2.5 INTEGERS AND ALGORITHMS

We accelerate evaluation of gcd’s, of arithmetic operations, and of monomials and polynomials.

POSITIONAL REPRESENTATION of INTEGERS

Although arithmetic algorithms are much more complicated for numbers in positional notation than for numbers in monadic notation, they pay benefits in execution time.

(1) Addition algorithm execution time decreases from \( \mathcal{O}(n) \) to \( \mathcal{O}(\log n) \).

(2) Multiplication algorithm execution time decreases from \( \mathcal{O}(nm) \) to \( \mathcal{O}(\log n \log m) \).
**Theorem 2.5.1.** Let \( b > 1 \) and \( n \geq 0 \) be integers. Let \( k \) be the maximum integer such that \( b^k \leq n \). Then there is a unique set of nonnegative integers \( a_k, a_{k-1}, \ldots, a_0 < b \) such that
\[
    n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b^1 + a_0
\]

**Proof:** Apply the division algorithm to \( n \) and \( b \) to obtain a quotient and remainder \( a_0 \). Then apply the division algorithm to that quotient and \( b \) to obtain a new quotient and remainder \( a_1 \). Etc.

\[\Diamond\]

**NUMBER BASE CONVERSION**

The algorithm in the proof of Theorem 2.5.1 provides a method to convert any positive integer from one base to another.

**Example 2.5.1:** Convert \( 1215_{10} \) to base-7.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( d )</th>
<th>( q )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1215</td>
<td>7</td>
<td>173</td>
<td>4</td>
</tr>
<tr>
<td>173</td>
<td>7</td>
<td>24</td>
<td>5</td>
</tr>
<tr>
<td>24</td>
<td>7</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Solution: \( 3354_7 \)
EVALUATION OF MONOMIALS

Example 2.5.2: Calculate $13^n$, e.g. $13^{19}$.

Usual method: $13 \times 13 \times 13 \times \cdots \times 13$

time = $\Theta(n)$.

Better method:
$13, 13^2, 13^4, 13^8, 13^{16}$ takes $\Theta(\log n)$ steps

$13 \times 13^2 \times 13^{16}$ takes $\Theta(\log n)$ steps

EVALUATION OF POLYNOMIALS

Evaluate $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

Usual method of evaluation takes $\Theta(n)$:

$n$ multiplications to calculate $n$ powers of $x$

$n$ multiplications by coefficients

$n$ additions

Horner’s method (due to _________):

$a_n x + a_{n-1}$

$(a_n x + a_{n-1}) x + a_{n-2}$ etc.

requires only $n$ multiplications and $n$ additions.
Lemma 2.5.2. Let \( d \mid m \) and \( d \mid n \). Then \( d \mid m - n \) and \( d \mid m + n \).

**Proof:** Suppose \( m = dp \) and \( n = dq \). Then \( m - n = d(p - q) \) and \( m + n = d(p + q) \). \( \diamond \)

Corollary 2.5.3. \( \gcd(m, n) = \gcd(m - n, n) \).

**Proof:** In three steps.

A1. \( \gcd(m, n) \) is a common div of \( m - n \) and \( n \), and \( \gcd(m - n, n) \) is a common div of \( m \) and \( n \).
   Pf. Both parts by Lemma 1.

A2. \( \gcd(m, n) \leq \gcd(m - n, n) \)
   and \( \gcd(m - n, n) \leq \gcd(m, n) \).
   Pf. Both parts by A1 and def of \( \gcd \) ("greatest").

A3. \( \gcd(m, n) = \gcd(m - n, n) \).
   Pf. Immediate from A2. \( \diamond \) Cor 2.5.3

Cor 2.5.4. \( \gcd(m, n) = \gcd(n, m \mod n) \).

**Proof:** The number \( m \mod n \) is obtained from \( m \) by subtracting a multiple of \( n \). Iteratively apply Cor 2.5.3. \( \diamond \)
Algorithm 2.5.1: Euclidean Algorithm

**Input:** positive integers $m \geq 0$, $n > 0$

**Output:** $\gcd(n, m)$

**If** $m = 0$ **then return** $(n)$

**else return** $\gcd(m, n \mod m)$

**Time-Complexity:** $\mathcal{O}(\log(\min(n, m)))$.

Much better than Naive GCD algorithm.

**Example 2.5.3:** Euclidean Algorithm

\[
\gcd(210, 111) = \gcd(111, 210 \mod 111) = \\
\gcd(111, 99) = \gcd(99, 111 \mod 99) = \\
\gcd(99, 12) = \gcd(12, 99 \mod 12) = \\
\gcd(12, 3) = \gcd(3, 12 \mod 3) = \\
\gcd(3, 0) = 3
\]
Example 2.5.4: Euclidean Algorithm

\[
\begin{align*}
gcd(42, 26) &= gcd(26, 42 \mod 26) = \\
gcd(26, 16) &= gcd(16, 26 \mod 16) = \\
gcd(16, 10) &= gcd(10, 16 \mod 10) = \\
gcd(10, 6) &= gcd(6, 10 \mod 6) = \\
gcd(6, 4) &= gcd(4, 6 \mod 4) = \\
gcd(4, 2) &= gcd(2, 4 \mod 2) = \\
gcd(2, 0) &= 2
\end{align*}
\]