V \) can be partitioned into two disjoint subsets \( V_1 \) and \( V_2 \) such that every edge connects a vertex in \( V_1 \) and a vertex in \( V_2 \).

Note: There are no edges which connect vertices in \( V_1 \) or in \( V_2 \).

A bipartite graph is *complete* if there is an edge from every vertex in \( V_1 \) to every vertex in \( V_2 \), denoted \( K_{m,n} \) where \( m = |V_1| \) and \( n = |V_2| \).

**Examples:**

- Suppose bigamy is permitted but not same sex marriages and males are in \( V_1 \) and females in \( V_2 \) and an edge represents a marriage. If every male is married to every female then the graph is complete.

- Supplier, warehouse transportation models are bipartite and an edge indicates that a given supplier sends inventory to a given warehouse.

- A Star network is a \( K_{1,n} \) bipartite graph.
- $C_k$ for $k$ even is a bipartite graph: even numbered vertices in $V_1$, odd numbered in $V_2$. 
• Is the following graph bipartite?

\[
\begin{array}{c}
a & b & c \\
d & e & a \\
\end{array}
\]

If \(a\) is in \(V1\) then \(e\), \(c\) and \(b\) must be in \(V1\) (why?). Then \(c\) is in \(V1\) and there is no inconsistency.

We rearrange the graph as follows:

\[
\begin{array}{c}
c & b \\
\end{array}
\]

\[
\begin{array}{c}
a & d \\
e & \end{array}
\]

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**New Graphs from Old**

**Definition:** \((W, F)\) is a subgraph of \(G = (V, E)\) if

\[W \subseteq V \text{ and } F \subseteq E.\]
**Definition:** If $G_1$ and $G_2$ are simple then

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

and the graph is simple.

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Examples:

- Find the subgraphs of $Q_1$:

  ![Graph](attachment:image.png)

- Count the number of subgraphs of a given graph.

- Find the union of the two graphs $G_1$ and $G_2$:
Note: The important properties of a graph do not depend on how we draw it. We want to be able to identify two graphs that are the same (up to labeling of the vertices).