51. \( f(x) = (x - 20)^2(x + 30) \)  
53. \( f(x) = (x - 20)^2(x + 30)^3x \)

52. \( f(x) = (x - 20)^2(x + 30)^2 \)  
54. \( f(x) = (x - 20)^2(x + 30)^3x^2 \)

10.4 Graphs of Rational Functions

We first studied rational expressions in Chapter 6. In this section we will study functions that are defined by rational expressions.

Domain

A rational expression was defined in Chapter 6 as a ratio of two polynomials. If a ratio of two polynomials is used to define a function, then the function is called a rational function.

Rational Function

If \( P(x) \) and \( Q(x) \) are polynomials with no common factor and \( f(x) = \frac{P(x)}{Q(x)} \) for \( Q(x) \neq 0 \), then \( f(x) \) is called a rational function.

The domain of a rational function is the set of all real numbers except those that cause the denominator to have a value of 0.

Example 1

Domain of a rational function

Find the domain of each rational function.

a) \( f(x) = \frac{x - 3}{x - 1} \)  
b) \( g(x) = \frac{2x - 3}{x^2 - 4} \)
**Solution**

a) Since $x - 1 = 0$ only for $x = 1$, the domain of $f$ is the set of all real numbers except 1, $\{x \mid x \neq 1\}$.

b) Since $x^2 - 4 = 0$ for $x = \pm 2$, the domain of $g$ is the set of all real numbers excluding 2 and $-2$, $\{x \mid x \neq 2 \text{ and } x \neq -2\}$.

**Horizontal and Vertical Asymptotes**

Consider the simplest rational function $f(x) = \frac{1}{x}$. Its domain does not include 0, but 0 is an important number for the graph of this function. The behavior of the graph of $f$ when $x$ is very close to 0 is what interests us. For this function the $y$-coordinate is the reciprocal of the $x$-coordinate. When the $x$-coordinate is close to 0, the $y$-coordinate is far from 0. Consider the following tables of ordered pairs that satisfy $f(x) = \frac{1}{x}$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>10</td>
<td>$-0.1$</td>
<td>$-10$</td>
</tr>
<tr>
<td>0.01</td>
<td>100</td>
<td>$-0.01$</td>
<td>$-100$</td>
</tr>
<tr>
<td>0.001</td>
<td>1000</td>
<td>$-0.001$</td>
<td>$-1000$</td>
</tr>
<tr>
<td>0.0001</td>
<td>10,000</td>
<td>$-0.0001$</td>
<td>$-10,000$</td>
</tr>
</tbody>
</table>

As $x$ gets closer and closer to 0 from above 0, the value of $y$ gets larger and larger. We say that $y$ goes to positive infinity. As $x$ gets closer and closer to 0 from below 0, the values of $y$ are negative but $|y|$ gets larger and larger. We say that $y$ goes to negative infinity. The graph of $f$ gets closer and closer to the vertical line $x = 0$, and so $x = 0$ is called a **vertical asymptote**. On the other hand, as $|x|$ gets larger and larger, $y$ gets closer and closer to 0. The graph approaches the $x$-axis as $x$ goes to infinity, and so the $x$-axis is a **horizontal asymptote** for the graph of $f$. See Fig. 10.5 for the graph of $f(x) = \frac{1}{x}$.

In general, a rational function has a vertical asymptote for every number excluded from the domain of the function. The horizontal asymptotes are determined by the behavior of the function when $|x|$ is large.

**Example 2**

Find the horizontal and vertical asymptotes for each rational function.

a) $f(x) = \frac{3}{x^2 - 1}$  
b) $g(x) = \frac{x}{x^2 - 4}$  
c) $h(x) = \frac{2x + 1}{x + 3}$

**Solution**

a) The denominator $x^2 - 1$ has a value of 0 if $x = \pm 1$. So the lines $x = 1$ and $x = -1$ are vertical asymptotes. If $|x|$ is very large, the value of $\frac{3}{x^2 - 1}$ is approximately 0. So the $x$-axis is a horizontal asymptote.

b) The denominator $x^2 - 4$ has a value of 0 if $x = \pm 2$. So the lines $x = 2$ and $x = -2$ are vertical asymptotes. If $|x|$ is very large, the value of $\frac{x}{x^2 - 4}$ is approximately 0. So the $x$-axis is a horizontal asymptote.
c) The denominator \( x + 3 \) has a value of 0 if \( x = -3 \). So the line \( x = -3 \) is a vertical asymptote. If \( |x| \) is very large, the value of \( h(x) \) is not approximately 0. To understand the value of \( h(x) \), we change the form of the rational expression by using long division:

\[
\frac{2}{x + 3} \div \frac{2x + 6}{-5}
\]

Writing the rational expression as quotient + remainder/divisor, we get

\[
h(x) = \frac{2x + 1}{x + 3} = 2 + \frac{-5}{x + 3}.
\]

If \( |x| \) is very large, \( \frac{-5}{x + 3} \) is approximately 0, and so the \( y \)-coordinate is approximately 2. The line \( y = 2 \) is a horizontal asymptote.

Example 2 illustrates two important facts about horizontal asymptotes. If the degree of the numerator is less than the degree of the denominator, then the \( x \)-axis is the horizontal asymptote. For example, \( y = \frac{x - 4}{x^2 - 7} \) has the \( x \)-axis as a horizontal asymptote. If the degree of the numerator is equal to the degree of the denominator, then the ratio of the leading coefficients determines the horizontal asymptote. For example, \( y = \frac{2x - 7}{3x - 5} \) has \( y = \frac{2}{3} \) as its horizontal asymptote. The remaining case is when the degree of the numerator is greater than the degree of the denominator. This case is discussed next.

**Oblique Asymptotes**

Each rational function of Example 2 had one horizontal asymptote and a vertical asymptote for each number that caused the denominator to be 0. The horizontal asymptote \( y = 0 \) occurs because as \( |x| \) gets larger and larger, the \( y \)-coordinate gets closer and closer to 0. Some rational functions have a nonhorizontal line for an asymptote. An asymptote that is neither horizontal nor vertical is called an **oblique asymptote** or **slant asymptote**.

**Finding an oblique asymptote**

Determine all of the asymptotes for

\[
g(x) = \frac{2x^2 + 3x - 5}{x + 2}.
\]

**Solution**

If \( x + 2 = 0 \), then \( x = -2 \). So the line \( x = -2 \) is a vertical asymptote. Use long division to rewrite the function as quotient + remainder/divisor:

\[
g(x) = \frac{2x^2 + 3x - 5}{x + 2} = 2x - 1 + \frac{-3}{x + 2}.
\]

If \( |x| \) is large, the value of \( \frac{-3}{x + 2} \) is approximately 0. So when \( |x| \) is large, the value of \( g(x) \) is approximately \( 2x - 1 \). The line \( y = 2x - 1 \) is an oblique asymptote for the graph of \( g \).

We can summarize this discussion of asymptotes with the following strategy for finding asymptotes for a rational function.
10.4 Graphs of Rational Functions

1. Solve the equation \( Q(x) = 0 \). The graph of \( f \) has a vertical asymptote corresponding to each solution to the equation.

2. If the degree of \( P(x) \) is less than the degree of \( Q(x) \), then the \( x \)-axis is a horizontal asymptote.

3. If the degree of \( P(x) \) is equal to the degree of \( Q(x) \), then find the ratio of the leading coefficients. The horizontal line through that ratio is the horizontal asymptote.

4. If the degree of \( P(x) \) is greater than the degree of \( Q(x) \), then use division to rewrite the function as

\[
\text{quotient} + \frac{\text{remainder}}{\text{divisor}}.
\]

The equation formed by setting \( y \) equal to the quotient gives us an oblique asymptote.

---

Machines that do computations have been around for thousands of years. The abacus, the slide rule, and the calculator have simplified computations. However, recent calculators have gone way beyond numerical computations. Graphing calculators now draw two- and three-dimensional graphs and even do symbolic computations. Modern calculators are great, but to use one effectively you must still learn the underlying principles of mathematics.

Consider the fairly simple process of drawing a graph of a function. The graph of a function is a picture of all ordered pairs of the function. When we graph a function, we typically plot a few ordered pairs and then use our knowledge about the function to draw a graph that shows all of the important features of the function. A typical graphing calculator plots 96 ordered pairs of a function on a screen that is not much bigger than a postage stamp. The calculator does not generalize and does not make conclusions. We are still responsible for looking at what the calculator shows and making conclusions. For example, if we graphed \( y = 1/(x - 100) \) with the window set for \(-10 \leq x \leq 10\), we would not see the vertical asymptote \( x = 100 \). Would we believe the calculator and conclude that the graph has no vertical asymptote? If we graph \( y = 1/x \) with the window set for \(-500 \leq x \leq 500\) and \(-500 \leq y \leq 500\), the graph will be too close to its horizontal and vertical asymptotes for us to see it. Would we conclude that there is no graph for \( y = 1/x \)?

If you have a graphing calculator, use it to help you graph the functions in this chapter and to reinforce your understanding of the properties of functions that we are learning. Do not attempt to use it as a substitute for learning.
**Sketching the Graphs**

We now use asymptotes and symmetry to help us sketch the graphs of some rational functions.

**Example 4**

**Graphing a rational function**

Sketch the graph of each rational function.

a) \( f(x) = \frac{3}{x^2 - 1} \)

b) \( g(x) = \frac{x}{x^2 - 4} \)

**Solution**

a) From Example 2(a) we know that the lines \( x = 1 \) and \( x = -1 \) are vertical asymptotes and the \( x \)-axis is a horizontal asymptote. Draw the vertical asymptotes on the graph with dashed lines. Since all of the powers of \( x \) are even, \( f(x) = f(-x) \), and the graph is symmetric about the \( y \)-axis. The ordered pairs \((0, -3), (0.9, -15.789), (1.1, 14.286), (2, 1), \) and \( \left( \frac{3}{2}, \frac{3}{8}\right) \) are also on the graph. Use the symmetry to sketch the graph shown in Fig. 10.6.

b) Draw the vertical asymptotes \( x = 2 \) and \( x = -2 \) from Example 2(b) as dashed lines. The \( x \)-axis is a horizontal asymptote. Because \( f(-x) = -f(x) \), the graph is symmetric about the origin. The ordered pairs \((0, 0), (1, \frac{-1}{3}), (1.9, -4.872), (2.1, 5.122), \left( \frac{3}{2}, \frac{3}{5}\right), \) and \( \left( 4, \frac{1}{3}\right) \) are on the graph. Use the symmetry to get the graph shown in Fig. 10.7.

**Example 5**

**Graphing a rational function**

Sketch the graph of each rational function.

a) \( h(x) = \frac{2x + 1}{x + 3} \)

b) \( g(x) = \frac{2x^2 + 3x - 5}{x + 2} \)
Solution

a) Draw the vertical asymptote \( x = -3 \) and the horizontal asymptote \( y = 2 \) from Example 2(c) as dashed lines. The points \((-2, -3), \left(0, \frac{1}{3}\right), \left(-\frac{1}{2}, 0\right), (7, 1.5), (-4, 7), \) and \((-13, 2.5)\) are on the graph shown in Fig. 10.8.

b) Draw the vertical asymptote \( x = -2 \) and the oblique asymptote \( y = 2x - 1 \) from Example 3 as dashed lines. The points \((-1, -6), \left(0, -\frac{5}{2}\right), (1, 0), (4, 6.5), \) and \((-2.5, 0)\) are on the graph shown in Fig. 10.9.

**WARM-UPS**

True or false? Explain your answer.

1. The domain of \(f(x) = \frac{1}{x - 9} \) is \( x = 9 \).
   - True

2. The domain of \(f(x) = \frac{x - 1}{x + 2} \) is \( \{x \mid x \neq 1 \text{ and } x \neq -2\} \).
   - True

3. The domain of \(f(x) = \frac{1}{x^2 + 1} \) is \( \{x \mid x \neq 1 \text{ and } x \neq -1\} \).
   - True

4. The line \( x = 2 \) is the only vertical asymptote for the graph of \(f(x) = \frac{1}{x^2 - 4} \).
   - False

5. The \(x\)-axis is a horizontal asymptote for the graph of \(f(x) = \frac{x^2 - 3x + 5}{x^2 - 9x} \).
   - True

6. The \(x\)-axis is a horizontal asymptote for the graph of \(f(x) = \frac{3x - 5}{x - 2} \).
   - False

7. The line \( y = 2x - 5 \) is an asymptote for the graph of \(f(x) = 2x - 5 + \frac{1}{x} \).
   - True

8. The line \( y = 2x - 5 \) is an asymptote for the graph of \(f(x) = 2x - 5 + x^2 \).
   - False

9. The graph of \(f(x) = \frac{x^2}{x^2 - 9} \) is symmetric about the \(y\)-axis.
   - True

10. The graph of \(f(x) = \frac{3x}{x^2 - 25} \) is symmetric about the origin.
    - True
Reading and Writing  After reading this section, write out the answers to these questions. Use complete sentences.

1. What is a rational function?

2. What is the domain of a rational function?

3. What is a vertical asymptote?

4. What is a horizontal asymptote?

5. What is an oblique asymptote?

6. What is a slant asymptote?

Find the domain of each rational function. See Example 1.

7. \( f(x) = \frac{2}{x-1} \)

8. \( f(x) = \frac{-2}{x+3} \)

9. \( f(x) = \frac{x^2 - 1}{x} \)

10. \( f(x) = \frac{-2x + 3}{x^2} \)

11. \( f(x) = \frac{5}{x^2 - 16} \)

12. \( f(x) = \frac{x + 12}{x^2 - x - 6} \)

Determine all asymptotes for the graph of each rational function. See Examples 2 and 3.

13. \( f(x) = \frac{7}{x + 4} \)

14. \( f(x) = \frac{-8}{x - 9} \)

15. \( f(x) = \frac{1}{x^2 - 16} \)

16. \( f(x) = \frac{-2}{x^2 - 5x + 6} \)

17. \( f(x) = \frac{5x}{x - 7} \)

18. \( f(x) = \frac{3x + 8}{x - 2} \)

19. \( f(x) = \frac{2x^2}{x - 3} \)

20. \( f(x) = \frac{3x^2 + 2}{x + 1} \)

Match each rational function with its graph a–h.

21. \( f(x) = \frac{-2}{x} \)

22. \( f(x) = -\frac{1}{x - 2} \)

23. \( f(x) = \frac{x}{x - 2} \)

24. \( f(x) = \frac{x - 2}{x} \)

25. \( f(x) = \frac{1}{x^2 - 2x} \)

26. \( f(x) = \frac{x^2}{x^2 - 4} \)

27. \( f(x) = \frac{x + 4}{2} \)

28. \( f(x) = \frac{x^2 + 2x + 1}{x} \)
Determine all asymptotes and sketch the graph of each function. See Examples 4 and 5.

29. \( f(x) = \frac{2}{x + 4} \)

30. \( f(x) = \frac{-3}{x - 1} \)

31. \( f(x) = \frac{x}{x^2 - 9} \)

32. \( f(x) = \frac{-2}{x^2 + x - 2} \)
33. \( f(x) = \frac{2x - 1}{x + 3} \)

34. \( f(x) = \frac{5 - 2x}{x - 2} \)

35. \( f(x) = \frac{x^2 - 3x + 1}{x} \)

36. \( f(x) = \frac{x^3 + 1}{x^2} \)

37. \( f(x) = \frac{3x^2 - 2x}{x - 1} \)

38. \( f(x) = \frac{-x^2 + 5x - 5}{x - 3} \)

Find all asymptotes, x-intercepts, and y-intercepts for the graph of each rational function and sketch the graph of the function.

39. \( f(x) = \frac{1}{x^2} \)

40. \( f(x) = \frac{2}{x^2 - 4x + 4} \)

41. \( f(x) = \frac{2x - 3}{x^2 + x - 6} \)
42. \( f(x) = \frac{x}{x^2 + 4x + 4} \)

43. \( f(x) = \frac{x + 1}{x^2} \)

44. \( f(x) = \frac{x - 1}{x^2} \)

45. \( f(x) = \frac{2x - 1}{x^3 - 9x} \)

46. \( f(x) = \frac{2x^2 + 1}{x^3 - x} \)

47. \( f(x) = \frac{x}{x^2 - 1} \)

48. \( f(x) = \frac{x}{x^2 + x - 2} \)

49. \( f(x) = \frac{2}{x^2 + 1} \)

50. \( f(x) = \frac{x}{x^2 + 1} \)

51. \( f(x) = \frac{x^2}{x + 1} \)
52. \[ f(x) = \frac{x^2}{x - 1} \]

Solve each problem.

53. **Oscillating modulators.** The number of oscillating modulators produced by a factory in \( t \) hours is given by the polynomial function \( n(t) = t^2 + 6t \) for \( t \geq 1 \). The cost in dollars of operating the factory for \( t \) hours is given by the function \( c(t) = 36t + 500 \) for \( t \geq 1 \). The average cost per modulator is given by the rational function \[ f(t) = \frac{36t + 500}{t^2 + 6t} \]
for \( t \geq 1 \). Graph the function \( f \). What is the average cost per modulator at time \( t = 20 \) and time \( t = 30 \)? What can you conclude about the average cost per modulator after a long period of time?

54. **Nonoscillating modulators.** The number of nonoscillating modulators produced by a factory in \( t \) hours is given by the polynomial function \( n(t) = 16t \) for \( t \geq 1 \). The cost in dollars of operating the factory for \( t \) hours is given by the function \( c(t) = 64t + 500 \) for \( t \geq 1 \). The average cost per modulator is given by the rational function \[ f(t) = \frac{64t + 500}{16t} \]
for \( t \geq 1 \). Graph the function \( f \). What is the average cost per modulator at time \( t = 10 \) and time \( t = 20 \)? What can you conclude about the average cost per modulator after a long period of time?

55. **Average cost of an SUV.** Mercedes-Benz spent $700 million to design its new 1998 SUV (Motor Trend, www.motortrend.com). If it costs $25,000 to manufacture each SUV, then the average cost per vehicle in dollars when \( x \) vehicles are manufactured is given by the rational function
\[
A(x) = \frac{25,000x + 700,000,000}{x}.
\]

a) What is the horizontal asymptote for the graph of this function?
b) What is the average cost per vehicle when 50,000 vehicles are made?
c) For what number of vehicles is the average cost $30,000?
d) Graph this function for \( x \) ranging from 0 to 100,000.

56. **Average cost of a pill.** Assuming Pfizer spent a typical $350 million to develop its latest miracle drug and $0.10 each to make the pills, then the average cost per pill in dollars when \( x \) pills are made is given by the rational function
\[
A(x) = \frac{0.10x + 350,000,000}{x}.
\]

FIGURE FOR EXERCISE 56
a) What is the horizontal asymptote for the graph of this function?
b) What is the average cost per pill when 100 million pills are made?
c) For what number of pills is the average cost per pill $2?
d) Graph this function for $x$ ranging from 0 to 100 million.

### Graphing Calculator Exercises

Sketch the graph of each pair of functions in the same coordinate system. What do you observe in each case?

57. $f(x) = x^2, g(x) = x^2 + 1/x$
58. $f(x) = x^2, g(x) = x^2 + 1/x^2$
59. $f(x) = |x|, g(x) = |x| + 1/x$
60. $f(x) = |x|, g(x) = |x| + 1/x^2$
61. $f(x) = \sqrt{x}, g(x) = \sqrt{x} + 1/x$
62. $f(x) = x^3, g(x) = x^3 + 1/x^2$

### 10.5 Transformations of Graphs

We can discover what the graph of almost any function looks like if we plot enough points. However, it is helpful to know something about a graph so that we do not have to plot very many points. For example, it is important to know that the graph of a function is symmetric about the $y$-axis before we begin calculating ordered pairs. In this section we will learn how one graph can be transformed into another by modifying the formula that defines the function.

#### Reflecting

Consider the graphs of $f(x) = x^2$ and $g(x) = -x^2$ shown in Fig. 10.10. Notice that the graph of $g$ is a mirror image of the graph of $f$. For any value of $x$ we compute the $y$-coordinate of an ordered pair of $f$ by squaring $x$. For an ordered pair of $g$ we square first and then find the opposite because of the order of operations. This gives a correspondence between the ordered pairs of $f$ and the ordered pairs of $g$. For every ordered pair on the graph of $f$ there is a corresponding ordered pair directly below it on the graph of $g$, and these ordered pairs are the same distance from the $x$-axis. We say that the graph of $g$ is obtained by reflecting the graph of $f$ in the $x$-axis or that $g$ is a reflection of the graph of $f$.

**Reflection**

The graph of $y = -f(x)$ is a **reflection** in the $x$-axis of the graph of $y = f(x)$.