9.4 The Factor Theorem

In Chapter 5 you learned to add, subtract, multiply, divide, and factor polynomials. In this section we study functions defined by polynomials and learn to solve some higher-degree polynomial equations.

The Factor Theorem

Consider the polynomial function

\[ P(x) = x^2 + 2x - 15. \]

The values of \( x \) for which \( P(x) = 0 \) are called the zeros or roots of the function. We can find the zeros of the function by solving the equation \( P(x) = 0 \):

\[ x^2 + 2x - 15 = 0 \]
\[ (x + 5)(x - 3) = 0 \]
\[ x + 5 = 0 \quad \text{or} \quad x - 3 = 0 \]
\[ x = -5 \quad \text{or} \quad x = 3 \]

Because \( x + 5 \) is a factor of \( x^2 + 2x - 15 \), \(-5\) is a solution to the equation \( x^2 + 2x - 15 = 0 \) and a zero of the function \( P(x) = x^2 + 2x - 15 \). We can check that \(-5\) is a zero of \( P(x) = x^2 + 2x - 15 \) as follows:

\[ P(-5) = (-5)^2 + 2(-5) - 15 \]
\[ = 25 - 10 - 15 \]
\[ = 0 \]

Because \( x - 3 \) is a factor of the polynomial, \( 3 \) is also a solution to the equation \( x^2 + 2x - 15 = 0 \) and a zero of the polynomial function. Check that \( P(3) = 0 \):

\[ P(3) = 3^2 + 2 \cdot 3 - 15 \]
\[ = 9 + 6 - 15 \]
\[ = 0 \]

Note that the zeros of the polynomial function are factors of the constant term 15.

62. To see the difference between direct and inverse variation, graph \( y_1 = 2x \) and \( y_2 = \frac{2}{x} \) using \( 0 \leq x \leq 5 \) and \( 0 \leq y \leq 10 \). Which of these functions is increasing and which is decreasing?

63. Graph \( y_1 = 2\sqrt{x} \) and \( y_2 = \frac{2}{\sqrt{x}} \) by using \( 0 \leq x \leq 5 \) and \( 0 \leq y \leq 10 \). At what point in the first quadrant do the curves cross? Which function is increasing and which is decreasing? Which represents direct variation and which represents inverse variation?
Every linear factor of the polynomial corresponds to a zero of the polynomial function, and every zero of the polynomial function corresponds to a linear factor.

Now suppose $P(x)$ represents an arbitrary polynomial. If $x - c$ is a factor of the polynomial $P(x)$, then $c$ is a solution to the equation $P(x) = 0$, and so $P(c) = 0$. If we divide $P(x)$ by $x - c$ and the remainder is 0, we must have

$$P(x) = (x - c)(\text{quotient}).$$

If the remainder is 0, then $x - c$ is a factor of $P(x)$.

The factor theorem summarizes these ideas.

**The Factor Theorem**

The following statements are equivalent for any polynomial $P(x)$.

1. The remainder is zero when $P(x)$ is divided by $x - c$.
2. $x - c$ is a factor of $P(x)$.
3. $c$ is a solution to $P(x) = 0$.
4. $c$ is a zero of the function $P(x)$, or $P(c) = 0$.

To say that statements are equivalent means that the truth of any one of them implies that the others are true.

According to the factor theorem, if we want to determine whether a given number $c$ is a zero of a polynomial function, we can divide the polynomial by $x - c$. The remainder is zero if and only if $c$ is a zero of the polynomial function. The quickest way to divide by $x - c$ is to use synthetic division from Section 5.5.

**Example 1**

**Using the factor theorem**

Use synthetic division to determine whether 2 is a zero of $P(x) = x^3 - 3x^2 + 5x - 2$.

**Solution**

By the factor theorem, 2 is a zero of the function if and only if the remainder is zero when $P(x)$ is divided by $x - 2$. We can use synthetic division to determine the remainder. If we divide by $x - 2$, we use 2 on the left in synthetic division along with the coefficients 1, -3, 5, -2 from the polynomial:

<table>
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<th></th>
<th>1</th>
<th>-3</th>
<th>5</th>
<th>-2</th>
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<tr>
<td>2</td>
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<td>2</td>
<td>-2</td>
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<tr>
<td></td>
<td>1</td>
<td>-1</td>
<td>3</td>
<td>4</td>
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</tbody>
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Because the remainder is 4, 2 is not a zero of the function.

**Example 2**

**Using the factor theorem**

Use synthetic division to determine whether $-4$ is a solution to the equation $2x^4 - 28x^2 + 14x - 8 = 0$. 
Solution
By the factor theorem, \(-4\) is a solution to the equation if and only if the remainder is zero when \(P(x)\) is divided by \(x + 4\). When dividing by \(x + 4\), we use \(-4\) in the synthetic division:

\[
\begin{array}{c|cccc}
-4 & 2 & 0 & -28 & 14 & -8 \\
 & & -8 & 32 & -16 & 8 \\
\hline
2 & -8 & 4 & -2 & 0 \\
\end{array}
\]

Because the remainder is zero, \(-4\) is a solution to \(2x^4 - 28x^2 + 14x - 8 = 0\).

In the next example we use the factor theorem to determine whether a given binomial is a factor of a polynomial.

**Example 3**

**Using the factor theorem**
Use synthetic division to determine whether \(x + 4\) is a factor of \(x^3 + 3x^2 + 16\).

**Solution**
According to the factor theorem, \(x + 4\) is a factor of \(x^3 + 3x^2 + 16\) if and only if the remainder is zero when the polynomial is divided by \(x + 4\). Use synthetic division to determine the remainder:

\[
\begin{array}{c|cccc}
-4 & 1 & 3 & 0 & 16 \\
 & & -4 & 4 & -16 \\
\hline
1 & 1 & -1 & 4 & 0 \\
\end{array}
\]

Because the remainder is zero, \(x + 4\) is a factor, and the polynomial can be written as

\(x^3 + 3x^2 + 16 = (x + 4)(x^2 - x + 4)\).

Because \(x^2 - x + 4\) is a prime polynomial, the factoring is complete.

**Solving Polynomial Equations**
The techniques used to solve polynomial equations of degree 3 or higher are not as straightforward as those used to solve linear equations and quadratic equations. The next example shows how the factor theorem can be used to solve a third-degree polynomial equation.

**Example 4**

**Solving a third-degree equation**
Suppose the equation \(x^3 - 4x^2 - 17x + 60 = 0\) is known to have a solution that is an integer between \(-3\) and \(3\) inclusive. Find the solution set.

**Solution**
Because one of the numbers \(-3, -2, -1, 0, 1, 2,\) and \(3\) is a solution to the equation, we can use synthetic division with these numbers until we discover which one is a solution. We arbitrarily select \(1\) to try first:

\[
\begin{array}{c|cccc}
1 & 1 & -4 & -17 & 60 \\
 & & 1 & -3 & -20 \\
\hline
1 & -3 & -20 & 40 \\
\end{array}
\]

Because the remainder is 40, \(1\) is not a solution to the equation. Next try 2:

\[
\begin{array}{c|cccc}
2 & 1 & -4 & -17 & 60 \\
 & & 2 & -4 & -42 \\
\hline
1 & -2 & -21 & 18 \\
\end{array}
\]

**Stay calm and confident. Take breaks when you study. Get 6 to 8 hours of sleep every night and keep reminding yourself that working hard all of the semester will really pay off.**
Because the remainder is not zero, 2 is not a solution to the equation. Next try 3:

\[
\begin{array}{c|ccc}
3 & 1 & -4 & -17 & 60 \\
 & & 3 & -3 & -60 \\
\hline
1 & -1 & -20 & 0
\end{array}
\]

The remainder is zero, so 3 is a solution to the equation, and \( x - 3 \) is a factor of the polynomial. (If 3 had not produced a remainder of zero, then we would have tried \(-3, -2, -1, \) and 0.) The other factor is the quotient, \( x^2 - x - 20 \).

\[
x^3 - 4x^2 - 17x + 60 = 0
\]

\[
(x - 3)(x^2 - x - 20) = 0 \quad \text{Use the results of synthetic division to factor.}
\]

\[
(x - 3)(x - 5)(x + 4) = 0 \quad \text{Factor completely.}
\]

\[
x - 3 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{or} \quad x + 4 = 0
\]

\[
x = 3 \quad \text{or} \quad x = 5 \quad \text{or} \quad x = -4
\]

Check each of these solutions in the original equation. The solution set is \{3, 5, -4\}.

**WARM-UPS**

**True or false? Explain your answers.**

1. To divide \( x^3 - 4x^2 - 3 \) by \( x - 5 \), use 5 in the synthetic division.
2. To divide \( 5x^4 - x^3 + x - 2 \) by \( x + 7 \), use -7 in the synthetic division.
3. The number 2 is a zero of \( P(x) = 3x^3 - 5x^2 - 2x + 2 \).
4. If \( x^3 - 8 \) is divided by \( x - 2 \), then \( R = 0 \).
5. If \( R = 0 \) when \( x^4 - 1 \) is divided by \( x - a \), then \( x - a \) is a factor of \( x^4 - 1 \).
6. If -2 satisfies \( x^4 + 8x = 0 \), then \( x + 2 \) is a factor of \( x^4 + 8x \).
7. The binomial \( x - 1 \) is a factor of \( x^{35} - 3x^{24} + 2x^{18} \).
8. The binomial \( x + 1 \) is a factor of \( x^3 - 3x^2 + x + 5 \).
9. If \( x^3 - 5x + 4 \) is divided by \( x - 1 \), then \( R = 0 \).
10. If \( R = 0 \) when \( P(x) = x^3 - 5x - 2 \) is divided by \( x + 2 \), then \( P(-2) = 0 \).

**Reading and Writing**  After reading this section, write out the answers to these questions. Use complete sentences.

1. What is a zero of a function?
2. What is a root of a function?
3. What does it mean that statements are equivalent?
4. What is the quickest way to divide a polynomial by \( x - c \)?
5. If the remainder is zero when you divide \( P(x) \) by \( x - c \), then what can you say about \( P(c) \)?
6. What are two ways to determine whether \( c \) is a zero of a polynomial?
Determine whether each given value of \( x \) is a zero of the given function. See Example 1.

1. \( x = 1, \ P(x) = x^3 - x^2 + x - 1 \)
2. \( x = -2, \ P(x) = -2x^3 - 5x^2 + 3x + 10 \)
3. \( x = -3, \ P(x) = -x^4 - 3x^3 - 2x^2 + 18 \)
4. \( x = 4, \ P(x) = x^4 - x^2 - 8x - 16 \)
5. \( x = 2, \ P(x) = 2x^3 - 4x^2 - 5x + 9 \)
6. \( x = -3, \ P(x) = x^3 + 5x^2 + 2x + 1 \)

Use synthetic division to determine whether each given value of \( x \) is a solution to the given equation. See Example 2.

7. \( x = -3, \ x^3 + 5x^2 + 2x - 12 = 0 \)
8. \( x = -5, \ x^2 - 3x - 40 = 0 \)
9. \( x = -2, \ x^3 + 3x^2 - 5x^2 - 10x + 5 = 0 \)
10. \( x = -3, \ -x^3 - 4x^2 + x + 12 = 0 \)
11. \( x = 4, \ -2x^4 + 30x^2 + 5x + 12 = 0 \)
12. \( x = 6, \ x^4 + x^3 - 40x^2 - 72 = 0 \)
13. \( x = 3, \ 0.8x^3 - 0.3x - 6.3 = 0 \)
14. \( x = 5, \ 6.2x^2 - 28.2x - 41.7 = 0 \)

Use synthetic division to determine whether the first polynomial is a factor of the second. If it is, then factor the polynomial completely. See Example 3.

15. \( x - 3, \ x^3 - 6x - 9 \)
16. \( x + 2, \ x^3 - 6x - 4 \)
17. \( x + 5, \ x^3 + 9x^2 + 23x + 15 \)
18. \( x - 3, \ x^4 - 9x^2 + x - 7 \)
19. \( x - 2, \ x^3 - 8x^2 + 4x - 6 \)
20. \( x + 5, \ x^3 + 125 \)
21. \( x + 1, \ x^4 + x^3 - 8x - 8 \)
22. \( x - 2, \ x^3 - 6x^2 + 12x - 8 \)
23. \( x - 0.5, \ 2x^3 - 3x^2 - 11x + 6 \)
24. \( x - \frac{1}{3}, \ 3x^3 - 10x^2 - 27x + 10 \)

Solve each equation, given that at least one of the solutions to each equation is an integer between \(-5\) and \(5\). See Example 4.

25. \( x^3 - 13x + 12 = 0 \)
26. \( x^3 + 2x^2 - 5x - 6 = 0 \)
27. \( 2x^3 - 9x^2 + 7x + 6 = 0 \)

28. \( 6x^3 + 13x^2 - 4 = 0 \)
29. \( 2x^3 - 3x^2 - 50x - 24 = 0 \)
30. \( x^3 - 4x^3 + 3x^2 + 4x - 4 = 0 \)
31. \( x^3 + 7x^2 + 2x + 40 = 0 \)
32. \( x^3 + 5x^2 + 3x - 9 = 0 \)
33. \( x^3 + 6x^2 + 12x + 8 = 0 \)
34. \( x^4 + x^3 - 7x^2 - x + 6 = 0 \)

GETTING MORE INVOLVED

41. **Exploration.** We can find the zeros of a polynomial function by solving a polynomial equation. We can also work backward to find a polynomial function that has given zeros.

a) Write a first-degree polynomial function whose zero is \(-2\).

b) Write a second-degree polynomial function whose zeros are \(5\) and \(-5\).

c) Write a third-degree polynomial function whose zeros are \(1, -3,\) and \(4\).

d) Is there a polynomial function with any given number of zeros? What is its degree?

**GRAPHING CALCULATOR EXERCISES**

42. The \( x \)-coordinate of each \( x \)-intercept on the graph of a polynomial function is a zero of the polynomial function. Find the zeros of each function from its graph. Use synthetic division to check that the zeros found on your calculator really are zeros of the function.

a) \( P(x) = x^3 - 2x^2 - 5x + 6 \)

b) \( P(x) = 12x^3 - 20x^2 + x + 3 \)

43. With a graphing calculator an equation can be solved without the kind of hint that was given for Exercises 31–40. Solve each of the following equations by examining the graph of a corresponding function. Use synthetic division to check.

a) \( x^3 - 4x^2 - 7x + 10 = 0 \)

b) \( 8x^3 - 20x^2 - 18x + 45 = 0 \)