We have seen *quadratic functions* on several occasions in this text, but we have not yet defined the term. In this section we study quadratic functions and their graphs.

**Definition**

In Section 9.1 you used the formula

$$s = -16t^2 + v_0t + s_0$$

to find the height $s$ (in feet) of an object that is projected upward with initial velocity $v_0$ ft/sec from an initial height of $s_0$ feet. This equation expresses $s$ as a quadratic function of $t$. In general, if $y$ is determined from $x$ by a formula involving a quadratic polynomial, then we say that $y$ is a *quadratic function of $x$*.

**Quadratic Function**

A *quadratic function* is a function of the form

$$y = ax^2 + bx + c,$$

where $a$, $b$, and $c$ are real numbers and $a \neq 0$.

Without the term $ax^2$, this function would be a linear function. That is why we specify that $a \neq 0$.

### Example 1

**Finding ordered pairs of a quadratic function**

Complete each ordered pair so that it satisfies the given equation.

a) $y = x^2 - x - 6; (2, ), ( , 0)$

b) $s = -16t^2 + 48t + 84; (0, ), ( , 20)$

**Solution**

a) If $x = 2$, then $y = 2^2 - 2 - 6 = -4$. So the ordered pair is $(2, -4)$. To find $x$ when $y = 0$, replace $y$ by 0 and solve the resulting quadratic equation:

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 3 \quad \text{or} \quad x = -2$$

The ordered pairs are $(-2, 0)$ and $(3, 0)$.

b) If $t = 0$, then $s = -16 \cdot 0^2 + 48 \cdot 0 + 84 = 84$. The ordered pair is $(0, 84)$. To find $t$ when $s = 20$, replace $s$ by 20 and solve the equation for $t$:

$$-16t^2 + 48t + 84 = 20$$

$$-16t^2 + 48t + 64 = 0$$

$$t^2 - 3t - 4 = 0$$

$$(t - 4)(t + 1) = 0$$

$$t - 4 = 0 \quad \text{or} \quad t + 1 = 0$$

Zero factor property

$$t = 4 \quad \text{or} \quad t = -1$$

The ordered pairs are $(-1, 20)$ and $(4, 20)$.
When variables other than $x$ and $y$ are used, the independent variable is the first coordinate of an ordered pair, and the dependent variable is the second coordinate. In Example 1(b), $t$ is the independent variable and first coordinate because $s$ depends on $t$ by the formula $s = -16t^2 + 48t + 84$.

**Graphing Quadratic Functions**

Any real number may be used for $x$ in the formula $y = ax^2 + bx + c$. So the domain (the set of $x$-coordinates) for any quadratic function is the set of all real numbers, $\mathbb{R}$. The range (the set of $y$-coordinates) can be determined from the graph. All quadratic functions have graphs that are similar in shape. The graph of any quadratic function is called a parabola.

**Example 2**

**Graphing the simplest quadratic function**

Graph the function $y = x^2$, and state the domain and range.

**Solution**

Make a table of values for $x$ and $y$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = x^2$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>-4</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

See Fig. 9.4 for the graph. The domain is the set of all real numbers, $\mathbb{R}$, because we can use any real number for $x$. From the graph we see that the smallest $y$-coordinate of the function is 0. So the range is the set of real numbers that are greater than or equal to 0, $\{y | y \geq 0\}$.

**Figure 9.4**

The parabola in Fig. 9.4 is said to **open upward**. In the next example we see a parabola that **opens downward**. If $a > 0$ in the equation $y = ax^2 + bx + c$, then the parabola opens upward. If $a < 0$, then the parabola opens downward.

**Example 3**

**A quadratic function**

Graph the function $y = 4 - x^2$, and state the domain and range.

**Solution**

We plot enough points to get the correct shape of the graph:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = 4 - x^2$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>-8</td>
<td>-3</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
See Fig. 9.5 for the graph. The domain is the set of all real numbers, \( R \). From the graph we see that the largest \( y \)-coordinate is 4. So the range is \( \{ y \mid y \leq 4 \} \).

**The Vertex and Intercepts**

The lowest point on a parabola that opens upward or the highest point on a parabola that opens downward is called the **vertex**. The \( y \)-coordinate of the vertex is the **minimum value** of the function if the parabola opens upward, and it is the **maximum value** of the function if the parabola opens downward. For \( y = x^2 \) the vertex is \((0, 0)\), and 0 is the minimum value of the function. For \( g(x) = 4 - x^2 \) the vertex is \((0, 4)\), and 4 is the maximum value of the function.

Because the vertex is either the highest or lowest point on a parabola, it is an important point to find before drawing the graph. The vertex can be found by using the following fact.

**Vertex of a Parabola**

The \( x \)-coordinate of the vertex of \( y = ax^2 + bx + c \) is \( \frac{-b}{2a} \), provided that \( a \neq 0 \).

You can remember \( \frac{-b}{2a} \) by observing that it is part of the quadratic formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

When you graph a parabola, you should always locate the vertex because it is the point at which the graph “turns around.” With the vertex and several nearby points you can see the correct shape of the parabola.

**Example 4**

Using the vertex in graphing a quadratic function

Graph \( y = -x^2 - x + 2 \), and state the domain and range.

**Solution**

First find the \( x \)-coordinate of the vertex:

\[
x = \frac{-b}{2a} = \frac{-(-1)}{2(-1)} = \frac{1}{-2} = -\frac{1}{2}
\]

Now find \( y \) for \( x = -\frac{1}{2} \):

\[
y = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 2 = -\frac{1}{4} + \frac{1}{2} + 2 = \frac{9}{4}
\]

The vertex is \( \left(-\frac{1}{2}, \frac{9}{4}\right) \). Now find a few points on either side of the vertex:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( -\frac{1}{2} )</th>
<th>( 0 )</th>
<th>( 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( 0 )</td>
<td>( 2 )</td>
<td>( \frac{9}{4} )</td>
<td>( 2 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

Sketch a parabola through these points as in Fig. 9.6. The domain is \( R \). Because the graph goes no higher than \( \frac{9}{4} \), the range is \( \{ y \mid y \leq \frac{9}{4} \} \).
The $y$-intercept of a parabola is the point that has 0 as the first coordinate. The $x$-intercepts are the points that have 0 as their second coordinates.

**Example 5**

Using the intercepts in graphing a quadratic function

Find the vertex and intercepts, and sketch the graph of each function.

a) $y = x^2 - 2x - 8$

b) $s = -16t^2 + 64t$

**Solution**

a) Use $x = \frac{-b}{2a}$ to get $x = 1$ as the $x$-coordinate of the vertex. If $x = 1$, then

$$y = 1^2 - 2 \cdot 1 - 8$$

$$= -9.$$  

So the vertex is $(1, -9)$. If $x = 0$, then

$$y = 0^2 - 2 \cdot 0 - 8$$

$$= -8.$$  

The $y$-intercept is $(0, -8)$. To find the $x$-intercepts, replace $y$ by 0:

$$x^2 - 2x - 8 = 0$$

$$(x - 4)(x + 2) = 0$$

$$x - 4 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 4 \quad \text{or} \quad x = -2$$

The $x$-intercepts are $(-2, 0)$ and $(4, 0)$. The graph is shown in Fig. 9.7.

b) Because $s$ is expressed as a function of $t$, the first coordinate is $t$. Use $t = \frac{-b}{2a}$ to get

$$t = \frac{-64}{2(-16)} = 2.$$  

If $t = 2$, then

$$s = -16 \cdot 2^2 + 64 \cdot 2$$

$$= 64.$$  

So the vertex is $(2, 64)$. If $t = 0$, then

$$s = -16 \cdot 0^2 + 64 \cdot 0$$

$$= 0.$$  

So the $s$-intercept is $(0, 0)$. To find the $t$-intercepts, replace $s$ by 0:

$$-16t^2 + 64t = 0$$

$$-16t(t - 4) = 0$$

$$-16t = 0 \quad \text{or} \quad t - 4 = 0$$

$$t = 0 \quad \text{or} \quad t = 4$$

The $t$-intercepts are $(0, 0)$ and $(4, 0)$. The graph is shown in Fig. 9.8.
Applications

In applications we are often interested in finding the maximum or minimum value of a variable. If the graph of a quadratic function opens downward, then the maximum value of the second coordinate is the second coordinate of the vertex. If the parabola opens upward, then the minimum value of the second coordinate is the second coordinate of the vertex.

**Example 6**

**Finding the maximum height**

A ball is tossed upward with a velocity of 64 feet per second from a height of 5 feet. What is the maximum height reached by the ball?

**Solution**

The height $s(t)$ of the ball for any time $t$ is given by $s(t) = -16t^2 + 64t + 5$. Because the maximum height occurs at the vertex of the parabola, we use $t = \frac{-b}{2a}$ to find the vertex:

$$t = \frac{-64}{2(-16)} = 2$$

Now use $t = 2$ to find the second coordinate of the vertex:

$$s(2) = -16(2)^2 + 64(2) + 5 = 69$$

The maximum height reached by the ball is 69 feet. See Fig. 9.9.
True or false? Explain your answer.

1. The ordered pair \((-2, -1)\) satisfies \(y = x^2 - 5\).
2. The \(y\)-intercept for \(y = x^2 - 3x + 9\) is \((9, 0)\).
3. The \(x\)-intercepts for \(y = x^2 - 5\) are \((\sqrt{5}, 0)\) and \((-\sqrt{5}, 0)\).
4. The graph of \(y = x^2 - 12\) opens upward.
5. The graph of \(y = 4 + x^2\) opens downward.
6. The vertex of \(y = x^2 + 2x\) is \((-1, -1)\).
7. The parabola \(y = x^2 + 1\) has no \(x\)-intercepts.
8. The \(y\)-intercept for \(y = ax^2 + bx + c\) is \((0, c)\).
9. If \(w = -2v^2 + 9\), then the maximum value of \(w\) is 9.
10. If \(y = 3x^2 - 7x + 9\), then the maximum value of \(y\) occurs when \(x = \frac{7}{6}\).

9.6 Exercises

Reading and Writing After reading this section, write out the answers to these questions. Use complete sentences.

1. What is a quadratic function?
2. What is a parabola?
3. When does a parabola open upward and when does a parabola open downward?
4. What is the domain of any quadratic function?
5. What is the vertex of a parabola?
6. How can you find the vertex of a parabola?

Graph each quadratic function, and state its domain and range. See Examples 2 and 3.

11. \(y = x^2 + 2\)  
12. \(y = x^2 - 4\)  
13. \(y = \frac{1}{2}x^2 - 4\)  
14. \(y = \frac{1}{3}x^2 - 6\)

Complete each ordered pair so that it satisfies the given equation. See Example 1.

7. \(y = x^2 - x - 12\) \((3, \ ), (\ , 0)\)
8. \(y = -\frac{1}{2}x^2 - x + 1\) \((0, \ ), (\ , -3)\)
9. \(s = -16t^2 + 32t\) \((4, \ ), (\ , 0)\)
15. \( y = -2x^2 + 5 \)  \hspace{1cm} 16. \( y = -x^2 - 1 \)  \hspace{1cm} 22. \( y = x^2 + 2x - 3 \)

23. \( y = x^2 + 2x - 8 \)

17. \( y = -\frac{1}{3}x^2 + 5 \)  \hspace{1cm} 18. \( y = -\frac{1}{2}x^2 + 3 \)

24. \( y = x^2 + x - 6 \)

19. \( y = (x - 2)^2 \)  \hspace{1cm} 20. \( y = (x + 3)^2 \)

25. \( y = -x^2 - 4x - 3 \)

26. \( y = -x^2 - 5x - 4 \)

Find the vertex and intercepts for each quadratic function. Sketch the graph, and state the domain and range. See Examples 4 and 5.

21. \( y = x^2 - x - 2 \)

27. \( y = -x^2 + 3x + 4 \)
28. \( y = -x^2 - 2x + 8 \)

29. \( a = b^2 - 6b - 16 \)

30. \( v = -u^2 - 8u + 9 \)

Find the maximum or minimum value of \( y \) for each function.

31. \( y = x^2 - 8 \)
32. \( y = 33 - x^2 \)
33. \( y = -3x^2 + 14 \)
34. \( y = 6 + 5x^2 \)
35. \( y = x^2 + 2x + 3 \)
36. \( y = x^2 - 2x + 5 \)
37. \( y = -2x^2 - 4x \)
38. \( y = -3x^2 + 24x \)

Solve each problem. See Example 6.

39. **Maximum height.** If a baseball is projected upward from ground level with an initial velocity of 64 feet per second, then its height is a function of time given by \( s(t) = -16t^2 + 32t \). Graph this function for \( 0 \leq t \leq 4 \). What is the maximum height reached by the ball?

40. **Maximum height.** If a soccer ball is kicked straight up with an initial velocity of 32 feet per second, then its height above the earth is a function of time given by \( s(t) = -16t^2 + 32t \). Graph this function for \( 0 \leq t \leq 2 \). What is the maximum height reached by this ball?

41. **Maximum area.** Jason plans to fence a rectangular area with 100 meters of fencing. He has written the formula \( A = w(50 - w) \) to express the area in terms of the width \( w \). What is the maximum possible area that he can enclose with his fencing?

42. **Minimizing cost.** A company uses the function \( C(x) = 0.02x^2 - 3.4x + 150 \) to model the unit cost in dollars for producing \( x \) stabilizer bars. For what number of bars is the unit cost at its minimum? What is the unit cost at that level of production?

43. **Air pollution.** The amount of nitrogen dioxide \( A \) in parts per million (ppm) that was present in the air in the city of Homer on a certain day in June is modeled by the function \( A(t) = -2t^2 + 32t + 12 \), where \( t \) is the number of hours after 6:00 A.M. Use this function to find the time at which the nitrogen dioxide level was at its maximum.

44. **Stabilization ratio.** The stabilization ratio (births/deaths) for South and Central America can be modeled by the function

\[ y = -0.0012x^2 + 0.074x + 2.69 \]

where \( y \) is the number of births divided by the number of deaths in the year \( 1950 + x \) (World Resources Institute, www.wri.org).
46. Exploration.
   a) Write the function \( y = 3(x - 2)^2 + 6 \) in the form \( y = ax^2 + bx + c \), and find the vertex of the parabola using the formula \( x = \frac{-b}{2a} \).
   b) Repeat part (a) with the functions
      \( y = -4(x - 5)^2 - 9 \) and \( y = 3(x + 2)^2 - 6 \).
   c) What is the vertex for a parabola that is written in the form \( y = a(x - h)^2 + k \)? Explain your answer.

45. Suspension bridge. The cable of the suspension bridge shown in the accompanying figure hangs in the shape of a parabola with equation \( y = 0.0375x^2 \), where \( x \) and \( y \) are in meters. What is the height of each tower above the roadway? What is the length \( z \) for the cable bracing the tower?

47. Graph \( y = x^2 \), \( y = \frac{1}{2}x^2 \), and \( y = 2x^2 \) on the same coordinate system. What can you say about the graph of \( y = kx^2 \)?

48. Graph \( y = x^2 \), \( y = (x - 3)^2 \), and \( y = (x + 3)^2 \) on the same coordinate system. How does the graph of \( y = (x - k)^2 \) compare to the graph of \( y = x^2 \)?

49. The equation \( x = y^2 \) is equivalent to \( y = \pm \sqrt{x} \). Graph both \( y = \sqrt{x} \) and \( y = -\sqrt{x} \) on a graphing calculator. How does the graph of \( x = y^2 \) compare to the graph of \( y = x^2 \)?

50. Graph each of the following equations by solving for \( y \).
   a) \( x = y^2 - 1 \)
   b) \( x = -y^2 \)
   c) \( x^2 + y^2 = 4 \)

51. Determine the approximate vertex, domain, range, and \( x \)-intercepts for each quadratic function.
   a) \( y = 3.2x^2 - 5.4x + 1.6 \)
   b) \( y = -1.09x^2 + 13x + 7.5 \)