4.4 Special Products

In Section 4.3 you learned the FOIL method to make multiplying binomials simpler. In this section you will learn rules for squaring binomials and for finding the product of a sum and a difference. These products are called special products.

The Square of a Binomial

To compute \((a + b)^2\), the square of a binomial, we can write it as \((a + b)(a + b)\) and use FOIL:

\[
(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2
\]

So to square \(a + b\), we square the first term \((a^2)\), add twice the product of the two terms \((2ab)\), then add the square of the last term \((b^2)\). The square of a binomial occurs so frequently that it is helpful to learn this new rule to find it. The rule for squaring a sum is given symbolically as follows.

The Square of a Sum

\[
(a + b)^2 = a^2 + 2ab + b^2
\]

Example 1

Using the rule for squaring a sum

Find the square of each sum.

a) \((x + 3)^2\)  

b) \((2a + 5)^2\)
Solution

a) \((x + 3)^2 = x^2 + 2(x)(3) + 3^2 = x^2 + 6x + 9\)

\[\text{Square} \quad \uparrow \quad \text{Twice of} \quad \uparrow \quad \text{Square of} \quad \text{first} \quad \text{product} \quad \text{the} \quad \text{last} \quad \text{term}\]

b) \((2a + 5)^2 = (2a)^2 + 2(2a)(5) + 5^2 = 4a^2 + 20a + 25\]

\[\text{Square} \quad \uparrow \quad \text{Twice of} \quad \uparrow \quad \text{Square of} \quad \text{first} \quad \text{product} \quad \text{the} \quad \text{last} \quad \text{term}\]

\[\text{C A U T I O N}\]

Do not forget the middle term when squaring a sum. The equation \((x + 3)^2 = x^2 + 6x + 9\) is an identity, but \((x + 3)^2 = x^2 + 9\) is not an identity. For example, if \(x = 1\) in \((x + 3)^2 = x^2 + 9\), then we get \(4^2 = 1^2 + 9\), which is false.

When we use FOIL to find \((a - b)^2\), we see that

\[(a - b)^2 = (a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2.\]

So to square \(a - b\), we square the first term \(a^2\), subtract twice the product of the two terms \((-2ab)\), and add the square of the last term \(b^2\). The rule for squaring a difference is given symbolically as follows.

\[\text{The Square of a Difference}\]

\[(a - b)^2 = a^2 - 2ab + b^2\]

**Example 2**

Using the rule for squaring a difference

Find the square of each difference.

a) \((x - 4)^2\)

b) \((4b - 5y)^2\)

\[\text{Solution}\]

a) \((x - 4)^2 = x^2 - 2(x)(4) + 4^2 = x^2 - 8x + 16\]

b) \((4b - 5y)^2 = (4b)^2 - 2(4b)(5y) + (5y)^2 = 16b^2 - 40by + 25y^2\]

**Product of a Sum and a Difference**

If we multiply the sum \(a + b\) and the difference \(a - b\) by using FOIL, we get

\[(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2.\]

The inner and outer products have a sum of 0. So the product of a sum and a difference of the same two terms is equal to the difference of two squares.
**Example 3**

Product of a sum and a difference

Find each product.

a) \((x + 2)(x - 2)\)  
b) \((b + 7)(b - 7)\)  
c) \((3x - 5)(3x + 5)\)

**Solution**

a) \((x + 2)(x - 2) = x^2 - 4\)

b) \((b + 7)(b - 7) = b^2 - 49\)

c) \((3x - 5)(3x + 5) = 9x^2 - 25\)

**Higher Powers of Binomials**

To find a power of a binomial that is higher than 2, we can use the rule for squaring a binomial along with the method of multiplying binomials using the distributive property. Finding the second or higher power of a binomial is called expanding the binomial because the result has more terms than the original.

**Example 4**

Higher powers of a binomial

Expand each binomial.

a) \((x + 4)^3\)  
b) \((y - 2)^4\)

**Solution**

a) \((x + 4)^3 = (x + 4)(x + 4)(x + 4)\)

= \((x^2 + 8x + 16)(x + 4)\)

= \((x^2 + 8x + 16)x + (x^2 + 8x + 16)4\)

= \(x^3 + 8x^2 + 16x + 4x^2 + 32x + 64\)

= \(x^3 + 12x^2 + 48x + 64\)

b) \((y - 2)^4 = (y - 2)(y - 2)(y - 2)(y - 2)\)

= \((y^2 - 4y + 4)(y^2 - 4y + 4)\)

= \((y^2 - 4y + 4)(y^2) + (y^2 - 4y + 4)(-4y) + (y^2 - 4y + 4)(4)\)

= \(y^4 - 4y^3 + 4y^2 - 4y^3 + 16y^2 - 16y + 16y^2 - 16y + 16\)

= \(y^4 - 8y^3 + 24y^2 - 32y + 16\)

**Example 5**

Area of a pizza

A pizza parlor saves money by making all of its round pizzas one inch smaller in radius than advertised. Write a trinomial for the actual area of a pizza with an advertised radius of \(r\) inches.

**Solution**

A pizza advertised as \(r\) inches has an actual radius of \(r - 1\) inches. The actual area is \(\pi (r - 1)^2\):

\[\pi (r - 1)^2 = \pi (r^2 - 2r + 1) = \pi r^2 - 2\pi r + \pi.\]

So \(\pi r^2 - 2\pi r + \pi\) is a trinomial representing the actual area.
**WARM-UPS**

**True or false? Explain your answer.**

1. $(2 + 3)^2 = 2^2 + 3^2$
2. $(x + 3)^2 = x^2 + 6x + 9$ for any value of $x$.
3. $(3 + 5)^2 = 9 + 25 + 30$
4. $(2x + 7)^2 = 4x^2 + 28x + 49$ for any value of $x$.
5. $(y + 8)^2 = y^2 + 64$ for any value of $y$.
6. The product of a sum and a difference of the same two terms is equal to the difference of two squares.
7. $(40 - 1)(40 + 1) = 1599$
8. $49 \cdot 51 = 2499$
9. $(x - 3)^2 = x^2 - 3x + 9$ for any value of $x$.
10. The square of a sum is equal to a sum of two squares.

**4.4 EXERCISES**

**Reading and Writing** After reading this section, write out the answers to these questions. Use complete sentences.

1. What are the special products?
2. What is the rule for squaring a sum?
3. Why do we need a new rule to find the square of a sum when we already have FOIL?
4. What happens to the inner and outer products in the product of a sum and a difference?
5. What is the rule for finding the product of a sum and a difference?
6. How can you find higher powers of binomials?

**Square each binomial. See Example 1.**

7. $(x + 1)^2$
8. $(y + 2)^2$
9. $(y + 4)^2$
10. $(z + 3)^2$
11. $(3x + 8)^2$
12. $(2m + 7)^2$
13. $(s + 1)^2$
14. $(x + 2)^2$
15. $(2x + y)^2$
16. $(3t + v)^2$
17. $(2t + 3h)^2$
18. $(3z + 5k)^2$
19. $(a - 3)^2$
20. $(w - 4)^2$
21. $(t - 1)^2$
22. $(t - 6)^2$
23. $(3t - 2)^2$
24. $(5a - 6)^2$
25. $(s - t)^2$
26. $(r - w)^2$
27. $(3a - b)^2$
28. $(4w - 7)^2$
29. $(3z - 5y)^2$
30. $(2z - 3w)^2$
31. $(a - 5)(a + 5)$
32. $(x - 6)(x + 6)$
33. $(y - 1)(y + 1)$
34. $(p + 2)(p - 2)$
35. $(3x - 8)(3x + 8)$
36. $(6x + 1)(6x - 1)$
37. $(r + s)(r - s)$
38. $(b - y)(b + y)$
39. $(8y - 3a)(8y + 3a)$
40. $(4u - 9v)(4u + 9v)$
41. $(5x^2 - 2)(5x^2 + 2)$
42. $(3z^2 + 1)(3z^2 - 1)$

**Find each product. See Example 3.**

43. $(x + 1)^3$
44. $(y - 1)^3$
45. $(2a - 3)^3$
46. $(3w - 1)^3$
47. $(a - 3)^4$
48. \((2b + 1)^4\)
49. \((a + b)^4\)
50. \((2a - 3b)^4\)

Find each product.

51. \((a - 20)(a + 20)\)
52. \((1 - x)(1 + x)\)
53. \((x + 8)(x + 7)\)
54. \((x - 9)(x + 5)\)
55. \((4x - 1)(4x + 1)\)
56. \((9y - 1)(9y + 1)\)
57. \((9y - 1)^2\)
58. \((4x - 1)^2\)
59. \((2t - 5)(3t + 4)\)
60. \((2t + 5)(3t - 4)\)
61. \((2t - 5)^2\)
62. \((2t + 5)^2\)
63. \((2t + 5)(2t - 5)\)
64. \((3t - 4)(3t + 4)\)
65. \((x^2 - 1)(x^2 + 1)\)
66. \((y^3 - 1)(y^3 + 1)\)
67. \((2y^3 - 9)^2\)
68. \((3z^4 - 8)^2\)
69. \((2x^3 + 3y^3)^2\)
70. \((4y^5 + 2w^3)^2\)
71. \(\left(\frac{1}{2x^2} + \frac{1}{3}\right)^2\)
72. \(\left(\frac{2}{3y^2} - \frac{1}{2}\right)^2\)

73. \((0.2x - 0.1)^2\)
74. \((0.1y + 0.5)^2\)
75. \((a + b)^3\)
76. \((2a - 3b)^3\)
77. \((1.5x + 3.8)^2\)
78. \((3.45a - 2.3)^2\)
79. \((3.5t - 2.5)(3.5t + 2.5)\)
80. \((4.5h + 5.7)(4.5h - 5.7)\)

In Exercises 81–90, solve each problem.

81. **Shrinking garden.** Rose’s garden is a square with sides of length \(x\) feet. Next spring she plans to make it rectangular by lengthening one side 5 feet and shortening the other side by 5 feet. What polynomial represents the new area? By how much will the area of the new garden differ from that of the old garden?

82. **Square lot.** Sam lives on a lot that he thought was a square, 157 feet by 157 feet. When he had it surveyed, he discovered that one side was actually 2 feet longer than he thought and the other was actually 2 feet shorter than he thought. How much less area does he have than he thought he had?

83. **Area of a circle.** Find a polynomial that represents the area of a circle whose radius is \(b + 1\) meters. Use the value 3.14 for \(\pi\).

84. **Comparing dart boards.** A toy store sells two sizes of circular dartboards. The larger of the two has a radius that is 3 inches greater than that of the other. The radius of the smaller dartboard is \(t\) inches. Find a polynomial that represents the difference in area between the two dartboards.

85. **Poiseuille’s law.** According to the nineteenth-century physician Poiseuille, the velocity (in centimeters per second) of blood \(r\) centimeters from the center of an artery of radius \(R\) centimeters is given by

\[ v = k(R - r)(R + r), \]

where \(k\) is a constant. Rewrite the formula using a special product rule.

86. **Going in circles.** A promoter is planning a circular race track with an inside radius of \(r\) feet and a width of \(w\) feet.
89. **Investing in treasury bills.** An investment advisor uses the polynomial $P(1 + r)^{10}$ to predict the value in 10 years of a client’s investment of $P$ dollars with an average annual return $r$. The accompanying graph shows historic average annual returns for the last 20 years for various asset classes (T. Rowe Price, www.troweprice.com). Use the historical average return to predict the value in 10 years of an investment of $10,000 in U.S. treasury bills?

$20,230.06

87. **Compounded annually.** $P$ dollars is invested at annual interest rate $r$ for 2 years. If the interest is compounded annually, then the polynomial $P(1 + r)^2$ represents the value of the investment after 2 years. Rewrite this expression without parentheses. Evaluate the polynomial if $P = 200$ and $r = 10%$.

\[P(1 + r)^2 = 200(1 + 0.10)^2 = 260.40\]

88. **Compounded semiannually.** $P$ dollars is invested at annual interest rate $r$ for 1 year. If the interest is compounded semiannually, then the polynomial $P\left(1 + \frac{r}{2}\right)^2$ represents the value of the investment after 1 year. Rewrite this expression without parentheses. Evaluate the polynomial if $P = 200$ and $r = 10%$.

\[P\left(1 + \frac{r}{2}\right)^2 = 200\left(1 + 0.10\right)^2 = 220.50\]

90. **Comparing investments.** How much more would the investment in Exercise 89 be worth in 10 years if the client invests in large company stocks rather than U.S. treasury bills?

$26,619.83

**GETTING MORE INVOLVED**

91. **Writing.** What is the difference between the equations \((x + 5)^2 = x^2 + 10x + 25\) and \((x + 5)^2 = x^2 + 25\)?

The first is an identity and the second is a conditional equation.

92. **Writing.** Is it possible to square a sum or a difference without using the rules presented in this section? Why should you learn the rules given in this section?

A sum or difference can be squared with the distributive property, FOIL, or the special product rules. It is easier with the special product rules.

### 4.5 Division of Polynomials

You multiplied polynomials in Section 4.2. In this section you will learn to divide polynomials.

**Dividing Monomials Using the Quotient Rule**

In Chapter 1 we used the definition of division to divide signed numbers. Because the definition of division applies to any division, we restate it here.

**Division of Real Numbers**

If $a$, $b$, and $c$ are any numbers with $b \neq 0$, then

\[a \div b = c\] provided that \[c \cdot b = a.\]

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**FIGURE FOR EXERCISE 86**

The cost in dollars for paving the track is given by the formula

\[C = 1.2\pi[(r + w)^2 - r^2].\]

Use a special product rule to simplify this formula. What is the cost of paving the track if the inside radius is 1000 feet and the width of the track is 40 feet?

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**FIGURE FOR EXERCISES 89 AND 90**

**90. Comparing investments.** How much more would the investment in Exercise 89 be worth in 10 years if the client invests in large company stocks rather than U.S. treasury bills?