1.3 PREDICATES AND QUANTIFIERS

DEF: Informally, a *predicate* is a statement about a (possibly empty) collection of variables over various domains. Its truth value depends on the values of the variables in their respective domains.

DEF: Formally, a *predicate* is a function from the cartesian product of the domains of the variables to the boolean set \{T, F\}.

**Example 1.3.1:** \( x + 2 = 5 \).

**Example 1.3.2:** \( 4x - 3y > 2x \).

DEF: The *universal quantification* (over \( x \)) of a predicate \( P(x) \) is the predicate \((\forall x)[P(x)]\).

**Example 1.3.3:** \((\forall x)[x + 2 = 5]\).

**Example 1.3.4:** \((\forall x)[4x - 3y > 2x]\).
DEF: The **existential quantification (over x)** of a predicate $P(x)$ is the predicate $(\exists x)[P(x)]$.

**Example 1.3.5:** $(\exists x)[x + 2 = 5]$.

**Example 1.3.6:** $(\exists x)[4x - 3y > 2x]$.

**Remark:** Observe that the result of quantifying a predicate is still a predicate. Moreover, when propositional operators are applied to predicates, the results are predicates.

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**VARYING THE DOMAIN**

**Example 1.3.7:** $(\forall x)[x^2 = 1]$ is FALSE over the domain of integers, but TRUE over the domain $\{-1, 1\}$.

**Example 1.3.8:** $(\exists x)[x^2 = -9]$ is FALSE over the integers, but TRUE over the domain of complex numbers.
CLASSROOM EXERCISE

Consider these two condition statements.

1. \((\forall x)[P(x)] \rightarrow (\exists x)[P(x)]\).
   Over the domain of people, this would mean “If something is good for everybody, then it’s good for somebody.”.

2. \((\exists x)[P(x)] \rightarrow (\forall x)[P(x)]\).
   Over the domain of people, this could mean “What’s good for me is good for everybody.”.

Try to think of a general property of a domain under which statement (1) is necessarily FALSE.

Try to think of a general property of a domain under which statement (2) is necessarily TRUE.

Hint: These general properties are based solely on the number of elements in the domain.
SCOPE of QUANTIFIERS

DEF: The **scope** of a quantifier is the clause to which it applies.

**Example 1.3.9:** Let \( x \) range over the integers.

\[ P(x) : x > 2 \quad Q(x) : x < 2 \]

Compare these two non-equivalent propositions:

A. \((\exists x)[P(x) \leftrightarrow Q(x)]\)  \( B. (\exists x)[P(x)] \leftrightarrow (\exists x)[Q(x)]\)

A is FALSE, but B is TRUE.

DEF: An **unbound variable** in a predicate is a variable not within the scope of any quantifier.

**Example 1.3.10:** \( x \) is an unbound variable.

\[ x + 4 > 2 \]

**Example 1.3.11:** \( x \) is an unbound variable.

\[ (\forall y)[2x + 3y = 7] \]

**Remark:** A predicate with no unbound variables is a proposition.
NEGATION with QUANTIFIERS

\( p \): There exists some input data for which this program will crash.

\( \neg p \): No matter what input data you supply to this program, it will not crash.

Rule 1: \( \neg (\exists x)P(x) \iff (\forall x)\neg P(x) \)
Rule 2: \( \neg (\forall x)P(x) \iff (\exists x)\neg P(x) \)

CLASSROOM EXERCISE

On a New Jersey Transit commuter run, the conductor announces:

At the next stop, all doors will not be open.

Express this in symbolic logic.
Explain what his words mean.
What words accurately express what he probably intended?