ne statistic that can be used to measure the general health of a nation or group within a nation is life expectancy. This data is considered more accurate than many other statistics because it is easy to determine the precise number of years in a person’s lifetime.

According to the National Center for Health Statistics, an American born in 1996 has a life expectancy of 76.1 years. However, an American male born in 1996 has a life expectancy of only 73.0 years, whereas a female can expect 79.0 years. A male who makes it to 65 can expect to live 15.7 more years, whereas a female who makes it to 65 can expect 18.9 more years. In the next few years, thanks in part to advances in health care and science, longevity is expected to increase significantly worldwide. In fact, the World Health Organization predicts that by 2025 no country will have a life expectancy of less than 50 years.

In this chapter we study algebraic expressions involving exponents. In Exercises 95 and 96 of Section 5.2 you will see how formulas involving exponents can be used to find the life expectancies of men and women.
In Chapter 1 we defined positive integral exponents and learned to evaluate expressions involving exponents. In this section we will extend the definition of exponents to include all integers and to learn some rules for working with integral exponents. In Chapter 7 we will see that any rational number can be used as an exponent.

**Positive and Negative Exponents**

Positive integral exponents provide a convenient way to write repeated multiplication or very large numbers. For example,

\[2 \cdot 2 \cdot 2 = 2^3, \quad y \cdot y \cdot y \cdot y = y^4, \quad \text{and} \quad 1,000,000,000 = 10^9.\]

We refer to \(2^3\) as “2 cubed,” “2 raised to the third power,” or “a power of 2.”

**Positive Integral Exponents**

If \(a\) is a nonzero real number and \(n\) is a positive integer, then

\[a^n = a \cdot a \cdot a \cdot \ldots \cdot a.\]

In the exponential expression \(a^n\), the base is \(a\), and the exponent is \(n\).

We use \(2^{-3}\) to represent the reciprocal of \(2^3\). Because \(2^3 = 8\), we have \(2^{-3} = \frac{1}{8}\).

In general, \(a^{-n}\) is defined as the reciprocal of \(a^n\).

**Negative Integral Exponents**

If \(a\) is a nonzero real number and \(n\) is a positive integer, then

\[a^{-n} = \frac{1}{a^n}. \quad \text{(If} \ n \ \text{is positive,} \ -n \ \text{is negative.)}\]

To evaluate \(2^{-3}\), you can first cube 2 to get 8 and then find the reciprocal to get \(\frac{1}{8}\), or you can first find the reciprocal of 2 \(\left(\text{which is} \ \frac{1}{2}\right)\) and then cube \(\frac{1}{2}\) to get \(\frac{1}{8}\). So

\[2^{-3} = \left(\frac{1}{2}\right)^3.\]

The power and the reciprocal can be found in either order. If the exponent is \(-1\), we simply find the reciprocal. For example,

\[2^{-1} = \frac{1}{2}, \quad \left(\frac{2}{3}\right)^{-1} = \frac{3}{2}, \quad \text{and} \quad \left(-\frac{1}{4}\right)^{-1} = -4.\]

Because \(2^3\) and \(2^{-3}\) are reciprocals of each other, we have

\[2^{-3} = \frac{1}{2^3} \quad \text{and} \quad \frac{1}{2^{-3}} = 2^3.\]

These examples illustrate the following rules.
Rules for Negative Exponents

If \( a \) is a nonzero real number and \( n \) is a positive integer, then

\[ a^{-n} = \frac{1}{a^n}, \quad a^{-1} = \frac{1}{a}, \quad \text{and} \quad \frac{1}{a^{-n}} = a^n. \]

**Example 1**

Evaluate each expression.

a) \(3^{-2}\)  

b) \((-3)^{-2}\)  

c) \(-3^{-2}\)  

d) \(\left(\frac{3}{4}\right)^{-3}\)  

e) \(\frac{1}{5^{-3}}\)

**Solution**

a) \(3^{-2} = \frac{1}{3^2} = \frac{1}{9}\)  

b) \((-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}\)  

c) \(-3^{-2} = -\frac{1}{3^2} = -\frac{1}{9}\)  

Evaluate \(3^{-2}\), then take the opposite.  

d) \(\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^3\)  

The reciprocal of \(\frac{3}{4}\) is \(\frac{4}{3}\).  

The cube of \(\frac{4}{3}\) is \(\frac{64}{27}\).  

e) \(\frac{1}{5^{-3}} = 5^3 = 125\)  

The reciprocal of \(5^{-3}\) is \(5^3\).  

**CAUTION**  
In Chapter 1 we agreed to evaluate \(-3^2\) by squaring \(3\) first and then taking the opposite. So \(-3^2 = -9\), whereas \((-3)^2 = 9\). The same agreement also holds for negative exponents. That is why the answer to Example 1(c) is negative.

**Product Rule**

We can simplify an expression such as \(2^3 \cdot 2^5\) using the definition of exponents.

\[2^3 \cdot 2^5 = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2) = 2^8\]

Notice that the exponent 8 is the sum of the exponents 3 and 5. This example illustrates the **product rule for exponents**.

\[a^m \cdot a^n = a^{m+n}.\]

**Example 2**

Using the product rule

Simplify each expression. Write answers with positive exponents and assume all variables represent nonzero real numbers.

a) \(3^4 \cdot 3^6\)  

b) \(4x^{-3} \cdot 5x\)  

c) \(-2y^{-3}(-5y^{-4})\)
**Solution**

a) \(3^4 \cdot 3^6 = 3^{4+6} = 3^{10}\)  Product rule

b) \(4x^{-3} \cdot 5x = 4 \cdot 5 \cdot x^{-3} \cdot x^1\)
   \[= 20x^{-2}\]  Product rule: \(x^{-3} \cdot x^1 = x^{-3+1} = x^{-2}\)
   \[= \frac{20}{x^2}\]  Definition of negative exponent

c) \(-2y^{-3}(-5y^{-4}) = (-2)(-5)y^{-3-4}\)
   \[= 10y^{-7}\]  Product rule: \(-3 + (-4) = -7\)
   \[= \frac{10}{y^7}\]  Definition of negative exponent

**CAUTION**  The product rule cannot be applied to \(2^3 \cdot 3^2\) because the bases are not identical. Even when the bases are identical, we do not multiply the bases. For example, \(2^5 \cdot 2^4 \neq 4^9\). Using the rule correctly, we get \(2^5 \cdot 2^4 = 2^9\).

**Zero Exponent**

We have used positive and negative integral exponents, but we have not yet seen the integer 0 used as an exponent. Note that the product rule was stated to hold for any integers \(m\) and \(n\). If we use the product rule on \(2^3 \cdot 2^{-3}\), we get

\[2^3 \cdot 2^{-3} = 2^0.\]

Because \(2^3 = 8\) and \(2^{-3} = \frac{1}{8}\), we must have \(2^3 \cdot 2^{-3} = 1\). So for consistency we define \(2^0\) and the zero power of any nonzero number to be 1.

**Example 3**

Using zero as an exponent

Simplify each expression. Write answers with positive exponents and assume all variables represent nonzero real numbers.

a) \(-3^0\)

b) \((\frac{1}{4} - \frac{3}{2})^0\)

c) \(-2a^5b^{-2} \cdot 3a^{-5}b^2\)

**Solution**

a) To evaluate \(-3^0\), we find \(3^0\) and then take the opposite. So \(-3^0 = -1\).

b) \((\frac{1}{4} - \frac{3}{2})^0 = 1\)  Definition of zero exponent

c) \(-2a^5b^{-2} \cdot 3a^{-5}b^2 = -6a^5 \cdot a^{-5} \cdot b^{-2} \cdot b^2\)
   \[= -6a^0b^{-4}\]  Product rule
   \[= \frac{6}{b^4}\]  Definitions of negative and zero exponent

**Changing the Sign of an Exponent**

Because \(a^{-n}\) and \(a^n\) are reciprocals of each other, we know that

\[a^{-n} = \frac{1}{a^n}\]  and  \[\frac{1}{a^{-n}} = a^n.\]
So a negative exponent in the numerator or denominator can be changed to positive by relocating the exponential expression. In the next example we use these facts to remove negative exponents from exponential expressions.

**Example 4**

Simplifying expressions with negative exponents

Write each expression without negative exponents and simplify. All variables represent nonzero real numbers.

\[ \frac{5a^{-3}}{a^2 \cdot 2^{-2}} \quad \text{and} \quad \frac{-2x^{-3}}{y^{-2}z^3} \]

**Solution**

\[ \frac{5a^{-3}}{a^2 \cdot 2^{-2}} = 5 \cdot a^{-3} \cdot \frac{1}{a^2} \cdot \frac{1}{2^{-2}} \]

Rewrite division as multiplication.

\[ = 5 \cdot \frac{1}{a^2} \cdot \frac{1}{a^2} \cdot 2^{2} \]

Change the signs of the negative exponents.

\[ = \frac{20}{a^4} \]

Product rule: \(a^3 \cdot a^2 = a^5\)

Note that in \(5a^{-3}\) the negative exponent applies only to \(a\).

\[ \frac{-2x^{-3}}{y^{-2}z^3} = -2 \cdot x^{-3} \cdot \frac{1}{y^{-2}} \cdot \frac{1}{z^3} \]

Rewrite as multiplication.

\[ = -2 \cdot \frac{1}{x^3} \cdot y^2 \cdot \frac{1}{z^3} \]

Definition of negative exponent

\[ = -\frac{2y^2}{x^3z^3} \]

Simplify.

In Example 4 we showed more steps than are necessary. For instance, in part (b) we could simply write

\[ \frac{-2x^{-3}}{y^{-2}z^3} = \frac{-2y^2}{x^3z^3}. \]

Exponential expressions (that are factors) can be moved from numerator to denominator (or vice versa) as long as we change the sign of the exponent.

**Caution** If an exponential expression is *not* a factor, you *cannot* move it from numerator to denominator (or vice versa). For example,

\[ \frac{2^{-1} + 1^{-1}}{1^{-1}} = \frac{\frac{1}{2} + 1}{1} = \frac{\frac{3}{2}}{1} = \frac{3}{2} \]

Because \(2^{-1} = \frac{1}{2}\) and \(1^{-1} = 1\), we get

\[ \frac{2^{-1} + 1^{-1}}{1^{-1}} = \frac{\frac{1}{2} + 1}{1} = \frac{\frac{3}{2}}{1} = \frac{3}{2} \]

not \(\frac{1}{2} + 1 = \frac{1}{3}\).

**Quotient Rule**

We can use arithmetic to simplify the quotient of two exponential expressions. For example,

\[ \frac{2^5}{2^3} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} = 2^2. \]
There are five 2’s in the numerator and three 2’s in the denominator. After divid-
ing, two 2’s remain. The exponent in $2^2$ can be obtained by subtracting the exponents 3 and 5. This example illustrates the quotient rule for exponents.

**Quotient Rule for Exponents**

If $m$ and $n$ are any integers and $a \neq 0$, then

\[
\frac{a^m}{a^n} = a^{m-n}.
\]

**Example 5**

Using the quotient rule

Simplify each expression. Write answers with positive exponents only. All variables represent nonzero real numbers.

a) \( \frac{2^9}{2^4} \)

b) \( \frac{m^5}{m^3} \)

c) \( \frac{y^{-4}}{y^2} \)

**Solution**

a) \( \frac{2^9}{2^4} = 2^{9-4} = 2^5 \)

b) \( \frac{m^5}{m^3} = m^{5-3} = m^2 \)

c) \( \frac{y^{-4}}{y^2} = y^{-4-2} = y^{-6} = \frac{1}{y^6} \)

The next example further illustrates the rules of exponents. Remember that the bases must be identical for the quotient rule or the product rule.

**Example 6**

Using the product and quotient rules

Use the rules of exponents to simplify each expression. Write answers with positive exponents only. All variables represent nonzero real numbers.

a) \( \frac{2x^{-7}}{x^{-7}} \)

b) \( \frac{w(2w^{-4})}{3w^{-2}} \)

c) \( \frac{x^{-1}x^{-3}y^5}{x^{-2}y^2} \)

**Solution**

a) \( \frac{2x^{-7}}{x^{-7}} = 2x^{0} \) Quotient rule: \(-7 - (-7) = 0\)

\( = 2 \) Definition of zero exponent

b) \( \frac{w(2w^{-4})}{3w^{-2}} = \frac{2w^{-3}}{3w^{-2}} \) Product rule: \(w^1 \cdot w^{-4} = w^{-3}\)

\( = \frac{2w^{-1}}{3} \) Quotient rule: \(-3 - (-2) = -1\)

\( = \frac{2}{3w} \) Definition of negative exponent

c) \( \frac{x^{-1}x^{-3}y^5}{x^{-2}y^2} = \frac{x^{-4}y^5}{x^{-2}y^2} = x^{-2}y^3 = \frac{y^3}{x^2} \)
Scientific Notation

Many of the numbers that are encountered in science are either very large or very small. For example, the distance from the earth to the sun is 93,000,000 miles, and a hydrogen atom weighs 0.0000000000000000000000017 gram. Scientific notation provides a convenient way of writing very large and very small numbers. In scientific notation the distance from the earth to the sun is \(9.3 \times 10^7\) miles and a hydrogen atom weighs \(1.7 \times 10^{-24}\) gram. In scientific notation the times symbol, \(\times\), is used to indicate multiplication. Converting a number from scientific notation to standard notation is simply a matter of multiplication.

**Example 7**

Scientific notation to standard notation

Write each number using standard notation.

\[
a) \ 7.62 \times 10^5 \\
b) \ 6.35 \times 10^{-4}
\]

**Solution**

\[
a) \ 7.62 \times 10^5 = 762000. \\
b) \ 6.35 \times 10^{-4} = 0.000635
\]

The procedure for converting a number from scientific notation to standard notation is summarized as follows.

**Strategy for Converting from Scientific Notation to Standard Notation**

1. Determine the number of places to move the decimal point by examining the exponent on the 10.
2. Move to the right for a positive exponent and to the left for a negative exponent.

A positive number in scientific notation is written as a product of a number between 1 and 10, and a power of 10. Numbers in scientific notation are written with only one digit to the left of the decimal point. A number larger than 10 is written with a positive power of 10, and a positive number smaller than 1 is written with a negative power of 10. Numbers between 1 and 10 are usually not written in scientific notation. To convert to scientific notation, we reverse the strategy for converting from scientific notation.

**Strategy for Converting from Standard Notation to Scientific Notation**

1. Count the number of places \((n)\) that the decimal point must be moved so that it will follow the first nonzero digit of the number.
2. If the original number was larger than 10, use \(10^n\).
3. If the original number was smaller than 1, use \(10^{-n}\).
E X A M P L E 8

Standard notation to scientific notation
Convert each number to scientific notation.

a) 934,000,000

b) 0.0000025

Solution

a) In 934,000,000 the decimal point must be moved eight places to the left to get it to follow 9, the first nonzero digit.

\[ 934,000,000 = 9.34 \times 10^8 \]

Use 8 because 934,000,000 > 10.

b) The decimal point in 0.0000025 must be moved six places to the right to get the 2 to the left of the decimal point.

\[ 0.0000025 = 2.5 \times 10^{-6} \]

Use -6 because 0.0000025 < 1.

We can perform computations with numbers in scientific notation by using the rules of exponents on the powers of 10.

E X A M P L E 9

Using scientific notation in computations
Evaluate \( \frac{(10,000)(0.000025)}{0.000005} \) by first converting each number to scientific notation.

Solution

\[
\frac{(10,000)(0.000025)}{0.000005} = \frac{(1 \times 10^4)(2.5 \times 10^{-5})}{5 \times 10^{-6}}
\]

\[
= \frac{2.5 \cdot 10^4 \cdot 10^{-5}}{5 \cdot 10^{-6}} \quad \text{Commutative and associative properties}
\]

\[
= 0.5 \times 10^3 \quad \text{Write 0.5 in scientific notation.}
\]

\[
= 5 \times 10^4
\]
**Example 10**  

**Counting hydrogen atoms**  

If the weight of hydrogen is $1.7 \times 10^{-24}$ gram per atom, then how many hydrogen atoms are there in one kilogram of hydrogen?

**Solution**  

There are 1000 or $1 \times 10^3$ grams in one kilogram. So to find the number of hydrogen atoms in one kilogram of hydrogen, we divide $1 \times 10^3$ by $1.7 \times 10^{-24}$:

$$\frac{1 \times 10^3 	ext{ g/kg}}{1.7 \times 10^{-24} \text{ g/atom}} \approx 5.9 \times 10^{26} \text{ atom per kilogram (atom/kg)}$$

To divide by grams per atom, we invert and multiply:  

$$\frac{k \text{ g}}{\text{kg}} \times \frac{\text{atom}}{g} = \frac{\text{atom}}{\text{kg}}$$

Keeping track of the units as we did here helps us to be sure that we performed the correct operation. So there are approximately $5.9 \times 10^{26}$ hydrogen atoms in one kilogram of hydrogen.

---

**True or false? Explain your answer.**

1. $3^5 \cdot 3^4 = 3^9$  
2. $3 \cdot 3 \cdot 3^{-1} = \frac{1}{27}$  
3. $10^{-3} = 0.0001$  
4. $1^{-1} = 1$  
5. $\frac{2^5}{2^{-2}} = 2^7$  
6. $2^3 \cdot 5^2 = 10^5$  
7. $-2^{-2} = -\frac{1}{4}$  
8. $46.7 \times 10^5 = 4.67 \times 10^6$  
9. $0.512 \times 10^{-3} = 5.12 \times 10^{-4}$  
10. $\frac{8 \times 10^{30}}{2 \times 10^{-5}} = 4 \times 10^{25}$

---

**5.1 Exercises**

**Reading and Writing**  

After reading this section, write out the answers to these questions. Use complete sentences.

1. What is an exponential expression?  
2. What is the meaning of a negative exponent?  
3. What is the product rule?  
4. What is the quotient rule?  
5. How do you convert a number from scientific notation to standard notation?  
6. How do you convert a number from standard notation to scientific notation?

**For all exercises in this section, assume that the variables represent nonzero real numbers and use only positive exponents in your answers.**

**Evaluate each expression. See Example 1.**

7. $4^2$  
8. $3^3$  
9. $4^{-2}$  
10. $3^{-3}$  
11. $(\frac{1}{4})^{-2}$  
12. $(\frac{3}{5})^{-1}$
### Chapter 5 Exponents and Polynomials

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<td>(\frac{4^{-2}x^3y^{-6}}{3x^{-3}y^2})</td>
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<td>(\frac{3y^{-2}y^{-6}}{2^{-3}x^2y^7})</td>
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<td><strong>Simplify each expression.</strong> See Examples 5 and 6.</td>
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<td>(\frac{3w^{-2}w^5}{3^{-2}w^3})</td>
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<td>48.</td>
<td>(\frac{2^{-3}w^5}{2w^3w^{-7}})</td>
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<td>49.</td>
<td>(\frac{3x^6 \cdot x^4y^{-1}}{6x^3y^{-2}})</td>
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<td>50.</td>
<td>(\frac{2r^{-1}y^{-1}}{10r^2y^3})</td>
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<tr>
<td><strong>Use the rules of exponents to simplify each expression.</strong></td>
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<td>57.</td>
<td>((-2)^{-3} \cdot 2^{-1})</td>
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<td>58.</td>
<td>((-3)^{-1} \cdot 9^{-1})</td>
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<td>59.</td>
<td>((1 + 2^{-1})^{-2})</td>
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<td>60.</td>
<td>((2^{-1} + 2^{-1})^{-2})</td>
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<td>61.</td>
<td>(-5a^2 \cdot 3a^2)</td>
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<td>62.</td>
<td>(2x^2 \cdot 5x^{-5})</td>
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<td>63.</td>
<td>(5a^2 \cdot a^2 + 3a^{-2} \cdot a^3)</td>
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<tr>
<td>64.</td>
<td>(2x^2 \cdot 5y^{-3})</td>
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<td>65.</td>
<td>(\frac{-3a^4(-2a^2)}{6a^3})</td>
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<td>66.</td>
<td>(\frac{6a(-ab^{-2})}{-2a^2b^{-3}})</td>
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<td>67.</td>
<td>(\frac{(-3x^y)(-2xy^{-3})}{-9x^y^3})</td>
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<td>68.</td>
<td>(\frac{(-2x^{-5}y)(-3xy^6)}{-6x^{-8}y^2})</td>
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<td>69.</td>
<td>(\frac{2^{-1} + 3^{-1}}{2^{-1}})</td>
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<td>70.</td>
<td>(\frac{3^{-1} + 4^{-1}}{12^{-1}})</td>
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<td>71.</td>
<td>(\frac{(2 + 3)^{-1}}{2^{-1} + 3^{-1}})</td>
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<td>72.</td>
<td>(\frac{(3 - 4)^{-1}}{3 \cdot 2^{-1}})</td>
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<tr>
<td><strong>For each equation, find the integer that can be used as the exponent to make the equation correct.</strong></td>
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<tr>
<td>73.</td>
<td>(8 = 2^3)</td>
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<td>74.</td>
<td>(27 = 3^3)</td>
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<td>75.</td>
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<td>76.</td>
<td>(\frac{1}{125} = 5^3)</td>
</tr>
<tr>
<td>77.</td>
<td>(16 = \left( \frac{1}{2} \right)^{-2})</td>
</tr>
<tr>
<td>78.</td>
<td>(81 = \left( \frac{1}{3} \right)^{-4})</td>
</tr>
<tr>
<td>79.</td>
<td>(10^3 = 10,000)</td>
</tr>
<tr>
<td>80.</td>
<td>(10^7 = 100,000)</td>
</tr>
<tr>
<td><strong>Write each number in standard notation.</strong> See Example 7.</td>
<td></td>
</tr>
<tr>
<td>81.</td>
<td>(4.86 \times 10^5)</td>
</tr>
<tr>
<td>82.</td>
<td>(3.80 \times 10^2)</td>
</tr>
<tr>
<td>83.</td>
<td>(2.37 \times 10^{-6})</td>
</tr>
<tr>
<td>84.</td>
<td>(1.62 \times 10^{-3})</td>
</tr>
<tr>
<td>85.</td>
<td>(4 \times 10^6)</td>
</tr>
<tr>
<td>86.</td>
<td>(496 \times 10^7)</td>
</tr>
<tr>
<td>87.</td>
<td>(5 \times 10^{-6})</td>
</tr>
<tr>
<td>88.</td>
<td>(48 \times 10^{-3})</td>
</tr>
<tr>
<td><strong>Write each number in scientific notation.</strong> See Example 8.</td>
<td></td>
</tr>
<tr>
<td>89.</td>
<td>(320,000)</td>
</tr>
<tr>
<td>90.</td>
<td>(43,298,000)</td>
</tr>
<tr>
<td>91.</td>
<td>(0.0000071)</td>
</tr>
<tr>
<td>92.</td>
<td>(0.00000894)</td>
</tr>
<tr>
<td>93.</td>
<td>(0.00007)</td>
</tr>
<tr>
<td>94.</td>
<td>(8,295,100)</td>
</tr>
<tr>
<td>95.</td>
<td>(235 \times 10^5)</td>
</tr>
<tr>
<td>96.</td>
<td>(0.43 \times 10^{-9})</td>
</tr>
<tr>
<td>97.</td>
<td>((5,000,000)(0.0003) \div 2000)</td>
</tr>
<tr>
<td>98.</td>
<td>((6000)(0.0004)(30,000)(0.002))</td>
</tr>
</tbody>
</table>
105. \(\frac{4.6 \times 10^{12}}{2.3 \times 10^5}\)

106. \(\frac{(-4 \times 10^5)(6 \times 10^{-6})}{2 \times 10^{-16}}\)

107. \((4.8 \times 10^{-5})(5 \times 10^{-8})\)

Evaluate each expression using a calculator. Write answers in scientific notation. Round the decimal part to three decimal places. See Example 10.

103. \((4.3 \times 10^9)(3.67 \times 10^{-5})\)

104. \((2.34 \times 10^6)(8.7 \times 10^5)\)

105. \((4.37 \times 10^{-6}) + (8.75 \times 10^{-5})\)

106. \((6.72 \times 10^5) \div (8.98 \times 10^6)\)

107. \((9.27 \times 10^{10})(6.43 \times 10^7)\)

108. \((1.35 \times 10^{46})(2.7 \times 10^{45})\)

109. \((5.6 \times 10^{13})(3.2 \times 10^{-6})\)

110. \((3.51 \times 10^{-6})^3(4000)^5\)

Solve each problem.

111. **Distance to the sun.** The distance from the earth to the sun is 93 million miles. Express this distance in feet using scientific notation (1 mile = 5,280 feet).

112. **Traveling time.** The speed of light is \(9.83569 \times 10^8\) feet per second. How long does it take light to get from the sun to the earth? (See Exercise 111.)

113. **Space travel.** How long does it take a spacecraft traveling \(1.2 \times 10^7\) kilometers per second to travel \(4.6 \times 10^{12}\) kilometers?

114. **Diameter of a dot.** If the circumference of a very small circle is \(2.35 \times 10^{-8}\) meter, then what is the diameter of the circle?

115. **Solid waste per person.** In 1960 the \(1.80863 \times 10^8\) people in the United States generated \(8.71 \times 10^7\) tons of municipal solid waste (Environmental Protection Agency, www.epa.gov). How many pounds per person per day were generated in 1960?

116. **An increasing problem.** According to the EPA, in 1998 the \(2.70058 \times 10^9\) people in the United States generated \(4.324 \times 10^{11}\) pounds of solid municipal waste.

a) How many pounds per person per day were generated in 1998?

b) Use the graph to predict the number of pounds per person per day that will be generated in the year 2010.

**Figure for Exercises 115 and 116**

117. **Exploration.** a) Using pairs of integers, find values for \(m\) and \(n\) for which \(2^m \cdot 3^n = 6^{m+n}\). b) For which values of \(m\) and \(n\) is it true that \(2^m \cdot 3^n \neq 6^{m+n}\)?

118. **Cooperative learning.** Work in a group to find the units digit of \(3^n\) and explain how you found it.

119. **Discussion.** What is the difference between \(-a^n\) and \((-a)^n\), where \(n\) is an integer? For which values of \(a\) and \(n\) do they have the same value, and for which values do they have different values?

120. **Exploration.** If \(a + b = a\), then what can you conclude about \(b\)? Use scientific notation on your calculator to find \(5 \times 10^{20} + 3 \times 10^6\). Explain why your calculator displays the answer that it gets.
In Section 5.1 you learned some of the basic rules for working with exponents. All of the rules of exponents are designed to make it easier to work with exponential expressions. In this section we will extend our list of rules to include three new ones.

### Raising an Exponential Expression to a Power

An expression such as \((x^3)^2\) consists of the exponential expression \(x^3\) raised to the power 2. We can use known rules to simplify this expression.

\[
(x^3)^2 = x^{3 \cdot 2} = x^6
\]

Note that the exponent 6 is the product of the exponents 2 and 3. This example illustrates the **power of a power rule**.

#### Power of a Power Rule

If \(m\) and \(n\) are any integers and \(a \neq 0\), then

\[
(a^m)^n = a^{mn}.
\]

### Example 1

Using the **power of a power rule**

Use the rules of exponents to simplify each expression. Write the answer with positive exponents only. Assume all variables represent nonzero real numbers.

- **a)** \((2^3)^5\)
- **b)** \((x^2)^{-6}\)
- **c)** \(3(y^{-3})^{-2}y^{-5}\)
- **d)** \(\frac{(x^2)^{-1}}{(x^{-3})^3}\)

#### Solution

- **a)** \((2^3)^5 = 2^{15}\)  
  Power of a power rule

- **b)** \((x^2)^{-6} = x^{-12}\)  
  Power of a power rule

- **c)** \(3(y^{-3})^{-2}y^{-5} = 3y^6y^{-5}\)  
  Power of a power rule

- **d)** \(\frac{(x^2)^{-1}}{(x^{-3})^3} = \frac{x^{-2}}{x^{-9}}\)  
  Power of a power rule

- **e)** \(\frac{x^{-2}}{x^{-9}} = x^7\)  
  Quotient rule
Raising a Product to a Power

Consider how we would simplify a product raised to a positive power and a product raised to a negative power using known rules.

\[
(2x)^3 = 2x \cdot 2x \cdot 2x = 2^3 \cdot x^3 = 8x^3
\]

\[
(ay)^{-3} = \frac{1}{(ay)(ay)(ay)} = \frac{1}{a^3y^3} = a^{-3}y^{-3}
\]

In each of these cases the original exponent is applied to each factor of the product. These examples illustrate the power of a product rule.

**Power of a Product Rule**

If \(a \) and \(b \) are nonzero real numbers and \(n \) is any integer, then

\[
(ab)^n = a^n \cdot b^n.
\]

**Example 2**

Using the power of a product rule

Simplify. Assume the variables represent nonzero real numbers. Write the answers with positive exponents only.

a) \((-3x)^4\)  

b) \((-2x^2)^3\)  

c) \((3x^{-2}y^3)^{-2}\)

**Solution**

a) \((-3x)^4 = (-3)^4x^4\) Power of a product rule

\[= 81x^4\]

b) \((-2x^2)^3 = (-2)^3(x^2)^3\) Power of a product rule

\[= -8x^6\]

Power of a power rule

c) \((3x^{-2}y^3)^{-2} = (3)^{-2}(x^{-2})^{-2}(y^3)^{-2} = \frac{1}{9}x^4y^{-6} = \frac{x^4}{9y^6}\)

Raising a Quotient to a Power

Now consider an example of applying known rules to a power of a quotient:

\[
\left(\frac{x}{5}\right)^3 = \frac{x \cdot x \cdot x}{5 \cdot 5 \cdot 5} = \frac{x^3}{5^3}
\]

We get a similar result with a negative power:

\[
\left(\frac{x}{5}\right)^{-3} = \left(\frac{5}{x}\right)^3 = \frac{5 \cdot 5 \cdot 5}{x \cdot x \cdot x} = \frac{5^3}{x^3} = \frac{x^{-3}}{5^{-3}}
\]

In each of these cases the original exponent applies to both the numerator and denominator. These examples illustrate the power of a quotient rule.

**Power of a Quotient Rule**

If \(a\) and \(b\) are nonzero real numbers and \(n\) is any integer, then

\[
\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}
\]
EXAMPLE 3
Using the power of a quotient rule
Use the rules of exponents to simplify each expression. Write your answers with positive exponents only. Assume the variables are nonzero real numbers.

a) \( \left( \frac{x}{2} \right)^3 \)  

b) \( \left( \frac{2x^3}{3y^2} \right)^3 \)  

c) \( \left( \frac{x^{-2}}{2^3} \right)^{-1} \)  

d) \( \left( \frac{-3}{4x^3} \right)^{-2} \)

Solution

The exponent rules in this section apply to expressions that involve only multiplication and division. This is not too surprising since exponents, multiplication, and division are closely related. Recall that \( a^3 = a \cdot a \cdot a \) and \( a \div b = a \cdot b^{-1} \).

a) \( \left( \frac{x}{2} \right)^3 = \frac{x^3}{2^3} = \frac{x^3}{8} \)  

b) \( \left( \frac{2x^3}{3y^2} \right)^3 = \frac{(-2)^3x^9}{3^3y^6} = \frac{-8x^9}{27y^6} \)  

Because \( (x^3)^3 = x^9 \) and \( (y^2)^3 = y^6 \)

c) \( \left( \frac{x^{-2}}{2^3} \right)^{-1} = \frac{x^2}{2^{-3}} = 8x^2 \)  

d) \( \left( \frac{-3}{4x^3} \right)^{-2} = \frac{(-3)^{-2}}{4^{-2}x^{-6}} = \frac{4^2x^6}{(-3)^2} = \frac{16x^6}{9} \)

A fraction to a negative power can be simplified by using the power of a quotient rule as in Example 3. Another method is to first find the reciprocal of the fraction first, then use the power of a quotient rule as shown in the next example.

EXAMPLE 4
Negative powers of fractions
Simplify. Assume the variables are nonzero real numbers and write the answers with positive exponents only.

a) \( \left( \frac{3}{4} \right)^{-3} \)  

b) \( \left( \frac{x^2}{5} \right)^{-2} \)  

c) \( \left( \frac{-2y^3}{3} \right)^{-2} \)

Solution

a) \( \left( \frac{3}{4} \right)^{-3} = \left( \frac{4}{3} \right)^3 \)  

The reciprocal of \( \frac{3}{4} \) is \( \frac{4}{3} \).

\( = \frac{4^3}{3^3} \)  

Power of a quotient rule

\( = \frac{64}{27} \)

c) \( \left( \frac{-2y^3}{3} \right)^{-2} = \left( \frac{3}{2y^3} \right)^2 = \frac{25}{x^4} \)

Variable Exponents

So far, we have used the rules of exponents only on expressions with integral exponents. However, we can use the rules to simplify expressions having variable exponents that represent integers.

EXAMPLE 5
Expressions with variables as exponents
Simplify. Assume the variables represent integers.

a) \( 3^y \cdot 3^{5y} \)  

b) \( (5^{2x})^{3x} \)  

c) \( \left( \frac{2^m}{3^n} \right)^{5n} \)
Solution

a) \(3^{y} \cdot 3^{y} = 3^{2y}\)  
Product rule: \(4y + 5y = 9y\)

b) \((5^{x})^{3} = 5^{3x}\)
Power of a power rule: \(2x \cdot 3x = 6x^{2}\)

c) \(\left(\frac{2^{n}}{3^{m}}\right) = \frac{(2^{n})^{5n}}{(3^{m})^{5n}}\)
Power of a quotient rule
\[= \frac{2^{5n}}{3^{5mn}}\]  
Power of a power rule

Summary of the Rules

The definitions and rules that were introduced in the last two sections are summarized in the following box.

Rules for Integral Exponents

For these rules \(m\) and \(n\) are integers and \(a\) and \(b\) are nonzero real numbers.

1. \(a^{-n} = \frac{1}{a^n}\) Definition of negative exponent

2. \(a^{-n} = \left(\frac{1}{a}\right)^n, a^{-1} = \frac{1}{a}\), and \(\frac{1}{a^{-n}} = a^n\) Negative exponent rules

3. \(a^0 = 1\) Definition of zero exponent

4. \(a^{m}a^{n} = a^{m+n}\) Product rule

5. \(\frac{a^{m}}{a^{n}} = a^{m-n}\) Quotient rule

6. \((a^{m})^{n} = a^{mn}\) Power of a power rule

7. \((ab)^{n} = a^{n}b^{n}\) Power of a product rule

8. \(\left(\frac{a}{b}\right)^{n} = \frac{a^{n}}{b^{n}}\) Power of a quotient rule

Applications

Both positive and negative exponents occur in formulas used in investment situations. The amount of money invested is the principal, and the value of the principal after a certain time period is the amount. Interest rates are annual percentage rates.

Amount Formula

The amount \(A\) of an investment of \(P\) dollars with interest rate \(r\) compounded annually for \(n\) years is given by the formula

\[A = P(1 + r)^n.\]

Finding the amount

According to Fidelity Investments of Boston, U.S. common stocks have returned an average of 10% annually since 1926. If your great-grandfather had invested $100 in the stock market in 1926 and obtained the average increase each year, then how much would the investment be worth in the year 2006 after 80 years of growth?
Solution
Use \( n = 80, P = \$100, \) and \( r = 0.10 \) in the amount formula:
\[
A = P(1 + r)^n
\]
\[
A = 100(1 + 0.10)^{80}
\]
\[
= 100(1.1)^{80}
\]
\[
= 204,840.02
\]
So \$100 invested in 1926 would have amounted to \$204,840.02 in 2006.

When we are interested in the principal that must be invested today to grow to a certain amount, the principal is called the present value of the investment. We can find a formula for present value by solving the amount formula for \( P \):
\[
P = \frac{A}{(1 + r)^n}
\]
Divide each side by \((1 + r)^n\).

Present Value Formula
The present value \( P \) that will amount to \( A \) dollars after \( n \) years with interest compounded annually at annual interest rate \( r \) is given by
\[
P = A(1 + r)^{-n}.
\]

Example 7
Finding the present value
If your great-grandfather wanted you to have \$1,000,000 in 2006, then how much could he have invested in the stock market in 1926 to achieve this goal? Assume he could get the average annual return of 10% (from Example 6) for 80 years.

Solution
Use \( r = 0.10, n = 80, \) and \( A = 1,000,000 \) in the present value formula:
\[
P = A(1 + r)^{-n}
\]
\[
P = 1,000,000(1 + 0.10)^{-80}
\]
\[
P = 1,000,000(1.1)^{-80} \quad \text{Use a calculator with an exponent key.}
\]
\[
P = 488.19
\]
A deposit of \$488.19 in 1926 would have grown to \$1,000,000 in 80 years at a rate of 10% compounded annually.

Warm-Ups
True or false? Explain your answer. Assume all variables represent nonzero real numbers.

1. \((2^2)^3 = 2^5\)
2. \((2^{-3})^{-1} = 8\)
3. \((x^{-3})^3 = x^{-9}\)
4. \((2^3)^2 = 2^6\)
5. \((2x)^3 = 6x^3\)
6. \((-3y^3)^2 = 9y^9\)
7. \(\left(\frac{x}{3}\right)^{-1} = \frac{3}{x}\)
8. \(\left(\frac{2}{3}\right)^3 = \frac{8}{27}\)
9. \(\left(\frac{x^3}{2}\right)^3 = \frac{x^9}{8}\)
10. \(\left(\frac{y}{x}\right)^{-2} = \frac{x^2}{y^2}\)
**5.2 Exercises**

**Reading and Writing** After reading this section, write out the answers to these questions. Use complete sentences.

1. What is the power of a power rule?
2. What is the power of a product rule?
3. What is the power of a quotient rule?
4. What is principal?
5. What formula is used for computing the amount of an investment for which interest is compounded annually?
6. What formula is used for computing the present value of an amount in the future with interest compounded annually?

For all exercises in this section, assume the variables represent nonzero real numbers and use positive exponents only in your answers.

Use the rules of exponents to simplify each expression. See Example 1.

7. \((2^3)^3\) 8. \((3^2)^3\) 9. \((x^3)^5\)
10. \((x^5)^2\) 11. \((x^2)^{-4}\) 12. \((x^{-2})^3\)
13. \((m^{-3})^{-6}\) 14. \((a^{-3})^{-3}\) 15. \((x^{-2})^3(x^{-3})^{-2}\)
16. \((m^{-3})^{-1}(m^2)^{-4}\) 17. \((x^3)^{-4}\) 18. \((a^2)^{-3}\) \((a^{-3})^4\)

Simplify. See Example 2.

19. 
20. \((-2a)^3\)
21. 
22. \((-2w^{-5})^3\)
23. 
24. \((a^2b^{-3})^2\)
25. \((3ab^{-1})^{-2}\) 26. \((2x^{-1}y^2)^{-3}\)
27. \(\frac{2xy^{-2}}{(3x^2y)^{-1}}\) 28. \(\frac{3ab^{-1}}{(5ab^2)^{-1}}\)
29. \(\frac{(2ab)^{-2}}{2ab^2}\) 30. \(\frac{(3xy)^{-3}}{3xy^3}\)

Use the rules of exponents to simplify each expression. If possible, write down only the answer.

31. \(\left(\frac{w^3}{2}\right)^2\) 32. \(\left(\frac{m^2}{5}\right)^2\)
33. \(\left(-\frac{3a}{4}\right)^3\) 34. \(\left(-\frac{2}{3b}\right)^4\)
35. \(\left(\frac{2x^{-1}}{y}\right)^2\) 36. \(\left(\frac{2a^2b^3}{3}\right)^{-3}\)
37. \(\left(-\frac{3x^4}{y}\right)^{-2}\) 38. \(\left(-\frac{2y^7}{x}\right)^{-3}\)

Simplify. See Example 4.

39. \(\left(\frac{2}{3}\right)^{-2}\) 40. \(\left(\frac{3}{4}\right)^{-2}\)
41. \(\left(-\frac{1}{2}\right)^{-2}\) 42. \(\left(-\frac{2}{3}\right)^{-2}\)
43. \(\left(-\frac{2x^3}{3}\right)^{-3}\) 44. \(\left(-\frac{ab}{c}\right)^{-1}\)
45. \(\left(\frac{2x^5}{3y}\right)^{-3}\) 46. \(\left(\frac{ab^{-2}}{a^2b}\right)^{-2}\)

Simplify each expression. Assume that the variables represent integers. See Example 5.

47. \(5^2 \cdot 5^4\) 48. \(3^{2n-3} \cdot 3^{4-2n}\)
49. \((2^{3w})^{-2w}\) 50. \(6^{x^2} \cdot (6^{2x})^{-3}\)
51. \(\frac{7^{2m+6}}{7^{m+3}}\) 52. \(\frac{4^{-3p}}{4^{-4p}}\)
53. \(8^{n-1} \cdot (8^{n+4})^3\) 54. \((5^{1-3})^3(5^{3-x})^2\)

Use the rules of exponents to simplify each expression. If possible, write down only the answer.

55. \(3x^4 \cdot 2x^3\) 56. \((3x^2)^2\) 57. \((-2x^2)^3\)
58. \(3x^2 \cdot 2x^{-4}\) 59. \(\frac{3x^2y^{-1}}{z^{-1}}\) 60. \(\frac{2-1x^2}{y^2}\)

61. \(\left(-\frac{2}{3}\right)^{-1}\) 62. \(\left(-\frac{1}{5}\right)^{-1}\) 63. \(\left(\frac{2x^3}{3}\right)^2\)
64. \(\left(-\frac{2x^3}{3}\right)^{-1}\) 65. \((-2x^{-2})^{-1}\) 66. \((-3x^{-2})^3\)

Use the rules of exponents to simplify each expression.

67. \(\frac{2x^2y^3}{xy^2}\) 68. \(\left(\frac{2x^2y^3}{3xy^3}\right)^{-1}\)
69. \(\left(\frac{5a^{-1}b^3}{(5ab^{-2})^4}\right)^3\) 70. \(\left(\frac{2m^2n^{-3}y^4}{mn^5}\right)^{-1}\)
Comparing stocks and bonds. According to Fidelity Investments of Boston, throughout the 1990s annual returns on common stocks averaged 19%, whereas annual returns on bonds averaged 9%.

a) If you had invested $10,000 in bonds in 1990 and achieved the average return, then what would your investment be worth after 10 years in 2000?

b) How much more would your $10,000 investment be worth in 2000 if you had invested in stocks?

93. Saving for college. Mr. Watkins wants to have $10,000 in a savings account when his little Wanda is ready for college. How much must he deposit today in an account paying 7% compounded annually to have $10,000 in 18 years?

94. Saving for retirement. In the 1990s returns on Treasury Bills fell to an average of 4.5% per year (Fidelity Investments). Wilma wants to have $2,000,000 when she retires in 45 years. If she assumes an average annual return of 4.5%, then how much must she invest now in Treasury Bills to achieve her goal?

95. Life expectancy of white males. Strange as it may seem, your life expectancy increases as you get older. The function

\[ L = 72.2(1.002)^a \]

can be used to model life expectancy \( L \) for U.S. white males with present age \( a \) (National Center for Health Statistics, www.cdc.gov/nchswww).

a) To what age can a 20-year-old white male expect to live?

b) To what age can a 60-year-old white male expect to live? (See also Chapter Review Exercises 153 and 154.)

96. Life expectancy of white females. Life expectancy improved more for females than for males during the 1940s and 1950s due to a dramatic decrease in maternal mortality rates. The function

\[ L = 78.5(1.001)^a \]

can be used to model life expectancy \( L \) for U.S. white females with present age \( a \).

a) To what age can a 20-year-old white female expect to live?

b) Bob, 30, and Ashley, 26, are an average white couple. How many years can Ashley expect to live as a widow?

c) Why do the life expectancy curves intersect in the accompanying figure?
GETTING MORE INVOLVED

97. Discussion. For which values of $a$ and $b$ is it true that $(ab)^{-1} = a^{-1}b^{-1}$? Find a pair of nonzero values for $a$ and $b$ for which $(a + b)^{-1} \neq a^{-1} + b^{-1}$.

98. Writing. Explain how to evaluate $\left(-\frac{2}{3}\right)^{-3}$ in three different ways.

99. Discussion. Which of the following expressions has a value different from the others? Explain.
   a) $-1^{-1}$
   b) $-3^0$
   c) $-2^{-1} - 2^{-1}$
   d) $(−1)^{-2}$
   e) $(-1)^{-3}$

100. True or False? Explain your answer.
   a) The square of a product is the product of the squares.
   b) The square of a sum is the sum of the squares.

GRAPHING CALCULATOR EXERCISES

101. At 12% compounded annually the value of an investment of $10,000 after $x$ years is given by
   \[ y = 10,000(1.12)^x. \]

102. The function $y = 72.2(1.002)^x$ gives the life expectancy $y$ of a U.S. white male with present age $x$. (See Exercise 95.)
   a) Graph $y = 72.2(1.002)^x$ and $y = 86$ on a graphing calculator. Use a viewing window that shows the intersection of the two graphs.
   b) Use the intersect feature of your calculator to find the point of intersection.
   c) What does the $x$-coordinate of the point of intersection tell you?

5.3 Addition, Subtraction, and Multiplication of Polynomials

A polynomial is a particular type of algebraic expression that serves as a fundamental building block in algebra. We used polynomials in Chapters 1 and 2, but we did not identify them as polynomials. In this section you will learn to recognize polynomials and to add, subtract, and multiply them.

Polynomials

The expression $3x^3 - 15x^2 + 7x - 2$ is an example of a polynomial in one variable. Because this expression could be written as
\[ 3x^3 + (-15x^2) + 7x + (-2), \]
we say that this polynomial is a sum of four terms:
\[ 3x^3, \ -15x^2, \ 7x, \ \text{and} \ -2. \]

A term of a polynomial is a single number or the product of a number and one or more variables raised to whole number powers. The number preceding the variable in each term is called the coefficient of that variable. In $3x^3 - 15x^2 + 7x - 2$ the coefficient of $x^3$ is 3, the coefficient of $x^2$ is $-15$, and the coefficient of $x$ is 7. In algebra a number is frequently referred to as a constant, and so the last term $-2$ is called the constant term. A polynomial is defined as a single term or a sum of a finite number of terms.
EXAMPLE 1
Identifying polynomials

Determine whether each algebraic expression is a polynomial.

a) \(-3\)  
b) \(3x + 2^{-1}\)  
c) \(3x^{-2} + 4y^2\)  
d) \(\frac{1}{x} + \frac{1}{x^2}\)  
e) \(x^{49} - 8x^2 + 11x - 2\)

Solution

a) The number \(-3\) is a polynomial of one term, a constant term.

b) Since \(3x + 2^{-1}\) can be written as \(3x + \frac{1}{2}\), it is a polynomial of two terms.

c) The expression \(3x^{-2} + 4y^2\) is not a polynomial because \(x\) has a negative exponent.

d) If this expression is rewritten as \(x^{-1} + x^{-2}\), then it fails to be a polynomial because of the negative exponents. So a polynomial does not have variables in denominators, and

\[
\frac{1}{x} + \frac{1}{x^2}
\]

is not a polynomial.

e) The expression \(x^{49} - 8x^2 + 11x - 2\) is a polynomial.

For simplicity we usually write polynomials in one variable with the exponents in decreasing order from left to right. Thus we would write

\[
3x^3 - 15x^2 + 7x - 2 \quad \text{rather than} \quad -15x^2 - 2 + 7x + 3x^3.
\]

When a polynomial is written in decreasing order, the coefficient of the first term is called the leading coefficient.

Certain polynomials have special names depending on the number of terms. A monomial is a polynomial that has one term, a binomial is a polynomial that has two terms, and a trinomial is a polynomial that has three terms. The degree of a polynomial in one variable is the highest power of the variable in the polynomial. The number 0 is considered to be a monomial without degree because \(0 = 0x^n\), where \(n\) could be any number.

EXAMPLE 2
Identifying coefficients and degree

State the degree of each polynomial and the coefficient of \(x^2\). Determine whether the polynomial is monomial, binomial, or trinomial.

a) \(\frac{x^2}{3} - 5x^3 + 7\)  
b) \(x^{48} - x^2\)  
c) 6

Solution

a) The degree of this trinomial is 3, and the coefficient of \(x^2\) is \(\frac{1}{3}\).

b) The degree of this binomial is 48, and the coefficient of \(x^2\) is \(-1\).

c) Because \(6 = 6x^0\), the number 6 is a monomial with degree 0. Because \(x^2\) does not appear in this polynomial, the coefficient of \(x^2\) is 0.

Although we are mainly concerned here with polynomials in one variable, we will also encounter polynomials in more than one variable, such as

\[
4x^2 - 5xy + 6y^2, \quad x^2 + y^2 + z^2, \quad \text{and} \quad ab^2 - c^3.
\]
In a term containing more than one variable, the coefficient of a variable consists of all other numbers and variables in the term. For example, the coefficient of \( x \) in \(-5xy\) is \(-5y\), and the coefficient of \( y \) is \(-5x\).

### Evaluating Polynomial Functions

The formula \( D = -16t^2 + v_0t + s_0 \) is used to model the effect of gravity on an object tossed straight upward with initial velocity \( v_0 \) feet per second from an initial height of \( s_0 \) feet. For example, if a ball is tossed into the air at 64 feet per second from a height of 4 feet, then \( D = -16t^2 + 64t + 4 \) gives the ball’s distance above the ground in feet, \( t \) seconds after it is tossed. Because \( D \) is determined by \( t \), we say that \( D \) is a function of \( t \). The values of \( t \) range from \( t = 0 \) when the ball is tossed to the time when it hits the ground. To emphasize that the value of \( D \) depends on \( t \), we can use the function notation introduced in Chapter 3 and write

\[
D(t) = -16t^2 + 64t + 4.
\]

We read \( D(t) \) as “\( D \) of \( t \).” The expression \( D(t) \) is the value of the polynomial at time \( t \). To find the value when \( t = 2 \), replace \( t \) by 2:

\[
D(2) = -16 \cdot 2^2 + 64 \cdot 2 + 4
= -16 \cdot 4 + 128 + 4
= 68
\]

The statement \( D(2) = 68 \) means that the ball is 68 feet above the ground 2 seconds after the ball was tossed upward. Note that \( D(2) \) does not mean \( D \) times 2.

### Example 3

**Finding the value of a polynomial**

Suppose \( Q(x) = 2x^3 - 3x^2 - 7x - 6 \). Find \( Q(3) \) and \( Q(-1) \).

**Solution**

To find \( Q(3) \), replace \( x \) by 3 in \( Q(x) = 2x^3 - 3x^2 - 7x - 6 \):

\[
Q(3) = 2 \cdot 3^3 - 3 \cdot 3^2 - 7 \cdot 3 - 6
= 54 - 27 - 21 - 6
= 0
\]

To find \( Q(-1) \), replace \( x \) by \(-1\) in \( Q(x) = 2x^3 - 3x^2 - 7x - 6 \):

\[
Q(-1) = 2(-1)^3 - 3(-1)^2 - 7(-1) - 6
= -2 - 3 + 7 - 6
= -4
\]

So \( Q(3) = 0 \) and \( Q(-1) = -4 \).

### Addition and Subtraction of Polynomials

When evaluating a polynomial, we get a real number. So the operations that we perform with real numbers can be performed with polynomials. Actually, we have been adding and subtracting polynomials since Chapter 1. To add two polynomials, we simply add the like terms.
Adding polynomials

Find the sums.

a) \((x^2 - 5x - 7) + (7x^2 - 4x + 10)\)

b) \((3x^3 - 5x^2 - 7) + (4x^2 - 2x + 3)\)

**Solution**

a) \((x^2 - 5x - 7) + (7x^2 - 4x + 10) = 8x^2 - 9x + 3\) Combine like terms.

b) For illustration we will write this addition vertically:

\[
\begin{align*}
3x^3 & - 5x^2 & & - 7 \\
4x^2 & - 2x & + & 3 \\
\underline{3x^3} & - x^2 & - 2x & - 4 \\
\end{align*}
\]

Line up like terms. Add.

When we subtract polynomials, we subtract like terms. Because \(a - b = a + (-b)\), we often perform subtraction by changing signs and adding.

Subtracting polynomials

Find the differences.

a) \((x^2 - 7x - 2) - (5x^2 + 6x - 4)\)

b) \((6y^3z - 5yz + 7) - (4y^2z - 3yz - 9)\)

**Solution**

a) We find the first difference horizontally:

\((x^2 - 7x - 2) - (5x^2 + 6x - 4) = x^2 - 7x - 2 - 5x^2 - 6x + 4\) Change signs. Combine like terms.

\[= -4x^2 - 13x + 2\]

b) For illustration we write \((6y^3z - 5yz + 7) - (4y^2z - 3yz - 9)\) vertically:

\[
\begin{align*}
6y^3z & & & - 5yz & + & 7 \\
& & -4y^2z & + & 3yz & + & 9 \\
\underline{6y^3z} & - 4y^2z & - 2yz & + & 16 \\
\end{align*}
\]

Change signs. Add.

It is certainly not necessary to write out all of the steps shown in Examples 4 and 5, but we must use the following rule.

**Addition and Subtraction of Polynomials**

To add two polynomials, add the like terms.

To subtract two polynomials, subtract the like terms.

**Multiplication of Polynomials**

We learned how to multiply monomials when we learned the product rule in Section 5.1. For example,

\[-2x^3 \cdot 4x^2 = -8x^5.\]

To multiply a monomial and a polynomial of two or more terms, we apply the distributive property. For example,

\[3x(x^3 - 5) = 3x^4 - 15x.\]
EXAMPLE 6  
Multiplying a monomial and a polynomial
Find the products.

a) \(2ab^2 \cdot 3a^2b\)
b) \((-1)(5 - x)\)
c) \((x^3 - 5x + 2)(-3x)\)

Solution

a) \(2ab^2 \cdot 3a^2b = 6a^3b^3\)
b) \((-1)(5 - x) = -5 + x = x - 5\)
c) Each term of \(x^3 - 5x + 2\) is multiplied by \(-3x\):
\[(x^3 - 5x + 2)(-3x) = -3x^4 + 15x^2 - 6x\]

Note what happened to the binomial in Example 6(b) when we multiplied it by \(-1\). If we multiply any difference by \(-1\), we get the same type of result:

\[-1(a - b) = -a + b = b - a.\]

Because multiplying by \(-1\) is the same as taking the opposite, we can write this equation as

\[-(a - b) = b - a.\]

This equation says that \(a - b\) and \(b - a\) are opposites or additive inverses of each other. Note that the opposite of \(a + b\) is \(-a - b\), not \(a - b\).

To multiply a binomial and a trinomial, we can use the distributive property or set it up like multiplication of whole numbers.

EXAMPLE 7  
Multiplying a binomial and a trinomial
Find the product \((x + 2)(x^2 + 3x - 5)\).

Solution

We can find this product by applying the distributive property twice. First we multiply the binomial and each term of the trinomial:

\[(x + 2)(x^2 + 3x - 5) = (x + 2)x^2 + (x + 2)3x + (x + 2)(-5)\]

\[= x^3 + 2x^2 + 3x^2 + 6x - 5x - 10\]

\[= x^3 + 5x^2 + x - 10\]

We could have found this product vertically:

\[
\begin{array}{c|cccc}
& x^2 & 3x & -5 \\
\hline
x & 2x^2 & 6x & -10 \\
\hline
& x^3 & 3x^2 & -5x \\
\hline
\end{array}
\]

\[
\frac{2x^2 + 6x - 10}{x^3 + 3x^2 - 5x} \quad \frac{2(x^2 + 3x - 5)}{2x^2 + 6x - 10} \quad \frac{x(x^2 + 3x - 5)}{x^3 + 3x^2 - 5x} \quad \text{Add.}
\]

Multiplication of Polynomials

To multiply polynomials, multiply each term of the first polynomial by each term of the second polynomial and then combine like terms.

In the next example we multiply binomials.
Example 8

Multiplying binomials

Find the products.

a) \((x + y)(z + 4)\)

\[ \text{Solution} \]

\[ (x + y)(z + 4) = (x + y)z + (x + y)4 \]
\[ = xz + yz + 4x + 4y \]

Notice that this product does not have any like terms to combine.

b) Multiply:

\[ \begin{align*}
2x + 5 \\
5x - 15 \\
\frac{2x^2 - 6x}{2x^2 - x - 15}
\end{align*} \]

Warm-ups

True or false? Explain your answers.

1. The expression \(3x^{-2} - 5x + 2\) is a trinomial.
2. In the polynomial \(3x^2 - 5x + 3\) the coefficient of \(x\) is 5.
3. The degree of the polynomial \(x^2 + 3x - 5x^3 + 4\) is 2.
4. If \(C(x) = x^2 - 3\), then \(C(5) = 22\).
5. If \(P(t) = 30t + 10\), then \(P(0) = 40\).
6. \((2x^2 - 3x + 5) + (x^2 + 5x - 7) = 3x^2 + 2x - 2\) for any value of \(x\).
7. \((x^2 - 5x) - (x^2 - 3x) = -8x\) for any value of \(x\).
8. \(-2x(3x - 4x^2) = 8x^3 - 6x^2\) for any value of \(x\).
9. \(-(x - 7) = 7 - x\) for any value of \(x\).
10. The opposite of \(y + 5\) is \(y - 5\) for any value of \(y\).

5.3 Exercises

Reading and Writing

After reading this section, write out the answers to these questions. Use complete sentences.

1. What is a term of a polynomial?
2. What is a coefficient?
3. What is a constant?
4. What is a polynomial?
5. What is the degree of a polynomial?
6. What property is used when multiplying a binomial and a trinomial?

Determine whether each algebraic expression is a polynomial. See Example 1.

7. \(3x\)
8. \(-9\)
9. \(x^{-1} + 4\)
10. \(3x^{-3} + 4x - 1\)
11. \(x^2 - 3x + 1\)
12. \(\frac{x^3}{3} - \frac{3x^2}{2} + 4\)
13. \(\frac{1}{x} + y - 3\)
14. \(x^{50} - \frac{9}{y^2}\)

State the degree of each polynomial and the coefficient of \(x^3\). Determine whether each polynomial is a monomial, binomial, or trinomial. See Example 2.

15. \(x^4 - 8x^3\)
\[
\begin{align*}
16. \quad 15 - x^3 \\
17. \quad -8 \\
18. \quad 17 \\
19. \quad \frac{x^2}{15} \\
20. \quad 5x^4 \\
21. \quad x^3 + 3x^4 - 5x^6 \\
22. \quad \frac{x^3 + 5x}{2} - 7
\end{align*}
\]

For each given polynomial, find the indicated value of the polynomial. See Example 3.

\[
\begin{align*}
23. \quad P(x) = x^2 - 4, \quad P(3) \\
24. \quad P(x) = x^2 - x - 2, \quad P(-1) \\
25. \quad M(x) = -3x^2 + 4x - 9, \quad M(-2) \\
26. \quad C(w) = 3w^2 - w, \quad C(0) \\
27. \quad R(x) = x^3 - x^4 + x^3 - x^2 + x - 1, \quad R(1) \\
28. \quad T(a) = a^2 + a^3, \quad T(-1)
\end{align*}
\]

Perform the indicated operations. See Examples 4 and 5.

\[
\begin{align*}
29. \quad (2a - 3) + (a + 5) \\
30. \quad (2w - 6) + (w + 5) \\
31. \quad (7xy + 30) - (2xy + 5) \\
32. \quad (5ab + 7) - (3ab + 6) \\
33. \quad (x^2 - 3x) + (-x^2 + 5x - 9) \\
34. \quad (2y^2 - 3y - 8) + (y^2 + 4y - 1) \\
35. \quad (2x^3 - 4x - 3) - (x^2 - 2x + 5) \\
36. \quad (2x - 5) - (x^2 - 3x + 2)
\end{align*}
\]

Perform the indicated operations vertically. See Examples 4 and 5.

\[
\begin{align*}
37. \quad \text{Add} & \quad x^2 + 3x^2 - 5x - 2 \\
& \quad -x^2 + 8x^2 + 3x - 7 \\
38. \quad \text{Add} & \quad x^2 - 3x + 7 \\
& \quad -2x^2 - 5x + 2
\end{align*}
\]

\[
\begin{align*}
39. \quad \text{Subtract} & \quad 5x + 2 \\
& \quad 4x + 3 \\
40. \quad \text{Subtract} & \quad 4x - 3 \\
& \quad 2x - 6
\end{align*}
\]

\[
\begin{align*}
41. \quad \text{Subtract} & \quad -x^2 + 3x - 5 \\
& \quad 5x^2 - 2x - 7 \\
42. \quad \text{Subtract} & \quad -3x^2 + 5x - 2 \\
& \quad x^2 - 5x - 6
\end{align*}
\]

\[
\begin{align*}
43. \quad \text{Add} & \quad x - y \\
& \quad x + y \\
44. \quad \text{Add} & \quad -w + 4 \\
& \quad 2w - 3
\end{align*}
\]

Find each product. See Examples 6–8.

\[
\begin{align*}
45. \quad -3x^2 \cdot 5x^3 \\
46. \quad (-5ab^2)(-2a^2b) \\
47. \quad 1(3x - 2) \\
48. \quad -1(-x^2 + 3x - 9) \\
49. \quad 5x^2y^3(3x^3y - 4x) \\
50. \quad 3y^2z(8y^3z^2 - 3yz + 2y)
\end{align*}
\]

\[
\begin{align*}
51. \quad (x - 2)(x + 2) \\
52. \quad (x - 1)(x + 1) \\
53. \quad (x^2 + x + 2)(2x - 3) \\
54. \quad (x^2 - 3x + 2)(x - 4)
\end{align*}
\]

Find each product vertically. See Examples 6–8.

\[
\begin{align*}
55. \quad \text{Multiply} & \quad 2x - 3 \\
& \quad -5x \\
56. \quad \text{Multiply} & \quad 3a^2 - 5a^2 + 7 \\
& \quad -2a
\end{align*}
\]

\[
\begin{align*}
57. \quad \text{Multiply} & \quad x + 5 \\
& \quad a + b \\
58. \quad \text{Multiply} & \quad x + 5 \\
& \quad a - b
\end{align*}
\]

\[
\begin{align*}
59. \quad \text{Multiply} & \quad x + 6 \\
& \quad 3x^2 + 2 \\
60. \quad \text{Multiply} & \quad 2x - 3 \\
& \quad 2x^2 - 5
\end{align*}
\]

\[
\begin{align*}
61. \quad \text{Multiply} & \quad x^2 + xy + y^2 \\
& \quad x - y \\
62. \quad \text{Multiply} & \quad a^2 - ab + b^2 \\
& \quad a + b
\end{align*}
\]

Perform the indicated operations.

\[
\begin{align*}
63. \quad (x - 7) + (2x - 3) + (5 - x) \\
64. \quad (5x - 3) + (x^3 + 3x - 2) + (-2x - 3) \\
65. \quad (a^2 - 5a + 3) + (3a^2 - 6a - 7) \\
66. \quad (w^2 - 3w + 2) + (2w + 3 + w^2) \\
67. \quad (w^2 - 7w - 2) - (w - 3w^2 + 5) \\
68. \quad (a^2 - 3a) - (1 - a - 2a^2)
\end{align*}
\]

\[
\begin{align*}
69. \quad (x - 2)(x^2 + 2x + 4) \\
70. \quad (a - 3)(a^2 + 3a + 9) \\
71. \quad (x - w)(z + 2w) \\
72. \quad (w^2 - a)(t^2 + 3) \\
73. \quad (2xy - 1)(3xy + 5) \\
74. \quad (3ab - 4)(ab + 8)
\end{align*}
\]

Perform the following operations using a calculator.

\[
\begin{align*}
75. \quad (2.31x - 5.4)(6.25x + 1.8) \\
76. \quad (x - 0.28)(x^2 - 34.6x + 21.2) \\
77. \quad (3.759x^2 - 4.71x + 2.85) + (11.61x^2 + 6.59x - 3.716) \\
78. \quad (43.19x^3 - 3.7x^2 - 5.42x + 3.1) - (62.7x^3 - 7.36x - 12.3)
\end{align*}
\]

Perform the indicated operations.

\[
\begin{align*}
79. \quad \left(\frac{1}{2}x + 2\right) + \left(\frac{1}{4}x - \frac{1}{2}\right) \\
80. \quad \left(\frac{1}{3}x + 1\right) + \left(\frac{1}{3}x - \frac{3}{2}\right)
\end{align*}
\]
81. \( \left( \frac{1}{2} x^2 + \frac{1}{3} x - \frac{1}{5} \right) - \left( x^2 - \frac{2}{3} x - \frac{1}{5} \right) \)
82. \( \left( \frac{2}{3} x^2 - \frac{1}{3} x + \frac{1}{6} \right) - \left( \frac{1}{3} x^2 + x + 1 \right) \)
83. \( [x^2 - 3 - (x^2 + 5x - 4)] - [x - 3(x^2 - 5x)] \)
84. \( [x^2 - 4(x^2 - 3x + 2) - 5x] + [x^2 - 5(4 - x^2) + 3] \)
85. \( [5x - 4(x - 3)][3x - 7(x + 2)] \)
86. \( [x^2 - (5x - 2)][x^2 + (5x - 2)] \)
87. \( [x^2 - (5x + 2)][x^2 + (m + 2)] \)
88. \( [3x^2 - (x - 2)][3x^2 + (x + 2)] \)
89. \( 2x(5x - 4) - 3x[5x^2 - 3x(x - 7)] \)
90. \( -3x(x - 2) - 5[2x - 4(x + 6)] \)

Perform the indicated operations. A variable used in an exponent represents an integer; a variable used as a base represents a nonzero real number.
91. \( (a^{2m} + 3a^n - 3) + (-5a^{2m} - 7a^n + 8) \)
92. \( (b^3 - 6) - (4b^3 - b^2 - 7) \)
93. \( (x^n - 1)(x^n + 3) \)
94. \( (2y^4 - 3)(4y^4 + 7) \)
95. \( z^{3m} - z^{2m}(z^{1-t} - 4z^m) \)
96. \( (w^p - 1)(w^{2p} + w^p + 1) \)
97. \( (x^2 + y)(x^4 - x^2y + y^2) \)
98. \( (2x^a - z)(2x^a + z) \)

Solve each problem. See Exercises 99–104.

99. Cost of gravel. The cost in dollars of \( x \) cubic yards of gravel is given by the formula

\[ C(x) = 20x + 15. \]

Find \( C(3) \), the cost of 3 cubic yards of gravel.

100. Annual bonus. Sheila’s annual bonus in dollars for selling \( n \) life insurance policies is given by the formula

\[ B(n) = 0.1n^2 + 3n + 50. \]

Find \( B(20) \), her bonus for selling 20 policies.

101. Marginal cost. A company uses the formula \( C(n) = 50n - 0.01n^2 \) to find the daily cost in dollars of manufacturing \( n \) aluminum windows. The marginal cost of the \( n \)th window is the additional cost incurred for manufacturing that window. For example, the marginal cost of the third window is \( C(3) - C(2) \). Find the marginal cost for manufacturing the third window. What is the marginal cost for manufacturing the tenth window?

102. Marginal profit. A company uses the formula \( P(n) = 4n + 0.9n^2 \) to estimate its daily profit in dollars for producing \( n \) automatic garage door openers. The marginal profit of the \( n \)th opener is the amount of additional profit made for that opener. For example, the marginal profit for the fourth opener is \( P(4) - P(3) \). Find the marginal profit for the fourth opener. What is the marginal profit for the tenth opener? Use the bar graph to explain why the marginal profit increases as production goes up.

103. Male and female life expectancy. Since 1950 the life expectancies of U.S. males and females born in year \( y \) can be modeled by the formulas

\[ M(y) = 0.16252y - 251.91 \]

and

\[ F(y) = 0.18268y - 284.98, \]


a) How much greater was the life expectancy of a female born in 1950 than a male born in 1950?
b) Are the lines in the accompanying figure parallel?
c) In what year will female life expectancy be 8 years greater than male life expectancy?

104. More life expectancy. Use the functions from the previous exercise for these questions.

a) A male born in 1975 does not want his future wife to outlive him. What should be the year of birth for his
wif[. . .] same year? 

\( b) \) Find \( \frac{M(y) + F(y)}{2} \) to get a formula for the life expectancy of a person born in year \( y \).

106. Discussion. Give an example of two fourth-degree trinomials whose sum is a third-degree binomial.

107. Cooperative learning. Work in a group to find the product \((a + b)(c + d)\). How many terms does it have? Find the product \((a + b)(c + d)(e + f)\). How many terms does it have? How many terms are there in a product of four binomials in which there are no like terms to combine? How many terms are there in a product of \( n \) binomials in which there are no like terms?

### GETTING MORE INVOLVED

105. Discussion. Is it possible for a binomial to have degree 4? If so, give an example.

### 5.4 M U L T I P L Y I N G B I N O M I A L S

In Section 5.3 you learned to multiply polynomials. In this section you will learn rules to make multiplication of binomials simpler.

#### The FOIL Method

Consider how we find the product of two binomials \( x + 3 \) and \( x + 5 \) using the distributive property twice:

\[
(x + 3)(x + 5) = (x + 3)x + (x + 3)5 \\
= x^2 + 3x + 5x + 15 \\
= x^2 + 8x + 15
\]

There are four terms in the product. The term \( x^2 \) is the product of the first term of each binomial. The term \( 5x \) is the product of the two outer terms, 5 and \( x \). The term \( 3x \) is the product of the two inner terms, 3 and \( x \). The term 15 is the product of the last two terms in each binomial, 3 and 5. It may be helpful to connect the terms multiplied by lines.

![FOIL diagram]

So instead of writing out all of the steps in using the distributive property, we can get the result by finding the products of the first, outer, inner, and last terms. This method is called the FOIL method.

For example, let’s apply FOIL to the product \((x - 3)(x + 4)\):

\[
(x - 3)(x + 4) = x^2 + 4x - 3x - 12 = x^2 + x - 12
\]

If the outer and inner products are like terms, you can save a step by writing down only their sum.
EXAMPLE 1

Multiplying binomials

Use FOIL to find the products of the binomials.

\[ \text{a) } (2x - 3)(3x + 4) \quad \text{b) } (2x^3 + 5)(2x^3 - 5) \]
\[ \text{c) } (m + w)(2m - w) \quad \text{d) } (a + b)(a - 3) \]

**Solution**

\[ \text{a) } (2x - 3)(3x + 4) = 6x^2 + 8x - 9x - 12 = 6x^2 - x - 12 \]
\[ \text{b) } (2x^3 + 5)(2x^3 - 5) = 4x^6 - 10x^3 + 10x^3 - 25 = 4x^6 - 25 \]
\[ \text{c) } (m + w)(2m - w) = 2m^2 - mw + 2mw - w^2 = 2m^2 + mw - w^2 \]
\[ \text{d) } (a + b)(a - 3) = a^2 - 3a + ab - 3b \quad \text{There are no like terms.} \]

The Square of a Binomial

To find \((a + b)^2\), the square of a sum, we can use FOIL on \((a + b)(a + b)\):

\[ (a + b)(a + b) = a^2 + ab + ab + b^2 \]
\[ = a^2 + 2ab + b^2 \]

You can use the result \(a^2 + 2ab + b^2\) that we obtained from FOIL to quickly find the square of any sum. To square a sum, we square the first term \(a^2\), add twice the product of the two terms \(2ab\), then add the square of the last term \(b^2\).

**Rule for the Square of a Sum**

\[ (a + b)^2 = a^2 + 2ab + b^2 \]

In general, the square of a sum \((a + b)^2\) is not equal to the sum of the squares \(a^2 + b^2\). The square of a sum has the middle term \(2ab\).

EXAMPLE 2

Squaring a binomial

Square each sum, using the new rule.

\[ \text{a) } (x + 5)^2 \quad \text{b) } (2w + 3)^2 \quad \text{c) } (2y^4 + 3)^2 \]

**Solution**

\[ \text{a) } (x + 5)^2 = x^2 + 2(x)(5) + 5^2 = x^2 + 10x + 25 \]
\[ \quad \text{Square Twice Square} \]
\[ \quad \text{of the first product last} \]
\[ \text{b) } (2w + 3)^2 = (2w)^2 + 2(2w)(3) + 3^2 = 4w^2 + 12w + 9 \]
\[ \text{c) } (2y^4 + 3)^2 = (2y^4)^2 + 2(2y^4)(3) + 3^2 = 4y^8 + 12y^4 + 9 \]

**CAUTION** Squaring \(x + 5\) correctly, as in Example 2(a), gives us the identity

\[ (x + 5)^2 = x^2 + 10x + 25, \]

which is satisfied by any \(x\). If you forget the middle term and write \((x + 5)^2 = x^2 + 25\), then you have an equation that is satisfied only if \(x = 0\).
To find \((a - b)^2\), the square of a difference, we can use FOIL:

\[
(a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2
\]

As in squaring a sum, it is simply better to remember the result of using FOIL. To square a difference, square the first term, subtract twice the product of the two terms, and add the square of the last term.

**Rule for the Square of a Difference**

\[
(a - b)^2 = a^2 - 2ab + b^2
\]

### Example 3

**Squaring a binomial**

Square each difference, using the new rule.

a) \((x - 6)^2\)  

**Solution**

\[
(x - 6)^2 = x^2 - 2(x)(6) + 36 = x^2 - 12x + 36
\]

b) \((3w - 5y)^2\)  

**Solution**

\[
(3w - 5y)^2 = (3w)^2 - 2(3w)(5y) + (5y)^2 = 9w^2 - 30wy + 25y^2
\]

c) \((-4 - st)^2\)  

**Solution**

\[
(-4 - st)^2 = (-4)^2 - 2(-4)(st) + (st)^2 = 16 + 8st + x^2t^2
\]

d) \((3 - 5a^3)^2\)  

**Solution**

\[
(3 - 5a^3)^2 = 3^2 - 2(3)(5a^3) + (5a^3)^2 = 9 - 30a^3 + 25a^6
\]

### Example 4

**Finding the product of a sum and a difference**

Find the products.

a) \((x + 3)(x - 3)\)  

**Solution**

\[
(x + 3)(x - 3) = x^2 - 9
\]

b) \((a^3 + 8)(a^3 - 8)\)  

**Solution**

\[
(a^3 + 8)(a^3 - 8) = a^6 - 64
\]

c) \((3x^2 - y^3)(3x^2 + y^3)\)  

**Solution**

\[
(3x^2 - y^3)(3x^2 + y^3) = 9x^4 - y^6
\]
The square of a sum, the square of a difference, and the product of a sum and a difference are referred to as **special products**. Although the special products can be found by using the distributive property or FOIL, they occur so frequently in algebra that it is essential to learn the new rules. In the next example we use the special product rules to multiply two trinomials and to square a trinomial.

**Example 5**

Using special product rules to multiply trinomials

Find the products.

a) \[(x + y + 3)(x + y - 3)\]

**Solution**

a) Use the rule \((a + b)(a - b) = a^2 - b^2\) with \(a = x + y\) and \(b = 3\):
\[
[(x + y) + 3][(x + y) - 3] = (x + y)^2 - 3^2
= x^2 + 2xy + y^2 - 9
\]

b) Use the rule \((a + b)^2 = a^2 + 2ab + b^2\) with \(a = m - n\) and \(b = 5\):
\[
[(m - n) + 5]^2 = (m - n)^2 + 2(m - n)5 + 5^2
= m^2 - 2mn + n^2 + 10m - 10n + 25
\]

**Warm-Ups**

True or false? Explain your answer.

1. \((x + 2)(x + 5) = x^2 + 7x + 10\) for any value of \(x\).
2. \((2x - 3)(3x + 5) = 6x^2 + x - 15\) for any value of \(x\).
3. \((2 + 3)^2 = 2^2 + 3^2\)
4. \((x + 7)^2 = x^2 + 14x + 49\) for any value of \(x\).
5. \((8 - 3)^2 = 64 - 9\)
6. The product of a sum and a difference of the same two terms is equal to the difference of two squares.
7. \((60 - 1)(60 + 1) = 3600 - 1\)
8. \((x - y)^2 = x^2 - 2xy + y^2\) for any values of \(x\) and \(y\).
9. \((x - 3)^2 = x^2 - 3x + 9\) for any value of \(x\).
10. The expression \(3x \cdot 5x\) is a product of two binomials.

**Exercises**

Reading and Writing After reading this section, write out the answers to these questions. Use complete sentences.

1. What property is used to multiply two binomials?
2. What does FOIL stand for?
3. What is the purpose of the FOIL method?
4. How do you square a sum?
5. How do you square a difference?
6. How do you find the product of a sum and a difference?

Find each product. When possible, write down only the answer. See Example 1.

7. \((x - 2)(x + 4)\)
8. \((x - 3)(x + 5)\)
Find the square of each sum or difference. When possible, write down only the answer. See Examples 2 and 3.

23. \((m + 3)^2\)
24. \((a + 2)^2\)
25. \((a - 4)^2\)
26. \((b - 3)^2\)
27. \((2w + 1)^2\)
28. \((3m + 4)^2\)
29. \((3t - 5u)^2\)
30. \((3w - 2x)^2\)
31. \((-x - 1)^2\)
32. \((-d - 5)^2\)
33. \((a - 3y)^2\)
34. \((3m - 5n^2)^2\)
35. \(\left(\frac{1}{2}x - 1\right)^2\)
36. \(\left(\frac{2}{3}x - \frac{1}{2}\right)^2\)

Find each product. See Example 4.

37. \((w - 9)(w + 9)\)
38. \((m - 4)(m + 4)\)
39. \((w^3 + y)(w^3 - y)\)
40. \((a^3 - x)(a^3 + x)\)
41. \((2x - 7)(2x + 7)\)
42. \((5x + 3)(5x - 3)\)
43. \((3x^2 - 2)(3x^2 + 2)\)
44. \((4y^2 + 1)(4y^2 - 1)\)

Use the special product rules to find each product. See Example 5.

45. \([(m + t) + 5][(m + t) - 5]\)
46. \([(2x + 3) - y][(2x + 3) + y]\)
47. \([y - (r + 5)][y + (r + 5)]\)
48. \([x + (3 - k)][x - (3 - k)]\)
49. \([(2y - t) + 3]^2\)
50. \([(u - 3v) - 4]^2\)

51. \([3h + (k - 1)]^2\)
52. \([2p - (3q + 6)]^2\)

Perform the operations and simplify.

53. \((x - 6)(x + 9)\)
54. \((2x^2 - 3)(3x^2 + 4)\)
55. \((5 - x)(5 + x)\)
56. \((4 - ab)(4 + ab)\)
57. \((3x - 4a)(2x + 5a)\)
58. \((x^3 + 2)(x^3 - 2)\)
59. \((2t - 3)(t + w)\)
60. \((5x - 9)(ax + b)\)
61. \((3x^2 + 2y)^2\)
62. \((5a^4 - 2b)^2\)
63. \((2y + 2)(3y - 5)^2\)
64. \((3b - 3)(2b + 3)^2\)
65. \((50 + 2)(50 - 2)\)
66. \((100 - 1)(100 + 1)\)
67. \((3 + 7a)^2\)
68. \((1 - pq)^2\)
69. \(4y\left(3y + \frac{1}{2}\right)^2\)
70. \(25y\left(2y - \frac{1}{5}\right)^2\)
71. \((a + h)^2 - a^2\)
72. \(\frac{(x + h)^2 - x^2}{h}\)
73. \((x + 2)(x + 2)^2\)
74. \((a + 1)^2(a + 1)^2\)
75. \((y - 3)^3\)
76. \((2b + 1)^4\)

Use a calculator to help you perform the following operations.

77. \((3.2x - 4.5)(5.1x + 3.9)\)
78. \((5.3x - 9.2)^2\)
79. \((3.6y + 4.4)^2\)
80. \((3.3a - 7.9b)(3.3a + 7.9b)\)

Find the products. Assume all variables are nonzero and variables used in exponents represent integers.

81. \((a^n + 2)(x^{2m} + 3)\)
82. \((a^n - b)(a^n + b)\)
83. \(a^{n+1}(a^n + a^n - 3)\)
84. \(x^{2b}(x^{-3b} + 3x^{-b} + 5)\)
85. \((a^m + a^n)^2\)
86. \((x^m - x^n)^2\)
87. \((5y^n + 8z^k)(3y^{2m} + 4z^{3-k})\)
88. \((4x^{n+1} + 3y^{b+5})(a^{2a-3} - 2x^{4-b})\)
Solve each problem.

89. Area of a room. Suppose the length of a rectangular room is $x + 3$ meters and the width is $x + 1$ meters. Find a trinomial that can be used to represent the area of the room.

90. House plans. Barbie and Ken planned to build a square house with area $x^2$ square feet. Then they revised the plan so that one side was lengthened by 20 feet and the other side was shortened by 6 feet. Find a trinomial that can be used to represent the area of the revised house.

91. Available habitat. A wild animal will generally stay more than $x$ kilometers from the edge of a forest preserve. So the available habitat for the animal excludes an area of uniform width $x$ on the edge of the rectangular forest preserve shown in the figure. The value of $x$ depends on the animal. Find a trinomial in $x$ that gives the area of the available habitat in square kilometers ($\text{km}^2$) for the forest preserve shown. What is the available habitat in this forest preserve for a bobcat for which $x = 0.4$ kilometers.

92. Cubic coating. A cubic metal box $x$ inches on each side is designed for transporting frozen specimens. The box is surrounded on all sides by a 2-inch-thick layer of styrofoam insulation. Find a polynomial that represents the total volume of the cube and styrofoam.

93. Overflow pan. An air conditioning contractor makes an overflow pan for a condenser by cutting squares with side of length $x$ feet from the corners of a 4 foot by 6 foot piece of galvanized sheet metal as shown in the figure. The sides are then folded up, and the corners are sealed. Find a polynomial that gives the volume of the pan in cubic feet ($\text{ft}^3$). What is the volume of the pan if $x = 4$ inches?

94. Energy efficient. A manufacturer of mobile homes makes a custom model that is $x$ feet long, 12 feet wide, and 8 feet high (all inside dimensions). The insulation is 3 inches thick in the walls, 6 inches thick in the floor, and 8 inches thick in the ceiling. Given that the insulation costs the manufacturer 25 cents per cubic foot and doors and windows take up 80 square feet of wall space, find a polynomial in $x$ that gives the cost in dollars for insulation in this model. State any assumptions that you are making to solve this problem.

**GETTING MORE INVOLVED**

95. Exploration. a) Find $(a + b)^3$ by multiplying $(a + b)^2$ by $a + b$.
   b) Next find $(a + b)^4$ and $(a + b)^5$.
   c) How many terms are in each of these powers of $a + b$ after combining like terms?
   d) Make a general statement about the number of terms in $(a + b)^n$.

96. Cooperative learning. Make a four-column table with columns for $a$, $b$, $(a + b)^2$, and $a^2 + b^2$. Work with a group to fill in the table with five pairs of numbers for $a$ and $b$ for which $(a + b)^2 \neq a^2 + b^2$. For what values of $a$ and $b$ does $(a + b)^2 = a^2 + b^2$?

97. Discussion. The area of the large square shown in the figure is $(a + b)^2$. Find the area of each of the four smaller regions in the figure, and then find the sum of those areas. What conclusion can you draw from these areas about $(a + b)^2$?
5.5 Division of Polynomials

We began our study of polynomials in Section 5.3 by learning how to add, subtract, and multiply polynomials. In this section we will study division of polynomials.

Dividing a Polynomial by a Monomial

You learned how to divide monomials in Section 5.1. For example,

\[ 6x^3 \div (3x) = \frac{6x^3}{3x} = 2x^2. \]

We check by multiplying. Because \( 2x^2 \cdot 3x = 6x^3 \), this answer is correct. Recall that \( a \div b = c \) if and only if \( c \cdot b = a \). We call \( a \) the **dividend**, \( b \) the **divisor**, and \( c \) the **quotient**. We may also refer to \( a \div b \) and \( \frac{a}{b} \) as quotients.

We can use the distributive property to find that

\[ 3x(2x^2 + 5x - 4) = 6x^3 + 15x^2 - 12x. \]

So if we divide \( 6x^3 + 15x^2 - 12x \) by the monomial \( 3x \), we must get \( 2x^2 + 5x - 4 \). We can perform this division by dividing \( 3x \) into each term of \( 6x^3 + 15x^2 - 12x \):

\[
\frac{6x^3 + 15x^2 - 12x}{3x} = \frac{6x^3}{3x} + \frac{15x^2}{3x} - \frac{12x}{3x}
\]

\[ = 2x^2 + 5x - 4. \]

In this case the divisor is \( 3x \), the dividend is \( 6x^3 + 15x^2 - 12x \), and the quotient is \( 2x^2 + 5x - 4 \).

**Example 1**

**Dividing polynomials**

Find the quotient.

**a)** \(-12x^5 \div (2x^3)\)

**b)** \((-20x^6 + 8x^4 - 4x^2) \div (4x^2)\)

**Solution**

**a)** When dividing \( x^5 \) by \( x^3 \), we subtract the exponents:

\[-12x^5 \div (2x^3) = \frac{-12x^5}{2x^3} = -6x^2\]

The quotient is \(-6x^2\). Check:

\[-6x^2 \cdot 2x^3 = -12x^5\]

**b)** Divide each term of \(-20x^6 + 8x^4 - 4x^2\) by \(4x^2\):

\[-20x^6 + 8x^4 - 4x^2 \div 4x^2 = -\frac{20x^6}{4x^2} + \frac{8x^4}{4x^2} - \frac{4x^2}{4x^2}\]

\[= -5x^4 + 2x^2 - 1\]

The quotient is \(-5x^4 + 2x^2 - 1\). Check:

\[4x^2(-5x^4 + 2x^2 - 1) = -20x^6 + 8x^4 - 4x^2\]

---

**helpful hint**

Recall that the order of operations gives multiplication and division an equal ranking and says to do them in order from left to right. So without parentheses,

\[-12x^5 + 2x^3\]

actually means

\[\frac{-12x^5}{2} \cdot x^3.\]
Dividing a Polynomial by a Binomial

We can multiply $x - 2$ and $x + 5$ to get

$$(x - 2)(x + 5) = x^2 + 3x - 10.$$ 

So if we divide $x^2 + 3x - 10$ by the factor $x - 2$, we should get the other factor $x + 5$. This division is not done like division by a monomial; it is done like long division of whole numbers. We get the first term of the quotient by dividing the first term of $x - 2$ into the first term of $x^2 + 3x - 10$. Divide $x^2$ by $x$ to get $x$.

$$x - 2 \left\| \begin{array}{c} x^2 + 3x - 10 \\ x^2 - 2x \\
\end{array} \right. \quad \frac{x^2 + x = x}{5x}$$

Multiply: $x(x - 2) = x^2 - 2x$.

Subtract: $3x - (-2x) = 5x$.

Now bring down $-10$. We get the second term of the quotient (below) by dividing the first term of $x - 2$ into the first term of $5x - 10$. Divide $5x$ by $x$ to get $5$.

$$x - 2 \left\| \begin{array}{c} x + 5 \\ x^2 + 3x - 10 \\ x^2 - 2x \\
\end{array} \right. \quad \frac{5x = 5}{5x - 10} \quad \frac{5x - 10}{0}$$

Multiply: $5(x - 2) = 5x - 10$.

Subtract: $-10 - (-10) = 0$.

So the quotient is $x + 5$ and the remainder is 0. If the remainder is not 0, then

$$\text{dividend} = (\text{divisor})(\text{quotient}) + (\text{remainder}).$$

If we divide each side of this equation by the divisor, we get

$$\frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient} + \text{remainder}}{\text{divisor}}.$$

When dividing polynomials, we must write the terms of the divisor and the dividend in descending order of the exponents. If any terms are missing, as in the next example, we insert terms with a coefficient of 0 as placeholders. When dividing polynomials, we stop the process when the degree of the remainder is smaller than the degree of the divisor.

**Example 2**

Dividing polynomials

Find the quotient and remainder for $(3x^4 - 2 - 5x) \div (x^2 - 3x)$.

**Solution**

Rearrange $3x^4 - 2 - 5x$ as $3x^4 - 5x - 2$ and insert the terms $0x^3$ and $0x^2$:

$$\begin{array}{c} \frac{3x^2 + 9x + 27}{x^2 - 3x} \left\| \begin{array}{c} 3x^4 + 0x^3 + 0x^2 - 5x - 2 \\ 3x^4 - 9x^3 \\
\end{array} \right. \quad \frac{9x^3 + 0x^2}{27x^2} \quad \frac{27x^2 - 5x}{76x - 2} \\
\end{array}$$

Students usually have the most difficulty with the subtraction part of long division. So pay particular attention to that step and double check your work.
The quotient is $3x^2 + 9x + 27$, and the remainder is $76x - 2$. Note that the degree of the remainder is 1, and the degree of the divisor is 2. To check, verify that 

$$(x^2 - 3x)(3x^2 + 9x + 27) + 76x - 2 = 3x^4 - 5x - 2.$$  

**Example 3**

Rewriting a ratio of two polynomials

Write $\frac{4x^3 - x - 9}{2x - 3}$ in the form 

quotient $+$ remainder $\div$ divisor.

**Solution**

Divide $4x^3 - x - 9$ by $2x - 3$. Insert $0 \cdot x^2$ for the missing term.

\[
\begin{array}{cccc}
2 & 2x^2 & +3x & +4 \\
2x - 3 & | & 4x^3 & +0x^2 & -x & -9 \\
\hline
 & 2x^2 & +3x & +4 & -9x & -27 \\
 & 6x^2 & -x & +9x & -27 & 0x^2 - (-6x^2) = 6x^2 \\
 & & 6x^2 & -9x & -27 & \frac{8x}{2} - 9 &= -x - (-9x) = 8x \\
 & & & & 8x & -12 \\
 & & & & 3 & -9 - (-12) = 3
\end{array}
\]

Since the quotient is $2x^2 + 3x + 4$ and the remainder is 3, we have

$$\frac{4x^3 - x - 9}{2x - 3} = 2x^2 + 3x + 4 + \frac{3}{2x - 3}.$$  

To check the answer, we must verify that 

$$(2x - 3)(2x^2 + 3x + 4) + 3 = 4x^3 - x - 9.$$  

**Synthetic Division**

When dividing a polynomial by a binomial of the form $x - c$, we can use **synthetic division** to speed up the process. For synthetic division we write only the essential parts of ordinary division. For example, to divide $x^3 - 5x^2 + 4x - 3$ by $x - 2$, we write only the coefficients of the dividend 1, -5, 4, and -3 in order of descending exponents. From the divisor $x - 2$ we use 2 and start with the following arrangement:

\[
\begin{array}{c|ccc}
2 & 1 & -5 & 4 & -3 \\
\hline
 & & & & (1 \cdot x^3 - 5x^2 + 4x - 3) \div (x - 2)
\end{array}
\]

Next we bring the first coefficient, 1, straight down:

\[
\begin{array}{c|ccc}
2 & 1 & -5 & 4 & -3 \\
\hline
 & & & & \\
 & & & & \downarrow \text{Bring down} \\
 & & & & 1
\end{array}
\]

We then multiply the 1 by the 2 from the divisor, place the answer under the -5, and then add that column. Using 2 for $x - 2$ allows us to add the column rather than subtract as in ordinary division:
We then repeat the multiply-and-add step for each of the remaining columns:

\[
\begin{array}{c|cccc}
2 & 1 & -5 & 4 & -3 \\
\hline \\
& 2 & -6 & -4 & \\
& 1 & -3 & -2 & -7 \\
\end{array}
\]

From the bottom row we can read the quotient and remainder. Since the degree of the quotient is one less than the degree of the dividend, the quotient is \(1x^2 - 3x - 2\). The remainder is \(-7\).

The strategy for getting the quotient \(Q(x)\) and remainder \(R\) by synthetic division can be stated as follows.

**Strategy for Using Synthetic Division**

1. List the coefficients of the polynomial (the dividend).
2. Be sure to include zeros for any missing terms in the dividend.
3. For dividing by \(x - c\), place \(c\) to the left.
4. Bring the first coefficient down.
5. Multiply by \(c\) and add for each column.
6. Read \(Q(x)\) and \(R\) from the bottom row.

**CAUTION** Synthetic division is used only for dividing a polynomial by the binomial \(x - c\), where \(c\) is a constant. If the binomial is \(x - 7\), then \(c = 7\). For the binomial \(x + 7\) we have \(x + 7 = x - (-7)\) and \(c = -7\).

**Example 4**

Using synthetic division

Find the quotient and remainder when \(2x^4 - 5x^2 + 6x - 9\) is divided by \(x + 2\).

**Solution**

Since \(x + 2 = x - (-2)\), we use \(-2\) for the divisor. Because \(x^3\) is missing in the dividend, use a zero for the coefficient of \(x^3\):

\[
\begin{array}{c|cccc}
-2 & 2 & 0 & -5 & 6 - 9 \\
\hline \\
& -4 & 8 & -6 & 0 \\
& 2 & -4 & 3 & 0 & -9 \\
\end{array}
\]

Multiply Add Quotient and remainder

\(-2x^4 + 0 \cdot x^3 - 5x^2 + 6x - 9\)

Because the degree of the dividend is 4, the degree of the quotient is 3. The quotient is \(2x^3 - 4x^2 + 3x\), and the remainder is \(-9\). We can also express the results of this division in the form quotient + \(\frac{\text{remainder}}{\text{divisor}}\):

\[
\frac{2x^4 - 5x^2 + 6x - 9}{x + 2} = 2x^3 - 4x^2 + 3x + \frac{-9}{x + 2}
\]

**Division and Factoring**

To factor a polynomial means to write it as a product of two or more simpler polynomials. If we divide two polynomials and get 0 remainder, then we can write

\[
\text{dividend} = (\text{divisor})(\text{quotient})
\]
and we have factored the dividend. The dividend factors as the divisor times the quotient if and only if the remainder is 0. We can use division to help us discover factors of polynomials. To use this idea, however, we must know a factor or a possible factor to use as the divisor.

**Example 5**

**Using synthetic division to determine factors**

Is \( x - 1 \) a factor of \( 6x^3 - 5x^2 - 4x + 3 \)?

**Solution**

We can use synthetic division to divide \( 6x^3 - 5x^2 - 4x + 3 \) by \( x - 1 \):

\[
\begin{array}{c|cccc}
  & 6 & -5 & -4 & 3 \\
\hline
 1 & 6 & 1 & -3 & 0 \\
 6 & 6 & 1 & -3 & 0 \\
\end{array}
\]

Because the remainder is 0, \( x - 1 \) is a factor, and

\[
6x^3 - 5x^2 - 4x + 3 = (x - 1)(6x^2 + x - 3).
\]

**Example 6**

**Using division to determine factors**

Is \( a - b \) a factor of \( a^3 - b^3 \)?

**Solution**

Divide \( a^3 - b^3 \) by \( a - b \). Insert zeros for the missing \( a^2b \)- and \( ab^2 \)-terms.

\[
\begin{array}{c|cccc}
  & a^2 & 0 + & 0 - b^3 \\
\hline
a - b & a^3 & -a^2b & \\
  & 0 & 0 & \\
& a^2b - ab^2 & & \\
  & ab^2 - b^3 & & \\
  & 0 & & \\
\end{array}
\]

Because the remainder is 0, \( a - b \) is a factor, and

\[
a^3 - b^3 = (a - b)(a^2 + ab + b^2).
\]

**Warm-Ups**

**True or false? Explain your answer.**

1. If \( a \div b = c \), then \( c \) is the dividend.
2. The quotient times the dividend plus the remainder equals the divisor.
3. \((x + 2)(x + 3) + 1 = x^2 + 5x + 7\) is true for any value of \( x \).
4. The quotient of \((x^2 + 5x + 7) \div (x + 3)\) is \( x + 2 \).
5. If \( x^2 + 5x + 7 \) is divided by \( x + 2 \), the remainder is 1.
6. To divide \( x^3 - 4x + 1 \) by \( x - 3 \), we use \( -3 \) in synthetic division.
7. We can use synthetic division to divide \( x^3 - 4x^2 - 6 \) by \( x^2 - 5 \).
8. If \( 3x^5 - 4x^2 - 3 \) is divided by \( x + 2 \), the quotient has degree 4.
9. If the remainder is zero, then the divisor is a factor of the dividend.
10. If the remainder is zero, then the quotient is a factor of the dividend.
5.5 EXERCISES

Reading and Writing  After reading this section, write out the answers to these questions. Use complete sentences.
1. What are the dividend, divisor, and quotient?
2. In what form should polynomials be written for long division?
3. What do you do about missing terms when dividing polynomials?
4. When do you stop the long division process for dividing polynomials?
5. What is synthetic division used for?
6. What is the relationship between division of polynomials and factoring polynomials?

Find the quotient. See Example 1.
7. $36x^7 \div (3x^3)$
8. $-30x^3 \div (-5x)$
9. $16x^2 \div (-8x^2)$
10. $-22a^3 \div (11a^2)$
11. $(6b - 9) \div 3$
12. $(8x^2 - 6x) \div 2$
13. $(3x^2 + 6x) \div (3x)$
14. $(5x^3 - 10x^2 + 20x) \div (5x)$
15. $(10x^4 - 8x^3 + 6x^2) \div (-2x^2)$
16. $(-9x^3 + 6x^2 - 12x) \div (-3x)$
17. $(7x^3 - 4x^2) \div (2x)$
18. $(6x^3 - 5x^2) \div (4x^2)$

Find the quotient and remainder as in Example 2. Check by using the formula
\[
\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}.
\]
19. $(x^2 + 8x + 13) \div (x + 3)$
20. $(x^2 + 5x + 7) \div (x + 3)$
21. $(x^2 - 2x) \div (x + 2)$
22. $(3x) \div (x - 1)$
23. $(x^3 + 8) \div (x + 2)$
24. $(y^3 - 1) \div (y - 1)$
25. $(a^3 + 4a - 5) \div (a - 2)$
26. $(w^3 + w^2 - 3) \div (w - 2)$
27. $(x^3 - x^2 + x - 3) \div (x + 1)$
28. $(a^3 - a^2 + a - 4) \div (a + 2)$
29. $(x^4 - x + x^3 - 1) \div (x - 2)$
30. $(3x^4 + 6 - x^3 + 3x) \div (x + 2)$
31. $(5x^2 - 3x^4 + x - 2) \div (x^2 - 2)$
32. $(x^4 - 2 + x^3) \div (x^2 + 3)$
33. $(x^3 - 4x + 2) \div (2x - 3)$
34. $(x^2 - 5x + 1) \div (3x + 6)$
35. $(2x^2 - x + 6) \div (3x - 2)$
36. $(3x^2 + 4x - 1) \div (2x + 1)$

Write each expression in the form
\[
\frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}.
\]
See Example 3.
37. $\frac{2x}{x - 5}$
38. $\frac{x}{x - 1}$
39. $\frac{x^2}{x + 1}$
40. $\frac{x^2 + 9}{x + 3}$
41. $\frac{x^3}{x + 2}$
42. $\frac{x^3 - 1}{x - 2}$
43. $\frac{x^3 + 2x}{x^2}$
44. $\frac{2x^2 + 3}{2x}$
45. $\frac{x^2 - 4x + 9}{x + 2}$
46. $\frac{x^2 - 5x - 10}{x - 3}$
47. $\frac{3x^3 - 4x^2 + 7}{x - 1}$
48. $\frac{-2x^3 + x^2 - 3}{x + 2}$

Use synthetic division to find the quotient and remainder when the first polynomial is divided by the second. See Example 4.
49. $x^3 - 5x^2 + 6x - 3, \quad x - 2$
50. $x^3 + 6x^2 - 3x - 5, \quad x - 3$
51. \(2x^2 - 4x + 5, \ x + 1\)
52. \(3x^3 - 7x + 4, \ x + 2\)
53. \(3x^4 - 15x^2 + 7x - 9, \ x - 3\)
54. \(-2x^4 + 3x^2 - 5, \ x - 2\)
55. \(x^5 - 1, \ x - 1\)
56. \(x^6 - 1, \ x + 1\)
57. \(x^3 - 5x + 6, \ x + 2\)
58. \(x^3 - 3x - 7, \ x - 4\)
59. \(2.3x^2 - 0.14x + 0.6, \ x - 0.32\)
60. \(1.6x^2 - 3.5x + 4.7, \ x + 1.8\)

For each pair of polynomials, determine whether the first polynomial is a factor of the second. Use synthetic division when possible. See Examples 5 and 6.

61. \(x + 4, \ x^3 + x^2 - 11x + 8\)
62. \(x + 4, \ x^3 + x^2 + x + 48\)
63. \(x - 4, \ x^3 - 13x - 12\)
64. \(x - 1, \ x^3 + 3x^2 - 5x\)
65. \(2x - 3, \ 2x^3 - 3x^2 - 4x + 6\)
66. \(3x - 5, \ 6x^2 - 7x - 6\)
67. \(3w + 1, \ 27w^3 + 1\)
68. \(2w + 3, \ 8w^3 + 27\)
69. \(a - 5, \ a^3 - 125\)
70. \(a - 2, \ a^6 - 64\)
71. \(x^2 - 2, \ x^4 + 3x^3 - 6x - 4\)
72. \(x^2 - 3, \ x^4 + 2x^3 - 4x^2 - 6x + 3\)

Factor each polynomial given that the binomial following each polynomial is a factor of the polynomial.

73. \(x^3 - 6x + 8, \ x - 4\)
74. \(x^3 + 3x - 40, \ x + 8\)
75. \(w^3 - 27, \ w - 3\)
76. \(w^3 + 125, \ w + 5\)
77. \(x^3 - 4x^2 + 6x - 4, \ x - 2\)
78. \(2x^3 + 5x + 7, \ x + 1\)
79. \(z^5 + 6z + 9, \ z + 3\)
80. \(4a^2 - 20a + 25, \ 2a - 5\)
81. \(6y^2 + 5y + 1, \ 2y + 1\)
82. \(12y^2 - y - 6, \ 4y - 3\)

Solve each problem.

83. **Average cost.** The total cost in dollars for manufacturing \(x\) professional racing bicycles in one week is given by the polynomial function

\[ C(x) = 0.03x^2 + 300x. \]

The average cost per bicycle is given by

\[ AC(x) = \frac{C(x)}{x}. \]

a) Find a formula for \(AC(x)\).

b) Is \(AC(x)\) a constant function?

c) Why does the average cost look constant in the accompanying figure?

84. **Average profit.** The weekly profit in dollars for manufacturing \(x\) bicycles is given by the polynomial \(P(x) = 100x + 2x^2\). The average profit per bicycle is given by \(AP(x) = \frac{P(x)}{x}\). Find \(AP(x)\). Find the average profit per bicycle when 12 bicycles are manufactured.

85. **Area of a poster.** The area of a rectangular poster advertising a Pearl Jam concert is \(x^2 - 1\) square feet. If the length is \(x + 1\) feet, then what is the width?

86. **Volume of a box.** The volume of a shipping crate is \(h^3 + 5h^2 + 6h\). If the height is \(h\) and the length is \(h + 2\), then what is the width?

87. **Volume of a pyramid.** Ancient Egyptian pyramid builders knew that the volume of the truncated pyramid shown in the figure on the next page is given by

\[ V = \frac{H(a^3 - b^3)}{3(a - b)}. \]
where $a^2$ is the area of the square base, $b^2$ is the area of the square top, and $H$ is the distance from the base to the top. Find the volume of a truncated pyramid that has a base of 900 square meters, a top of 400 square meters, and a height $H$ of 10 meters.

88. **Egyptian pyramid formula.** Rewrite the formula of the previous exercise so that the denominator contains the number 3 only.

GETTING MORE INVOLVED

89. **Discussion.** On a test a student divided $3x^3 - 5x^2 - 3x + 7$ by $x - 3$ and got a quotient of $3x^2 + 4x$ and remainder $9x + 7$. Verify that the divisor times the quotient plus the remainder is equal to the dividend. Why was the student’s answer incorrect?

90. **Exploration.** Use synthetic division to find the quotient when $x^5 - 1$ is divided by $x - 1$ and the quotient when $x^9 - 1$ is divided by $x - 1$. Observe the pattern in the first two quotients and then write the quotient for $x^9 - 1$ divided by $x - 1$ without dividing.

5.6 **Factoring Polynomials**

In Section 5.5 you learned that a polynomial could be factored by using division: If we know one factor of a polynomial, then we can use it as a divisor to obtain the other factor, the quotient. However, this technique is not very practical because the division process can be somewhat tedious, and it is not easy to obtain a factor to use as the divisor. In this section and the next two sections we will develop better techniques for factoring polynomials. These techniques will be used for solving equations and problems in the last section of this chapter.

**Factoring Out the Greatest Common Factor (GCF)**

A natural number larger than 1 that has no factors other than itself and 1 is called a **prime number.** The numbers

2, 3, 5, 7, 11, 13, 17, 19, 23

are the first nine prime numbers. There are infinitely many prime numbers.

To factor a natural number completely means to write it as a product of prime numbers. In factoring 12 we might write $12 = 4 \cdot 3$. However, 12 is not factored completely as $4 \cdot 3$ because 4 is not a prime. To factor 12 completely, we write $12 = 2 \cdot 2 \cdot 3$ (or $2^2 \cdot 3$).

We use the distributive property to multiply a monomial and a binomial:

$$6x(2x - 1) = 12x^2 - 6x$$

If we start with $12x^2 - 6x$, we can use the distributive property to get

$$12x^2 - 6x = 6x(2x - 1).$$

We have factored out $6x$, which is a common factor of $12x^2$ and $-6x$. We could have factored out just 3 to get

$$12x^2 - 6x = 3(4x^2 - 2x),$$

but this would not be factoring out the greatest common factor. The **greatest common factor** (GCF) is a monomial that includes every number or variable that is a factor of all of the terms of the polynomial.
We can use the following strategy for finding the greatest common factor of a group of terms.

**Strategy for Finding the Greatest Common Factor (GCF)**

1. Factor each term completely.
2. Write a product using each factor that is common to all of the terms.
3. On each of these factors, use an exponent equal to the smallest exponent that appears on that factor in any of the terms.

### Example 1

**The greatest common factor**

Find the greatest common factor (GCF) for each group of terms.

**a)** \(8x^2y, 20xy^3\)  
**b)** \(30a^2, 45a^3b^2, 75a^4b\)

**Solution**

**a)** First factor each term completely:

\[
8x^2y = 2^3x^2y \\
20xy^3 = 2^2 \cdot 5xy^3
\]

The factors common to both terms are 2, \(x\), and \(y\). In the GCF we use the smallest exponent that appears on each factor in either of the terms. So the GCF is \(2^2xy\) or \(4xy\).

**b)** First factor each term completely:

\[
30a^2 = 2 \cdot 3 \cdot 5a^2 \\
45a^3b^2 = 3^2 \cdot 5a^3b^2 \\
75a^4b = 3 \cdot 5^2a^4b
\]

The GCF is \(3 \cdot 5a^2\) or \(15a^2\).

To factor out the GCF from a polynomial, find the GCF for the terms, then use the distributive property to factor it out.

### Example 2

**Factoring out the greatest common factor**

Factor each polynomial by factoring out the GCF.

**a)** \(5x^4 - 10x^3 + 15x^2\)  
**b)** \(8xy^2 + 20x^2y\)  
**c)** \(60x^5 + 24x^3 + 36x^2\)

**Solution**

**a)** First factor each term completely:

\[
5x^4 = 5x^4, \quad 10x^3 = 2 \cdot 5x^3, \quad 15x^2 = 3 \cdot 5x^2.
\]

The GCF of the three terms is \(5x^2\). Now factor \(5x^2\) out of each term:

\[
5x^4 - 10x^3 + 15x^2 = 5x^2(x^2 - 2x + 3)
\]

**b)** The GCF for \(8xy^2\) and \(20x^2y\) is \(4xy\):

\[
8xy^2 + 20x^2y = 4xy(2y + 5x)
\]

**c)** First factor each coefficient in \(60x^5 + 24x^3 + 36x^2\):

\[
60 = 2^2 \cdot 3 \cdot 5, \quad 24 = 2^3 \cdot 3, \quad 36 = 2^2 \cdot 3^2.
\]

The GCF of the three terms is \(2^2 \cdot 3x^2\) or \(12x^2\):

\[
60x^5 + 24x^3 + 36x^2 = 12x^2(5x^3 + 2x + 3)
\]

In the next example the common factor in each term is a binomial.
**Example 3**

**Factoring out a binomial**

Factor.

a) \((x + 3)w + (x + 3)a\)

b) \(x(x - 9) - 4(x - 9)\)

**Solution**

a) We treat \(x + 3\) like a common monomial when factoring:

\[(x + 3)w + (x + 3)a = (x + 3)(w + a)\]

b) Factor out the common binomial \(x - 9\):

\[x(x - 9) - 4(x - 9) = (x - 4)(x - 9)\]

**Factoring Out the Opposite of the GCF**

The GCF, the greatest common factor, for \(x^2 - 4x\) is 2, but we can factor out either 2 or its opposite, \(-2\):

\[-6x^2 - 4x = 2x(-3x - 2)\]

\[= -2x(3x + 2)\]

In Example 8 of this section it will be necessary to factor out the opposite of the GCF.

**Example 4**

**Factoring out the opposite of the GCF**

Factor out the GCF, then factor out the opposite of the GCF.

a) \(5x - 5y\)

b) \(-x^2 - 3\)

c) \(-x^3 + 3x^2 - 5x\)

**Solution**

a) \(5x - 5y = 5(x - y)\) Factor out 5.

\[= -5(-x + y)\] Factor out \(-5\).

b) \(-x^2 - 3 = 1(-x^2 - 3)\) The GCF is 1.

\[= -1(x^2 + 3)\] Factor out \(-1\).

c) \(-x^3 + 3x^2 - 5x = x(-x^2 + 3x - 5)\) Factor out \(x\).

\[= -x(x^2 - 3x + 5)\] Factor out \(-x\).

**Factoring the Difference of Two Squares**

A first-degree polynomial in one variable, such as \(3x - 5\), is called a linear polynomial. (The equation \(3x - 5 = 0\) is a linear equation.)

**Helpful Hint**

The prefix "quad" means four. So why is a polynomial of three terms called quadratic? Perhaps it is because a quadratic polynomial can often be factored into a product of two binomials.

**Linear Polynomial**

If \(a\) and \(b\) are real numbers with \(a \neq 0\), then \(ax + b\) is called a linear polynomial.

A second-degree polynomial such as \(x^2 + 5x - 6\) is called a quadratic polynomial.

**Quadratic Polynomial**

If \(a, b,\) and \(c\) are real numbers with \(a \neq 0\), then \(ax^2 + bx + c\) is called a quadratic polynomial.
One of the main goals of this chapter is to write a quadratic polynomial (when possible) as a product of linear factors.

Consider the quadratic polynomial \( x^2 - 25 \). We recognize that \( x^2 - 25 \) is a difference of two squares, \( x^2 - 5^2 \). We recall that the product of a sum and a difference is a difference of two squares: \((a + b)(a - b) = a^2 - b^2\). If we reverse this special product rule, we get a rule for factoring the difference of two squares.

### Factoring the Difference of Two Squares

\[
a^2 - b^2 = (a + b)(a - b)
\]

The difference of two squares factors as the product of a sum and a difference. To factor \( x^2 - 25 \), we replace \( a \) by \( x \) and \( b \) by \( 5 \) to get

\[
x^2 - 25 = (x + 5)(x - 5).
\]

This equation expresses a quadratic polynomial as a product of two linear factors.

### Example 5

**Factoring the difference of two squares**

Factor each polynomial.

**a)** \( y^2 - 36 \)  
**b)** \( 9x^2 - 1 \)  
**c)** \( 4x^2 - y^2 \)

**Solution**

Each of these binomials is a difference of two squares. Each binomial factors into a product of a sum and a difference.

**a)** \( y^2 - 36 = (y + 6)(y - 6) \) We could also write \( (y - 6)(y + 6) \) because the factors can be written in any order.

**b)** \( 9x^2 - 1 = (3x + 1)(3x - 1) \)

**c)** \( 4x^2 - y^2 = (2x + y)(2x - y) \)

### Factoring Perfect Square Trinomials

The trinomial that results from squaring a binomial is called a perfect square trinomial. We can reverse the rules from Section 5.4 for the square of a sum or a difference to get rules for factoring.

### Factoring Perfect Square Trinomials

\[
a^2 + 2ab + b^2 = (a + b)^2 \\
(a - b)^2
\]

Consider the polynomial \( x^2 + 6x + 9 \). If we recognize that

\[
x^2 + 6x + 9 = x^2 + 2 \cdot x \cdot 3 + 3^2,
\]

then we can see that it is a perfect square trinomial. It fits the rule if \( a = x \) and \( b = 3 \):

\[
x^2 + 6x + 9 = (x + 3)^2
\]

Perfect square trinomials can be identified by using the following strategy.
We use this strategy in the next example.

**Example 6**

**Factoring perfect square trinomials**

Factor each polynomial.

a) $x^2 - 8x + 16$

b) $a^2 + 14a + 49$

c) $4x^2 + 12x + 9$

**Solution**

a) Because the first term is $x^2$, the last is $4^2$, and $-2(x)(4)$ is equal to the middle term $-8x$, the trinomial $x^2 - 8x + 16$ is a perfect square trinomial:

$$x^2 - 8x + 16 = (x - 4)^2$$

b) Because $49 = 7^2$ and $14a = 2(a)(7)$, we have a perfect square trinomial:

$$a^2 + 14a + 49 = (a + 7)^2$$

c) Because $4x^2 = (2x)^2$, $9 = 3^2$, and the middle term $12x$ is equal to $2(2x)(3)$, the trinomial $4x^2 + 12x + 9$ is a perfect square trinomial:

$$4x^2 + 12x + 9 = (2x + 3)^2$$

**Example 7**

**Factoring a difference or a sum of two cubes**

In Example 6 of Section 5.5 we divided $a^3 - b^3$ by $a - b$ to get the quotient $a^2 + ab + b^2$ and no remainder. So $a - b$ is a factor of $a^3 - b^3$, a difference of two cubes. If you divide $a^3 + b^3$ by $a + b$, you will get the quotient $a^2 - ab + b^2$ and no remainder. Try it. So $a + b$ is a factor of $a^3 + b^3$, a sum of two cubes. These results give us two more factoring rules.

**Factoring a Difference or a Sum of Two Cubes**

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
c) \(8z^3 - 27 = (2z)^3 - 3^3\)  
\[= (2z - 3)(4z^2 + 6z + 9)\]  
Recognize a difference of two cubes. 
Let \(a = 2z\) and \(b = 3\) in the formula for a difference of two cubes.

**Factoring a Polynomial Completely**

Polynomials that cannot be factored are called **prime polynomials**. Because binomials such as \(x + 5\), \(a - 6\), and \(3x + 1\) cannot be factored, they are prime polynomials. A polynomial is **factored completely** when it is written as a product of prime polynomials. To factor completely, always factor out the GCF (or its opposite) first. Then continue to factor until all of the factors are prime.

**Example 8**

**Factoring completely**

Factor each polynomial completely.

a) \(5x^2 - 20\)  
b) \(3a^3 - 30a^2 + 75a\)  
c) \(-2b^4 + 16b\)

**Solution**

a) \(5x^2 - 20 = 5(x^2 - 4)\)  
\[= 5(x - 2)(x + 2)\]  
Greatest common factor  
Difference of two squares

b) \(3a^3 - 30a^2 + 75a = 3a(a^2 - 10a + 25)\)  
\[= 3a(a - 5)^2\]  
Greatest common factor  
Perfect square trinomial

c) \(-2b^4 + 16b = -2b(b^3 - 8)\)  
\[= -2b(b - 2)(b^2 + 2b + 4)\]  
Factor out \(-2b\) to make the next step easier.  
Difference of two cubes

**Factoring by Substitution**

So far, the polynomials that we have factored, without common factors, have all been of degree 2 or 3. Some polynomials of higher degree can be factored by substituting a single variable for a variable with a higher power. After factoring, we replace the single variable by the higher-power variable. This method is called substitution.

**Example 9**

**Factoring by substitution**

Factor each polynomial.

a) \(x^4 - 9\)  
b) \(y^8 - 14y^4 + 49\)

**Solution**

a) We recognize \(x^4 - 9\) as a difference of two squares in which \(x^4 = (x^2)^2\) and \(9 = 3^2\). If we let \(w = x^2\), then \(w^2 = x^4\). So we can replace \(x^4\) by \(w^2\) and factor:
\[x^4 - 9 = w^2 - 9\]  
\[= (w + 3)(w - 3)\]  
Replace \(x^4\) by \(w^2\).  
Difference of two squares

b) We recognize \(y^8 - 14y^4 + 49\) as a perfect square trinomial in which \(y^8 = (y^4)^2\) and \(49 = 7^2\). We let \(w = y^4\) and \(w^2 = y^8\):
\[y^8 - 14y^4 + 49 = w^2 - 14w + 49\]  
\[= (w - 7)^2\]  
Replace \(y^8\) by \(w^2\) and \(y^4\) by \(w\).  
Perfect square trinomial

**Helpful Hint**

It is not actually necessary to perform the substitution step. If you can recognize that \(x^4 - 9 = (x^2 - 3)(x^2 + 3)\) then skip the substitution.
The polynomials that we factor by substitution must contain just the right powers of the variable. We can factor \( y^8 - 14y^4 + 49 \) because \((y^4)^2 = y^8\), but we cannot factor \( y^7 - 14y^4 + 49 \) by substitution.

In the next example we use substitution to factor polynomials that have variables as exponents.

**Example 10**

**Polynomials with variable exponents**

Factor completely. The variables used in the exponents represent positive integers.

a) \( x^{2m} - y^2 \)

b) \( z^{2n+1} - 6z^{n+1} + 9z \)

**Solution**

a) Notice that \( x^{2m} = (x^m)^2 \). So if we let \( w = x^m \), then \( w^2 = x^{2m} \):

\[
\begin{align*}
\text{Substitution} \\
x^{2m} - y^2 &= w^2 - y^2 \\
&= (w + y)(w - y) \\
&= (x^m + y)(x^m - y) \\
\text{Replace } w \text{ by } x^m.
\end{align*}
\]

b) First factor out the common factor \( z \):

\[
\begin{align*}
z^{2n+1} - 6z^{n+1} + 9z &= z(z^{2n} - 6z^n + 9) \\
&= z(a^2 - 6a + 9) \\
&= z(a - 3)^2 \\
&= z(z^n - 3)^2 \\
\text{Let } a &= z^2. \\
\text{Replace } a \text{ by } z^n.
\end{align*}
\]

**Warm-Ups**

**True or false? Explain your answer.**

1. For the polynomial \( 3x^2y - 6xy^2 \) we can factor out either \( 3xy \) or \( -3xy \).
2. The greatest common factor for the polynomial \( 8a^3 - 15b^2 \) is 1.
3. \( 2x - 4 = -2(2 - x) \) for any value of \( x \).
4. \( x^2 - 16 = (x - 4)(x + 4) \) for any value of \( x \).
5. The polynomial \( x^2 + 6x + 36 \) is a perfect square trinomial.
6. The polynomial \( y^2 + 16 \) is a perfect square trinomial.
7. \( 9x^2 + 21x + 49 = (3x + 7)^2 \) for any value of \( x \).
8. The polynomial \( x + 1 \) is a factor of \( x^3 + 1 \).
9. \( x^3 - 27 = (x - 3)(x^2 + 6x + 9) \) for any value of \( x \).
10. \( x^3 - 8 = (x - 2)^3 \) for any value of \( x \).

**5.6 Exercises**

**Reading and Writing**

After reading this section, write out the answers to these questions. Use complete sentences.

1. What is a prime number?
2. When is a natural number factored completely?
3. What is the greatest common factor for the terms of a polynomial?
4. What are the two ways to factor out the greatest common factor?
5. What is a linear polynomial?

6. What is a quadratic polynomial?

7. What is a prime polynomial?

8. When is a polynomial factored completely?

Find the greatest common factor for each group of terms. See Example 1.
9. 48, 36x
10. 42a, 28a²
11. 9wx, 21wy, 15xy
12. 70x², 84x, 42x³
13. 24x³y, 42xy², 66xy³
14. 60a²b², 140a²b⁵, 40a³b⁶

Factor out the greatest common factor in each expression. See Examples 2 and 3.
15. x³ - 5x
16. 10x² - 20y³
17. 48wx + 36wy
18. 42wz + 28wa
19. 2x³ - 4x² + 6x
20. 6x³ - 12x² + 18x
21. 36a²b² - 24ab² + 60a²b³
22. 44x³y²z - 110x⁵y²z²
23. (x - 6)a + (x - 6)b
24. (y - 4)³ + (y - 4)b
25. (y - 1)³y + (y - 1)²z
26. (w - 2)³w + (w - 2)²·3

Factor out the greatest common factor, then factor out the opposite of the greatest common factor. See Example 4.
27. 2x - 2y
28. -3x + 6
29. 6x² - 3x
30. 10x² + 5x
31. -w³ + 3w²
32. -2w³ + 6w³
33. -a³ + a² - 7a
34. -2a⁴ - 4a³ + 6a²

Factor each polynomial. See Example 5.
35. x² - 100
36. 81 - y²
37. 4y² - 49
38. 16b² - 1
39. 9x² - 25a²
40. 121a² - b²
41. 144w²z² - 1
42. x³y² - 9c²

Factor each polynomial. See Example 6.
43. x² - 20x + 100
44. y³ + 10y + 25
45. 4m² - 4m + 1
46. 9r³ + 30r + 25
47. w² - 2wr + r²
48. 4r² + 20rt + 25r²

Factor. See Example 7.
49. a³ - 1
50. w³ + 1
51. w³ + 27
52. x³ - 64
53. 8x³ - 1
54. 27x³ + 1
55. a³ + 8
56. m³ - 8

Factor each polynomial completely. See Example 8.
57. 2c² - 8
58. 3x³ - 27x
59. x³ + 10x² + 25x
60. 5a⁴m - 45a⁴m
61. 4x² + 4x + 1
62. ax² - 8ax + 
63. (x + 3)x + (x + 3)³
64. (x - 2)x - (x - 2)⁵
65. 6y³ + 3y
66. 4y² - y
67. 4x² - 20x + 25
68. a³x³ - 6a²x² + 9ax
69. 2m⁴ - 2mn³
70. 5x³y² - y³
71. (2x - 3)x - (2x - 3)²
72. (2x + 1)x + (2x + 1)³
73. 9a³ - aw²
74. 2bn² - 4b²n + 2b³
75. -5a² + 30a - 45
76. -2c² + 50
77. 16 - 54x³
78. 27x³y - 64xy²
79. -3y³ - 18y² - 27y
80. -2m²n - 8mn - 8n
81. -7a²b² + 7
82. -17a² - 17a
Factor each polynomial completely. See Example 9.
83. \(x^{10} - 9\)
84. \(y^8 - 4\)
85. \(z^{12} - 6z^6 + 9\)
86. \(a^6 + 10a^3 + 25\)
87. \(2x^2 + 8x^4 + 8x\)
88. \(x^{13} - 6x^7 + 9x\)
89. \(4x^5 + 4x^3 + x\)
90. \(18x^6 + 24x^3 + 8\)
91. \(x^6 - 8\)
92. \(y^6 - 27\)
93. \(2x^9 + 16\)
94. \(x^{13} + x\)

Factor each polynomial completely. The variables used as exponents represent positive integers. See Example 10.
95. \(a^{2n} - 1\)
96. \(b^{4n} - 9\)
97. \(a^{2r} + 6a^r + 9\)
98. \(a^{6n} - 4a^{3n} + 4\)
99. \(x^{3n} - 8\)
100. \(y^{3n} + 1\)
101. \(a^{3n} - b^3\)
102. \(r^{3n} + 8r^3\)
103. \(k^{2w+1} - 10k^{w+1} + 25k\)
104. \(4a^{3u+1} + 4a^{u+1} + a\)
105. \(u^{6k} - 2u^{4k} + u^{2k}\)
106. \(a^{2m} + 2a^{2m} - 1 + u^{6m}\)

Replace \(k\) in each trinomial by a number that makes the trinomial a perfect square trinomial.
107. \(x^2 + 6x + k\)
108. \(y^2 - 8y + k\)
109. \(4a^2 - ka + 25\)
110. \(9u^2 + kuv + 49v^2\)
111. \(km^2 - 24m + 9\)
112. \(kz^2 + 40z + 16\)
113. \(81y^2 - 180y + k\)
114. \(36a^2 + 60a + k\)

Solve each problem.

115. **Volume of a bird cage.** A company makes rectangular shaped bird cages with height \(b\) inches and square bottoms. The volume of these cages is given by the function 
\[V = b^3 - 6b^2 + 9b.\]

a) What is the length of a side of the square bottom?
b) Use the function to find the volume of a cage with a height of 18 inches.

e) Use the accompanying graph to estimate the height of a cage for which the volume is 20,000 cubic inches.

116. **Pyramid power.** A powerful crystal pyramid has a square base and a volume of \(3y^3 + 12y^2 + 12y\) cubic centimeters. If its height is \(y\) centimeters, then what polynomial represents the length of a side of the square base? (The volume of a pyramid with a square base of area \(a^2\) and height \(h\) is given by \(V = \frac{1}{3}ah^2\).)

\[V = 3y^3 + 12y^2 + 12y\]

117. **Cooperative learning.** List the perfect square trinomials corresponding to \((x + 1)^2\), \((x + 2)^2\), \((x + 3)^2\), \ldots, \((x + 12)^2\). Use your list to quiz a classmate. Read a perfect square trinomial at random from your list and ask your classmate to write its factored form. Repeat until both of you have mastered these 12 perfect square trinomials.
5.7 Factoring \( ax^2 + bx + c \)

In Section 5.5 you learned to factor certain special polynomials. In this section you will learn to factor general quadratic polynomials. We first factor \( ax^2 + bx + c \) with \( a = 1 \), and then we consider the case \( a \neq 1 \).

**Factoring Trinomials with Leading Coefficient 1**

Let’s look closely at an example of finding the product of two binomials using the distributive property:

\[
(x + 3)(x + 4) = (x + 3)x + (x + 3)4 \\
= x^2 + 3x + 4x + 12 \\
= x^2 + 7x + 12
\]

To factor \( x^2 + 7x + 12 \), we need to reverse these steps. First observe that the coefficient 7 is the sum of two numbers that have a product of 12. The only numbers that have a product of 12 and a sum of 7 are 3 and 4. So write 7 as \( 3x + 4x \):

\[
x^2 + 7x + 12 = x^2 + 3x + 4x + 12
\]

Now factor the common factor \( x \) out of the first two terms and the common factor 4 out of the last two terms. This method is called factoring by grouping.

\[
\begin{align*}
\text{Factor out} & \quad \text{Factor out} \\
3x & \quad 4 \\
x^2 + 7x + 12 & = x^2 + 3x + 4x + 12 \quad \text{Rewrite 7x as 3x + 4x.} \\
& = (x + 3)x + (x + 3)4 \quad \text{Factor out common factors.} \\
& = (x + 3)(x + 4) \quad \text{Factor out the common factor x + 3.}
\end{align*}
\]

**Example 1**

Factoring a trinomial by grouping

Factor each trinomial by grouping.

**a)** \( x^2 + 9x + 18 \)

**b)** \( x^2 - 2x - 24 \)

**Solution**

**a)** We need to find two integers with a product of 18 and a sum of 9. For a product of 18 we could use 1 and 18, 2 and 9, or 3 and 6. Only 3 and 6 have a sum of 9. So we replace 9x with 3x + 6x and factor by grouping:

\[
x^2 + 9x + 18 = x^2 + 3x + 6x + 18 \\
= (x + 3)x + (x + 3)6 \quad \text{Factor out common factors.} \\
= (x + 3)(x + 6) \quad \text{Check by using FOIL.}
\]

**b)** We need to find two integers with a product of −24 and a sum of −2. For a product of 24 we have 1 and 24, 2 and 12, 3 and 8, or 4 and 6. To get a product of −24 and a sum of −2, we must use 4 and −6:

\[
x^2 - 2x - 24 = x^2 - 6x + 4x - 24 \\
= (x - 6)x + (x - 6)4 \quad \text{Factor out common factors.} \\
= (x - 6)(x + 4) \quad \text{Check by using FOIL.}
\]

The method shown in Example 1 can be shortened greatly. Once we discover that 3 and 6 have a product of 18 and a sum of 9, we can simply write

\[
x^2 + 9x + 18 = (x + 3)(x + 6).
\]
Once we discover that 4 and \(-6\) have a product of \(-24\) and a sum of \(-2\), we can simply write
\[x^2 - 2x - 24 = (x - 6)(x + 4).\]

In the next example we use this shortcut.

**Example 2**

**Factoring \(ax^2 + bx + c\) with \(a = 1\)**

Factor each quadratic polynomial.

a) \(x^2 + 4x + 3\)  
b) \(x^2 + 3x - 10\)  
c) \(a^2 - 5a + 6\)

**Solution**

a) Two integers with a product of 3 and a sum of 4 are 1 and 3:
\[x^2 + 4x + 3 = (x + 1)(x + 3)\]
Check by using FOIL.

b) Two integers with a product of \(-10\) and a sum of 3 are 5 and \(-2\):
\[x^2 + 3x - 10 = (x + 5)(x - 2)\]
Check by using FOIL.

c) Two integers with a product of 6 and a sum of \(-5\) are \(-3\) and \(-2\):
\[a^2 - 5a + 6 = (a - 3)(a - 2)\]
Check by using FOIL.

**Factoring Trinomials with Leading Coefficient Not 1**

If the leading coefficient of a quadratic trinomial is not 1, we can again use grouping to factor the trinomial. However, the procedure is slightly different.

Consider the trinomial \(2x^2 + 11x + 12\), for which \(a = 2\), \(b = 11\), and \(c = 12\). First find \(ac\), the product of the leading coefficient and the constant term. In this case \(ac = 2 \cdot 12 = 24\). Now find two integers with a product of 24 and a sum of 11. The pairs of integers with a product of 24 are 1 and 24, 2 and 12, 3 and 8, and 4 and 6. Only 3 and 8 have a product of 24 and a sum of 11. Now replace 11x by \(3x + 8x\) and factor by grouping:
\[2x^2 + 11x + 12 = 2x^2 + 3x + 8x + 12\]
\[= (2x + 3)x + (2x + 3)4\]
\[= (2x + 3)(x + 4)\]

This strategy for factoring a quadratic trinomial, known as the **ac method**, is summarized in the following box. The **ac method** works also when \(a = 1\).

**Strategy for Factoring \(ax^2 + bx + c\) by the **ac** Method**

To factor the trinomial \(ax^2 + bx + c\)
1. find two integers that have a product equal to \(ac\) and a sum equal to \(b\),
2. replace \(bx\) by two terms using the two new integers as coefficients,
3. then factor the resulting four-term polynomial by grouping.
EXAMPLE 3

Factoring \( ax^2 + bx + c \) with \( a \neq 1 \)

Factor each trinomial.

a) \( 2x^2 + 9x + 4 \)

b) \( 2x^2 + 5x - 12 \)

Solution

a) Because \( 2 \cdot 4 = 8 \), we need two numbers with a product of 8 and a sum of 9. The numbers are 1 and 8. Replace 9x by \( x + 8x \) and factor by grouping:

\[
2x^2 + 9x + 4 = 2x^2 + x + 8x + 4 \\
= (2x + 1)x + (2x + 1)4 \\
= (2x + 1)(x + 4) \quad \text{Check by FOIL.}
\]

Note that if you start with \( 2x^2 + 8x + x + 4 \), and factor by grouping, you get the same result.

b) Because \( 2(-12) = -24 \), we need two numbers with a product of \(-24\) and a sum of 5. The pairs of numbers with a product of 24 are 1 and 24, 2 and 12, 3 and 8, and 4 and 6. To get a product of \(-24\), one of the numbers must be negative and the other positive. To get a sum of positive 5, we need \(-3\) and 8:

\[
2x^2 + 5x - 12 = 2x^2 - 3x + 8x - 12 \\
= (2x - 3)x + (2x - 3)4 \\
= (2x - 3)(x + 4) \quad \text{Check by FOIL.}
\]

**Trial and Error**

After we have gained some experience at factoring by grouping, we can often find the factors without going through the steps of grouping. Consider the polynomial

\( 2x^2 - 7x + 6 \).

The factors of \( 2x^2 \) can only be \( 2x \) and \( x \). The factors of 6 could be 2 and 3 or 1 and 6. We can list all of the possibilities that give the correct first and last terms without putting in the signs:

\[
\begin{align*}
(2x & \quad 2)(x & \quad 3) \quad (2x & \quad 6)(x & \quad 1) \\
(2x & \quad 3)(x & \quad 2) \quad (2x & \quad 1)(x & \quad 6)
\end{align*}
\]

Before actually trying these out, we make an important observation. If \( (2x \quad 2) \) or \( (2x \quad 6) \) were one of the factors, then there would be a common factor 2 in the original trinomial, but there is not. *If the original trinomial has no common factor, there can be no common factor in either of its linear factors.* Since 6 is positive and the middle term is \(-7x\), both of the missing signs must be negative. So the only possibilities are \( (2x - 1)(x - 6) \) and \( (2x - 3)(x - 2) \). The middle term of the first product is \(-13x\), and the middle term of the second product is \(-7x\). So we have found the factors:

\[
2x^2 - 7x + 6 = (2x - 3)(x - 2)
\]

Even though there may be many possibilities in some factoring problems, often we find the correct factors without writing down every possibility. We can use a bit of guesswork in factoring trinomials. Try whichever possibility you think might work. *Check* it by multiplying. If it is not right, then try again. That is why this method is called **trial and error**.
**Example 4**

**Trial and error**

Factor each quadratic trinomial using trial and error.

**Solution**

**a)** Because $2x^2$ factors only as $2x \cdot x$ and $3$ factors only as $1 \cdot 3$, there are only two possible ways to factor this trinomial to get the correct first and last terms:

$$(2x + 1)(x - 3) \quad \text{and} \quad (2x + 3)(x - 1)$$

Because the last term of the trinomial is negative, one of the missing signs must be $+$, and the other must be $-$. Now we try the various possibilities until we get the correct middle term:

$$(2x + 1)(x - 3) = 2x^2 - 5x - 3$$
$$(2x + 3)(x - 1) = 2x^2 + x - 3$$
$$(2x - 1)(x + 3) = 2x^2 + 5x - 3$$

Since the last product has the correct middle term, the trinomial is factored as $2x^2 + 5x - 3 = (2x - 1)(x + 3)$.

**b)** There are four possible ways to factor $3x^2 - 11x + 6$:

$$(3x + 1)(x - 6) \quad (3x - 2)(x + 3)$$
$$(3x + 6)(x - 1) \quad (3x - 3)(x + 2)$$

Because the last term is positive and the middle term is negative, both signs must be negative. Now try possible factors until we get the correct middle term:

$$(3x - 1)(x - 6) = 3x^2 - 19x + 6$$
$$(3x - 2)(x - 3) = 3x^2 - 11x + 6$$

The trinomial is factored correctly as $3x^2 - 11x + 6 = (3x - 2)(x - 3)$.

**Higher Degrees and Variable Exponents**

It is not necessary always to use substitution to factor polynomials with higher degrees or variable exponents as we did in Section 5.6. In the next example we use trial and error to factor two polynomials of higher degree and one with variable exponents. Remember that if there is a common factor to all terms, factor it out first.

**Example 5**

**Higher-degree and variable exponent trinomials**

Factor each polynomial completely. Variables used as exponents represent positive integers.

**Solution**

**a)** To factor by trial and error, notice that $x^8 = x^4 \cdot x^4$. Now $15$ is $3 \cdot 5$ or $1 \cdot 15$. Using $1$ and $15$ will not give the required $-2$ for the coefficient of the middle term. So choose $3$ and $-5$ to get the $-2$ in the middle term:

$$x^8 - 2x^4 - 15 = (x^4 - 5)(x^4 + 3)$$

**b)** $-18y^7 + 21y^4 + 15y = -3y(6y^6 - 7y^3 - 5)$ Factor out the common factor $-3y$ first.

$$= -3y(2y^3 + 1)(3y^3 - 5)$$ Factor the trinomial by trial and error.

**c)** Notice that $2u^{2m} = 2u^m \cdot u^m$ and $3 = 3 \cdot 1$. Using trial and error, we get

$$2u^{2m} - 5u^m - 3 = (2u^m + 1)(u^m - 3)$$. 

\[\text{\blacktriangleright}  \]
5.7 Factoring $ax^2 + bx + c$

**WARM-UPS**

True or false? Answer true if the polynomial is factored correctly and false otherwise.

1. $x^2 + 9x + 18 = (x + 3)(x + 6)$
2. $y^2 + 2y - 35 = (y + 5)(y - 7)$
3. $x^2 + 4 = (x + 2)(x + 2)$
4. $x^2 - 5x - 6 = (x - 3)(x - 2)$
5. $x^2 - 4x - 12 = (x - 6)(x + 2)$
6. $x^2 + 15x + 36 = (x + 4)(x + 9)$
7. $3x^2 + 4x - 15 = (3x + 5)(x - 3)$
8. $4x^2 + 4x - 3 = (4x - 1)(x + 3)$
9. $4x^2 - 4x - 3 = (2x + 1)(2x - 3)$
10. $4x^2 + 8x + 3 = (2x + 1)(2x + 3)$

**EXERCISES**

**Reading and Writing** After reading this section, write out the answers to these questions. Use complete sentences.

1. How do we factor trinomials that have a leading coefficient of 1?

2. How do we factor trinomials in which the leading coefficient is not 1?

3. What is trial-and-error factoring?

4. What should you always first look for when factoring a polynomial?

**Factor each polynomial. See Examples 1 and 2.**

5. $x^2 + 4x + 3$
6. $y^2 + 5y + 6$
7. $a^2 + 15a + 50$
8. $t^2 + 11t + 24$
9. $y^2 - 5y - 14$
10. $x^2 - 3x - 18$
11. $x^2 - 6x + 8$
12. $y^2 - 13y + 30$
13. $a^2 - 12a + 27$
14. $x^2 - x - 30$
15. $a^2 + 7a - 30$
16. $w^2 + 29w - 30$

**Factor each polynomial using the ac method. See Example 3.**

17. $6w^2 + 5w + 1$
18. $4x^2 + 11x + 6$
19. $2x^2 - 5x - 3$
20. $2a^2 + 3a - 2$
21. $4x^2 + 16x + 15$
22. $6y^2 + 17y + 12$
23. $6x^2 - 5x + 1$
24. $6m^2 - m - 12$
25. $12y^2 + y - 1$
26. $12x^2 + 5x - 2$
27. $6a^2 + a - 5$
28. $30b^2 - b - 3$

**Factor each polynomial using trial and error. See Example 4.**

29. $2x^2 + 15x - 8$
30. $3a^2 + 20a + 12$
31. $3b^2 - 16b - 35$
32. $2y^2 - 17y + 21$
33. $6w^2 - 35w + 36$
34. $15x^2 - x - 6$
35. $4x^2 - 5x + 1$
36. $4x^2 + 7x + 3$
37. $5m^2 + 13m - 6$
38. $5t^2 - 9t - 2$
39. $6y^2 - 7y - 20$
40. $7u^2 + 11u - 6$

**Factor each polynomial completely. See Example 5. The variables used in exponents represent positive integers.**

41. $x^6 - 2x^3 - 35$
42. $x^4 + 7x^2 - 30$
43. $a^{20} - 20a^{10} + 100$
44. $b^{16} + 22b^8 + 121$
Factor each polynomial completely.

45. \(-12a^5 - 10a^3 - 2a\)

46. \(-4b^7 + 4b^4 + 3b\)

47. \(x^2a + 2xa - 15\)

48. \(y^{2b} + y^b - 20\)

49. \(x^2a - y^{2b}\)

50. \(w^{4m} - a^2\)

51. \(x^5 - x^4 - 6\)

52. \(m^{10} - 5m^5 - 6\)

53. \(x^{a+2} - x^a\)

54. \(y^{2a+1} - y\)

55. \(x^2a + 6xa + 9\)

56. \(x^{2a} - 2x^ayb + y^{2b}\)

57. \(4y^3z^6 + 5z^2y^2 - 6y\)

58. \(2u^6v^6 + 5u^4v^3 - 12u^2\)
Both products have an $x$-term. Of course, $(x + 1)(x - 1)$ has no $x$-term, but
\[(x + 1)(x - 1) = x^2 - 1.\]
Because none of these possibilities results in $x^2 + 1$, the polynomial $x^2 + 1$ is a
prime polynomial. Note that $x^2 + 1$ is a sum of two squares. A sum of two squares of
the form $a^2 + b^2$ is always a prime polynomial.

**Example 1**

Prime polynomials

Determine whether the polynomial $x^2 + 3x + 4$ is a prime polynomial.

**Solution**

To factor $x^2 + 3x + 4$, we must find two integers with a product of 4 and a sum of 3.
The only pairs of positive integers with a product of 4 are 1 and 4, and 2 and 2. Because
the product is positive 4, both numbers must be negative or both positive. Under
these conditions it is impossible to get a sum of positive 3. The polynomial is prime.

**Factoring Polynomials Completely**

So far, a typical polynomial has been a product of two factors, with possibly a common
factor removed first. However, it is possible that the factors can still be factored again. A polynomial in a single variable may have as many factors as its degree. We have factored a polynomial completely when all of the factors are prime polynomials.

**Example 2**

Factoring higher-degree polynomials completely

Factor $x^4 + x^2 - 2$ completely.

**Solution**

Two numbers with a product of $-2$ and a sum of 1 are 2 and $-1$:
\[
x^4 + x^2 - 2 = (x^2 + 2)(x^2 - 1) = (x^2 + 2)(x - 1)(x + 1) \quad \text{Difference of two squares}
\]
Since $x^2 + 2$, $x - 1$, and $x + 1$ are prime, the polynomial is factored completely.

In the next example we factor a sixth-degree polynomial.

**Example 3**

Factoring completely

Factor $3x^6 - 3$ completely.

**Solution**

To factor $3x^6 - 3$, we must first factor out the common factor 3 and then recognize that $x^6$ is a perfect square: $x^6 = (x^3)^2$:
\[
3x^6 - 3 = 3(x^6 - 1) = 3((x^3)^2 - 1) = 3(x^3 - 1)(x^3 + 1) = 3(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) \quad \text{Difference of two cubes and sum of two cubes}
\]
Since $x^2 + x + 1$ and $x^2 - x + 1$ are prime, the polynomial is factored completely.
In Example 3 we recognized $x^6 - 1$ as a difference of two squares. However, $x^6 - 1$ is also a difference of two cubes, and we can factor it using the rule for the difference of two cubes:

$$x^6 - 1 = (x^2)^3 - 1 = (x^2 - 1)(x^4 + x^2 + 1)$$

Now we can factor $x^2 - 1$, but it is difficult to see how to factor $x^4 + x^2 + 1$. (It is not prime.) Although $x^6$ can be thought of as a perfect square or a perfect cube, in this case thinking of it as a perfect square is better.

In the next example we use substitution to simplify the polynomial before factoring. This fourth-degree polynomial has four factors.

**Example 4**

Using substitution to simplify

Factor $(w^2 - 1)^2 - 11(w^2 - 1) + 24$ completely.

**Solution**

Let $a = w^2 - 1$ to simplify the polynomial:

$$(w^2 - 1)^2 - 11(w^2 - 1) + 24 = a^2 - 11a + 24$$

Replace $w^2 - 1$ by $a$.

$$= (a - 8)(a - 3)$$

Replace $a$ by $w^2 - 1$.

$$= (w^2 - 1 - 8)(w^2 - 1 - 3)$$

$$= (w^2 - 9)(w^2 - 4)$$

$$= (w + 3)(w - 3)(w + 2)(w - 2)$$

**Factoring Polynomials with Four Terms**

In Section 5.6 we rewrote a trinomial as a polynomial with four terms and then used factoring by grouping. Factoring by grouping can also be used on other types of polynomials with four terms.

**Example 5**

Polynomials with four terms

Use grouping to factor each polynomial completely.

a) $x^3 + x^2 + 4x + 4$  

b) $3x^3 - x^2 - 27x + 9$  

c) $ax - bw + bx - aw$

**Solution**

a) Note that the first two terms of $x^3 + x^2 + 4x + 4$ have a common factor of $x^2$, and the last two terms have a common factor of 4.

$$x^3 + x^2 + 4x + 4 = x^2(x + 1) + 4(x + 1)$$

Factor by grouping.

$$= (x^2 + 4)(x + 1)$$

Factor out $x + 1$.

Since $x^2 + 4$ is a sum of two squares, it is prime and the polynomial is factored completely.

b) We can factor $x^2$ out of the first two terms of $3x^3 - x^2 - 27x + 9$ and 9 or $-9$ from the last two terms. We choose $-9$ to get the factor $3x - 1$ in each case.

$$3x^3 - x^2 - 27x + 9 = x^2(3x - 1) - 9(3x - 1)$$

Factor by grouping.

$$= (x^2 - 9)(3x - 1)$$

Factor out $3x - 1$.

$$= (x - 3)(x + 3)(3x - 1)$$

Difference of two squares

This third-degree polynomial has three factors.
c) First rearrange the terms so that the first two and the last two have common factors:

\[ ax - bw + bx - aw = ax + bx - aw - bw \]
\[ = x(a + b) - w(a + b) \]
\[ = (x - w)(a + b) \]

**Summary**

A strategy for factoring polynomials is given in the following box.

**Strategy for Factoring Polynomials**

1. If there are any common factors, factor them out first.
2. When factoring a binomial, look for the special cases: difference of two squares, difference of two cubes, and sum of two cubes. Remember that a sum of two squares \(a^2 + b^2\) is prime.
3. When factoring a trinomial, check to see whether it is a perfect square trinomial.
4. When factoring a trinomial that is not a perfect square, use grouping or trial and error.
5. When factoring a polynomial of high degree, use substitution to get a polynomial of degree 2 or 3, or use trial and error.
6. If the polynomial has four terms, try factoring by grouping.

**Example 6**

Using the factoring strategy

Factor each polynomial completely.

a) \(3w^3 - 3w^2 - 18w\)  
b) \(10x^2 + 160\)  
c) \(16a^2b - 80ab + 100b\)  
d) \(aw + mw + az + mz\)

**Solution**

a) The greatest common factor (GCF) for the three terms is \(3w\):

\[ 3w^3 - 3w^2 - 18w = 3w(w^2 - w - 6) \]
\[ = 3w(w - 3)(w + 2) \]

b) The GCF in \(10x^2 + 160\) is 10:

\[ 10x^2 + 160 = 10(x^2 + 16) \]

Because \(x^2 + 16\) is prime, the polynomial is factored completely.

c) The GCF in \(16a^2b - 80ab + 100b\) is \(4b\):

\[ 16a^2b - 80ab + 100b = 4b(4a^2 - 20a + 25) \]
\[ = 4b(2a - 5)^2 \]

d) The polynomial has four terms, and we can factor it by grouping:

\[ aw + mw + az + mz = w(a + m) + z(a + m) \]
\[ = (w + z)(a + m) \]
**WARM-UPS**

**True or false? Explain your answer.**

1. $x^2 - 9 = (x - 3)^2$ for any value of $x$.
2. The polynomial $4x^2 + 12x + 9$ is a perfect square trinomial.
3. The sum of two squares $a^2 + b^2$ is prime.
4. The polynomial $x^4 - 16$ is factored completely as $(x^2 - 4)(x^2 + 4)$.
5. $y^3 - 27 = (y + 3)(y^2 + 3y - 9)$ for any value of $y$.
6. The polynomial $y^6 - 1$ is a difference of two squares.
7. The polynomial $2x^2 + 2x - 12$ is factored completely as $(2x - 4)(x + 3)$.
8. The polynomial $x^2 - 4x - 4$ is a prime polynomial.
9. The polynomial $a^6 - 1$ is the difference of two cubes.
10. The polynomial $x^2 + 3x - ax + 3a$ can be factored by grouping.

@ http://www.sosmath.com/algebra/quadraticeq/sobyfactor/sobyfactor.html

**5.8 EXERCISES**

**Reading and Writing** After reading this section, write out the answers to these questions. Use complete sentences.

1. What should you do first when factoring a polynomial?
2. If you are factoring a binomial, then what should you look for?
3. When factoring a trinomial what should you look for?
4. What should you look for when factoring a four-term polynomial?

**Determine whether each polynomial is a prime polynomial. See Example 1.**

5. $y^2 + 100$
6. $3x^2 + 27$
7. $-9w^2 - 9$
8. $25y^2 + 36$
9. $x^2 - 2x - 3$
10. $x^2 - 2x + 3$
11. $x^2 + 2x + 3$
12. $x^2 + 4x + 3$
13. $x^2 - 4x - 3$
14. $x^2 + 4x - 3$
15. $6x^2 + 3x - 4$
16. $4x^2 - 5x - 3$

**Factor each polynomial completely. See Examples 2–4.**

17. $a^4 - 10a^2 + 25$
18. $9y^4 + 12y^2 + 4$
19. $x^4 - 6x^2 + 8$
20. $x^6 + 2x^3 - 3$
21. $(3x - 5)^2 - 1$
22. $(2x + 1)^2 - 4$
23. $2y^6 - 128$
24. $6 - 6y^6$
25. $32a^4 - 18$
26. $2a^4 - 32$
27. $x^4 - (x - 6)^2$
28. $y^4 - (2y + 1)^2$
29. $(m + 2)^2 + 2(m + 2) - 3$
30. $(2w - 3)^2 - 2(2w - 3) - 15$
31. $3(y^2 - 1)^2 + 11(y - 1) - 20$
32. $2(w + 2)^2 + 5(w + 2) - 3$
33. $(y^2 - 3)^2 - 4(y^2 - 3) - 12$
34. $(m^2 - 8)^2 - 4(m^2 - 8) - 32$

Use grouping to factor each polynomial completely. See Example 5.

35. $ax + ay + bx + by$
36. $7x^2 + 7z + kx + kz$
37. $x^3 + x^2 - 9x - 9$
38. $x^3 + x^2 - 25x - 25$
39. $aw - bw - 3a + 3b$
40. $wx - wy + 2x - 2y$
41. $a^4 + 3a^3 + 27a + 81$
42. $ac - be - 5a + 5b$
43. \( y^4 - 5y^3 + 8y - 40 \)
44. \( x^3 + ax - 3a - 3x^2 \)
45. \( ady + d - w - awy \)
46. \( xy + by + ax + ab \)
47. \( x^2y - a + ax^2 - y \)
48. \( a^2d - b^2c + a^2c - b^2d \)
49. \( y^3 + b + by^3 + y \)
50. \( ab + mw^2 + bm + aw^2 \)

Use the factoring strategy to factor each polynomial completely. See Example 6.
51. \( 9x^2 - 24x + 16 \)
52. \(-3x^2 + 18x + 48 \)
53. \( 12x^2 - 13x + 3 \)
54. \( 2x^2 - 3x - 6 \)
55. \( 3a^4 + 81a \)
56. \(-a^3 + 25a \)
57. \( 32 + 2x^3 \)
58. \( x^3 + 4x^2 + 4x \)
59. \( 6x^2 - 5x + 12 \)
60. \( x^4 + 2x^3 - x - 2 \)
61. \( (x + y)^2 - 1 \)
62. \( x^3 + 9x \)
63. \( a^3b - ab^3 \)
64. \( 2m^3 - 250 \)
65. \( x^4 + 2x^3 - 8x - 16 \)
66. \( (x + 5)^2 - 4 \)
67. \( m^2n + 2mn^2 + n^3 \)
68. \( a^2b - 6ab + 9b \)
69. \( 2m + wn + 2n + wm \)
70. \( aw - 5b + bw - 5a \)
71. \( 4w^2 + 4w - 3 \)
72. \( 4w^2 + 8w - 63 \)
73. \( t^4 + 4t^2 - 21 \)
74. \( m^4 + 5m^2 + 4 \)
75. \(-a^3 - 7a^2 + 30a \)
76. \( 2y^4 + 3y^3 - 20y^2 \)
77. \( (y + 5)^2 - 2(y + 5) - 3 \)
78. \( (2t - 1)^2 + 7(2t - 1) + 10 \)
79. \(-2w^4 + 1250 \)
80. \( 5a^3 - 5a \)
81. \( 8a^3 + 8a \)
82. \( awx + ax \)
83. \((w + 5)^2 - 9 \)
84. \( (a - 6)^2 - 1 \)
85. \( 4aw^2 - 12aw + 9a \)
86. \( 9an^3 + 15an^2 - 14an \)
87. \( x^2 - 6x + 9 \)
88. \( x^3 + 12x^2 + 36x \)
89. \( 3x^4 - 75x^2 \)
90. \( 3x^2 + 9x + 12 \)
91. \( m^3n - n \)
92. \( m^4 + 16m^2 \)
93. \( 12x^2 + 2x - 30 \)
94. \( 90x^2 + 3x - 60 \)
95. \( 2a^3 - 32 \)
96. \( 12x^2 - 28x + 15 \)

Factor completely. Assume variables used as exponents represent positive integers.
97. \( a^{3m} - 1 \)
98. \( x^{6a} + 8 \)
99. \( a^{3w} - b^{3w} \)
100. \( x^{2n} - 9 \)
101. \( t^{3m} - 16 \)
102. \( a^{3w+2} + a^2 \)
103. \( a^{2n+1} - 2a^{n+1} - 15a \)
104. \( x^{3m} + x^{2m} - 6x^m \)
105. \( a^{2n} - 3a^n + a^n b - 3b \)
106. \( x^{m_3} + 6z + x^{m+1} + 5x \)

GETTING MORE INVOLVED

107. **Cooperative learning.** Write down 10 trinomials of the form \( ax^2 + bx + c \) “at random” using integers for \( a, b, \) and \( c. \) What percent of your 10 trinomials are prime? Would you say that prime trinomials are the exception or the rule? Compare your results with those of your classmates.

108. **Writing.** The polynomial
\[ x^5 + x^4 - 9x^3 - 13x^2 + 8x + 12 \]
is a product of five factors of the form \( x \pm n, \) where \( n \) is a natural number smaller than 4. Factor this polynomial completely and explain your procedure.
5.9 SOLVING EQUATIONS BY FACTORING

The techniques of factoring can be used to solve equations involving polynomials that cannot be solved by the other methods that you have learned. After you learn to solve equations by factoring, we will use this technique to solve some new applied problems in this section and in Chapters 6 and 8.

The Zero Factor Property

The equation \( ab = 0 \) indicates that the product of two unknown numbers is 0. But the product of two real numbers is zero only when one or the other of the numbers is 0. So even though we do not know exactly the values of \( a \) and \( b \) from \( ab = 0 \), we do know that \( a = 0 \) or \( b = 0 \). This idea is called the zero factor property.

**Zero Factor Property**

The equation \( ab = 0 \) is equivalent to the compound equation

\[
\begin{align*}
    a = 0 & \quad \text{or} & \quad b = 0.
\end{align*}
\]

The next example shows how to use the zero factor property to solve an equation in one variable.

**Example 1**

Using the zero factor property

Solve \( x^2 + x - 12 = 0 \).

**Solution**

We factor the left-hand side of the equation to get a product of two factors that are equal to 0. Then we write an equivalent equation using the zero factor property.

\[
\begin{align*}
    x^2 + x - 12 &= 0 \\
    (x + 4)(x - 3) &= 0 \quad \text{Factor the left-hand side.} \\
    x + 4 &= 0 \quad \text{or} \quad x - 3 &= 0 \quad \text{Zero factor property} \\
    x &= -4 \quad \text{or} \quad x &= 3 \quad \text{Solve each part of the compound equation.}
\end{align*}
\]

Check that both \(-4\) and \(3\) satisfy \( x^2 + x - 12 = 0 \). If \( x = -4 \), we get

\[
(-4)^2 + (-4) - 12 = 16 - 4 - 12 = 0.
\]

If \( x = 3 \), we get

\[
(3)^2 + 3 - 12 = 9 + 3 - 12 = 0.
\]

So the solution set is \( \{ -4, 3 \} \).

The zero factor property is used only in solving polynomial equations that have zero on one side and a polynomial that can be factored on the other side. The polynomials that we factored most often were the quadratic polynomials. The equations that we will solve most often using the zero factor property will be quadratic equations.

**Quadratic Equation**

If \( a, \ b, \) and \( c \) are real numbers, with \( a \neq 0 \), then the equation

\[
ax^2 + bx + c = 0
\]

is called a quadratic equation.
In Chapter 8 we will study quadratic equations further and solve quadratic equations that cannot be solved by factoring. Keep the following strategy in mind when solving equations by factoring.

**Strategy for Solving Equations by Factoring**

1. Write the equation with 0 on the right-hand side.
2. Factor the left-hand side.
3. Use the zero factor property to get simpler equations. (Set each factor equal to 0.)
4. Solve the simpler equations.
5. Check the answers in the original equation.

### Example 2

Solving a quadratic equation by factoring

Solve each equation.

a) \[10x^2 = 5x\]  
b) \[3x - 6x^2 = -9\]

**Solution**

a) Use the steps in the strategy for solving equations by factoring:

\[
\begin{align*}
10x^2 &= 5x & \text{Original equation} \\
10x^2 - 5x &= 0 & \text{Rewrite with zero on the right-hand side.} \\
5x(2x - 1) &= 0 & \text{Factor the left-hand side.} \\
5x &= 0 & \text{Zero factor property} \\
2x - 1 &= 0 & \\
5x &= 0 \quad \text{or} \quad 2x - 1 = 0 & \text{Solve for } x. \\
x &= 0 & \quad x = \frac{1}{2} \\
\end{align*}
\]

The solution set is \(\{0, \frac{1}{2}\}\). Check each solution in the original equation.
b) First rewrite the equation with 0 on the right-hand side and the left-hand side in order of descending exponents:

\[
3x - 6x^2 = -9 \\
-6x^2 + 3x + 9 = 0 \\
2x^2 - x - 3 = 0 \\
(2x - 3)(x + 1) = 0 \\
2x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \\
\]

Zero factor property

\[
x = \frac{3}{2} \quad \text{or} \quad x = -1 \\
\]

Solve for \(x\).

The solution set is \([-1, \frac{3}{2}]\). Check each solution in the original equation.

**CAUTION** If we divide each side of \(10x^2 = 5x\) by \(5x\), we get \(2x = 1\), or \(x = \frac{1}{2}\). We do not get \(x = 0\). By dividing by \(5x\) we have lost one of the factors and one of the solutions.

In the next example there are more than two factors, but we can still write an equivalent equation by setting each factor equal to 0.

**Example 3**

**Solving a cubic equation by factoring**

Solve \(2x^3 - 3x^2 - 8x + 12 = 0\).

**Solution**

First notice that the first two terms have the common factor \(x^2\) and the last two terms have the common factor \(-4\).

\[
x^2(2x - 3) - 4(2x - 3) = 0 \\
(2x - 3)(x^2 - 4) = 0 \\
(2x - 3)(x - 2)(x + 2) = 0 \\
\]

Factor by grouping. Factor out \(2x - 3\). Factor completely.

\[
x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad 2x - 3 = 0 \\
\]

Set each factor equal to 0.

\[
x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = \frac{3}{2} \\
\]

The solution set is \([-2, \frac{3}{2}, 2]\). Check each solution in the original equation.

**Example 4**

**Solving an absolute value equation by factoring**

Solve \(|x^2 - 2x - 16| = 8\).

**Solution**

First write an equivalent compound equation without absolute value:

\[
x^2 - 2x - 16 = 8 \quad \text{or} \quad x^2 - 2x - 16 = -8 \\
x^2 - 2x - 24 = 0 \quad \text{or} \quad x^2 - 2x - 8 = 0 \\
(x - 6)(x + 4) = 0 \quad \text{or} \quad (x - 4)(x + 2) = 0 \\
x - 6 = 0 \quad \text{or} \quad x + 4 = 0 \quad \text{or} \quad x - 4 = 0 \quad \text{or} \quad x + 2 = 0 \\
x = 6 \quad \text{or} \quad x = -4 \quad \text{or} \quad x = 4 \quad \text{or} \quad x = -2 \\
\]

The solution set is \([-2, -4, 4, 6]\). Check each solution.
Applications

Many applied problems can be solved by using equations such as those we have been solving.

**Example 5**

Area of a room

Ronald’s living room is 2 feet longer than it is wide, and its area is 168 square feet. What are the dimensions of the room?

**Solution**

Let $x$ be the width and $x + 2$ be the length. See Figure 5.1. Because the area of a rectangle is the length times the width, we can write the equation

$$x(x + 2) = 168.$$  

We solve the equation by factoring:

$$x^2 + 2x - 168 = 0$$

$$(x - 12)(x + 14) = 0$$

$$x = 12 \quad \text{or} \quad x = -14$$  

Because the width of a room is a positive number, we disregard the solution $x = -14$. We use $x = 12$ and get a width of 12 feet and a length of 14 feet. Check this answer by multiplying 12 and 14 to get 168.

Applications involving quadratic equations often require a theorem called the **Pythagorean theorem**. This theorem states that *in any right triangle the sum of the squares of the lengths of the legs is equal to the length of the hypotenuse squared.*

The Pythagorean Theorem

The triangle shown is a right triangle if and only if

$$a^2 + b^2 = c^2.$$
Using the Pythagorean theorem

Shirley used 14 meters of fencing to enclose a rectangular region. To be sure that the region was a rectangle, she measured the diagonals and found that they were 5 meters each. (If the opposite sides of a quadrilateral are equal and the diagonals are equal, then the quadrilateral is a rectangle.) What are the length and width of the rectangle?

**Solution**

The perimeter of a rectangle is twice the length plus twice the width, \( P = 2L + 2W \). Because the perimeter is 14 meters, the sum of one length and one width is 7 meters. If we let \( x \) represent the width, then \( 7 - x \) is the length. We use the Pythagorean theorem to get a relationship among the length, width, and diagonal. See Figure 5.2.

\[
\begin{align*}
    x^2 + (7 - x)^2 &= 5^2 & \text{Pythagorean theorem} \\
    x^2 + 49 - 14x + x^2 &= 25 & \text{Simplify.} \\
    2x^2 - 14x + 24 &= 0 & \text{Simplify.} \\
    x^2 - 7x + 12 &= 0 & \text{Divide each side by 2.} \\
    (x - 3)(x - 4) &= 0 & \text{Factor the left-hand side.} \\
    x - 3 &= 0 & \text{or} & & x - 4 &= 0 & \text{Zero factor property} \\
    x &= 3 & \text{or} & & x &= 4 \\
    7 - x &= 4 & \text{or} & & 7 - x &= 3
\end{align*}
\]

Solving the equation gives two possible rectangles: a 3 by 4 rectangle or a 4 by 3 rectangle. However, those are identical rectangles. The rectangle is 3 meters by 4 meters.

True or false? Explain your answer.

1. The equation \((x - 1)(x + 3) = 12\) is equivalent to \(x - 1 = 3\) or \(x + 3 = 4\).

2. Equations solved by factoring may have two solutions.

3. The equation \(c \cdot d = 0\) is equivalent to \(c = 0\) or \(d = 0\).

4. The equation \(|x^2 + 4| = 5\) is equivalent to the compound equation \(x^2 + 4 = 5\) or \(x^2 - 4 = 5\).

5. The solution set to the equation \((2x - 1)(3x + 4) = 0\) is \(\left\{ \frac{1}{2}, -\frac{4}{3} \right\}\).

6. The Pythagorean theorem states that the sum of the squares of any two sides of any triangle is equal to the square of the third side.

7. If the perimeter of a rectangular room is 38 feet, then the sum of the length and width is 19 feet.

8. Two numbers that have a sum of 8 can be represented by \(x\) and \(8 - x\).

9. The solution set to the equation \(x(x - 1)(x - 2) = 0\) is \(\{1, 2\}\).

10. The solution set to the equation \(3(x + 2)(x - 5) = 0\) is \(\{3, -2, 5\}\).
5.9 Exercises

Reading and Writing. After reading this section, write out the answers to these questions. Use complete sentences.

1. What is the zero factor property?

2. What is a quadratic equation?

3. Where is the hypotenuse in a right triangle?

4. Where are the legs in a right triangle?

5. What is the Pythagorean theorem?

6. Where is the diagonal of a rectangle?

Solve each equation. See Examples 1–3.

7. \((x - 5)(x + 4) = 0\)
8. \((a - 6)(a + 5) = 0\)
9. \((2x - 5)(3x + 4) = 0\)

10. \((3k + 8)(4k - 3) = 0\)
11. \(w^2 + 5w - 14 = 0\)
12. \(t^2 - 6t - 27 = 0\)
13. \(m^2 - 7m = 0\)
14. \(h^2 - 5h = 0\)
15. \(a^2 - a = 20\)
16. \(p^2 - p = 42\)
17. \(3x^2 - 3x - 36 = 0\)
18. \(-2x^2 - 16x - 24 = 0\)
19. \(z^2 + \frac{3}{2}z = 10\)
20. \(m^2 + \frac{11}{3}m = -2\)
21. \(x^3 - 4x = 0\)
22. \(16x - x^3 = 0\)
23. \(w^3 + 4w^2 - 25w - 100 = 0\)
24. \(a^3 + 2a^2 - 16a - 32 = 0\)
25. \(n^3 - 2n^2 - n + 2 = 0\)
26. \(w^3 - w^2 - 25w + 25 = 0\)

Solve each equation. See Example 4.

27. \(|x^2 - 5| = 4\)
28. \(|x^2 - 17| = 8\)
29. \(|x^2 + 2x - 36| = 12\)
30. \(|x^2 + 2x - 19| = 16\)
31. \(|x^2 + 4x + 2| = 2\)
32. \(|x^2 + 8x + 8| = 8\)
33. \(|x^2 + 6x + 1| = 8\)
34. \(|x^2 - x - 21| = 9\)
35. \(2x^2 - x = 6\)
36. \(3x^2 + 14x = 5\)
37. \(|x^2 + 5x| = 6\)
38. \(|x^2 + 6x - 4| = 12\)
39. \(x^2 + 5x = 6\)
40. \(x + 5x = 6\)
41. \((x + 2)(x + 1) = 12\)
42. \((x + 2)(x + 3) = 20\)
43. \(y^3 + 9y^2 + 20y = 0\)
44. \(m^3 - 2m^2 - 3m = 0\)
45. \(5a^3 = 45a\)
46. \(5x^3 = 125x\)
47. \((2x - 1)(x^2 - 9) = 0\)
48. \((x - 1)(x + 3)(x - 9) = 0\)
49. \(4x^2 - 12x + 9 = 0\)
50. \(16x^2 + 8x + 1 = 0\)

Solve each equation for y. Assume a and b are positive numbers.

51. \(y^2 + by = 0\)
52. \(y^2 + ay + by + ab = 0\)
53. \(a^2y^2 - b^2 = 0\)
54. \(9y^2 + 6ay + a^2 = 0\)
55. \(4y^2 + 4by + b^2 = 0\)
56. \(y^2 - b^2 = 0\)
57. \(ay^2 + 3y - ay = 3\)
58. \(a^2y^2 + 2aby + b^2 = 0\)

Solve each problem. See Examples 5 and 6.

59. Color print. The length of a new “super size” color print is 2 inches more than the width. If the area is 24 square inches, what are the length and width?
60. **Tennis court dimensions.** In singles competition, each player plays on a rectangular area of 117 square yards. Given that the length of that area is 4 yards greater than its width, find the length and width.

61. **Missing numbers.** The sum of two numbers is 13 and their product is 36. Find the numbers.

62. **More missing numbers.** The sum of two numbers is 6.5, and their product is 9. Find the numbers.

63. **Bodyboarding.** The Seamas Channel pro bodyboard shown in the figure has a length that is 21 inches greater than its width. Any rider weighing up to 200 pounds can use it because its surface area is 946 square inches. Find the length and width.

64. **New dimensions in gardening.** Mary Gold has a rectangular flower bed that measures 4 feet by 6 feet. If she wants to increase the length and width by the same amount to have a flower bed of 48 square feet, then what will be the new dimensions?

65. **Shooting arrows.** An archer shoots an arrow straight upward at 64 feet per second. The height of the arrow \( h(t) \) (in feet) at time \( t \) seconds is given by the function \( h(t) = -16t^2 + 64t. \)

   a) Use the accompanying graph to estimate the amount of time that the arrow is in the air.

   b) Algebraically find the amount of time that the arrow is in the air.

   c) Use the accompanying graph to estimate the maximum height reached by the arrow.

   d) At what time does the arrow reach its maximum height?

66. **Time until impact.** If an object is dropped from a height of \( s_0 \) feet, then its altitude after \( t \) seconds is given by the formula \( S = -16t^2 + s_0. \) If a pack of emergency supplies is dropped from an airplane at a height of 1600 feet, then how long does it take for it to reach the ground?

67. **Yolanda’s closet.** The length of Yolanda’s closet is 2 feet longer than twice its width. If the diagonal measures 13 feet, then what are the length and width?

68. **Ski jump.** The base of a ski ramp forms a right triangle. One leg of the triangle is 2 meters longer than the other. If the hypotenuse is 10 meters, then what are the lengths of the legs?

69. **Trimming a gate.** A total of 34 feet of \( 1 \times 4 \) lumber is used around the perimeter of the gate shown in the figure on the next page. If the diagonal brace is 13 feet long, then what are the length and width of the gate?
70. **Perimeter of a rectangle.** The perimeter of a rectangle is 28 inches, and the diagonal measures 10 inches. What are the length and width of the rectangle?

71. **Consecutive integers.** The sum of the squares of two consecutive integers is 25. Find the integers.

72. **Pete’s garden.** Each row in Pete’s garden is 3 feet wide. If the rows run north and south, he can have two more rows than if they run east and west. If the area of Pete’s garden is 135 square feet, then what are the length and width?

73. **House plans.** In the plans for their dream house the Baileys have a master bedroom that is 240 square feet in area. If they increase the width by 3 feet, they must decrease the length by 4 feet to keep the original area. What are the original dimensions of the bedroom?

---

**Magic Tricks**

Jim and Sadar are talking one day after class.

**Sadar:** Jim, I have a trick for you. Think of a number between 1 and 10. I will ask you to do some things to this number. Then at the end tell me your result, and I will tell you your number.

**Jim:** Oh, yeah you probably rig it so the result is my number.

**Sadar:** Good, now write it down, and don’t let me see your paper. Now add x. Got that? Now multiply everything by 2.

**Jim:** Hey, I didn’t know you were going to make me think! This is algebra!

**Sadar:** I know, now just do it. Okay, now square the polynomial. Got that? Now subtract 4x².

**Jim:** How did you know I had a 4x²? I told you this was rigged!

**Sadar:** Of course it’s rigged, or it wouldn’t work. Do you want to finish or not?

**Jim:** Yeah, I guess so. Go ahead, what do I do next?

**Sadar:** Divide by 4. Okay, now subtract the x-term.

**Jim:** Just any old x-term? Got any particular coefficient in mind?

**Sadar:** Got that? Now multiply everything by 2. I know, now just do it. Okay, now square the polynomial. This is algebra!

**Jim:** Hey, I can’t do that! How did you do that?

**Sadar:** Aha, then the number you chose at the beginning was 5, and the coefficient was 10!

**Jim:** Hey, you’re right! How did you do that?

In your group, follow Sadar’s instructions and determine why she knew Jim’s number. Make up another set of instructions to use as a magic trick. Be sure to use variables and some of the exponent rules or rules for multiplying polynomials that you learned in this chapter. Exchange instructions with another group and see whether you can figure out how their trick works.

---

**GETTING MORE INVOLVED**

75. **Writing.** If you divide each side of \(x^2 = x\) by \(x\), you get \(x = 1\). If you subtract \(x\) from each side of \(x^2 = x\), you get \(x^2 - x = 0\), which has two solutions. Which method is correct? Explain.

76. **Cooperative learning.** Work with a group to examine the following solution to \(x^2 - 2x = -1\):

\[
\begin{align*}
x(x - 2) &= -1 \\
x &= -1 \quad \text{or} \quad x - 2 = -1 \\
x &= -1 \quad \text{or} \quad x = 1
\end{align*}
\]

Is this method correct? Explain.

77. **Cooperative learning.** Work with a group to examine the following steps in the solution to \(5x^2 - 5 = 0\):

\[
\begin{align*}
5(x^2 - 1) &= 0 \\
5(x - 1)(x + 1) &= 0 \\
x - 1 &= 0 \quad \text{or} \quad x + 1 &= 0 \\
x &= 1 \quad \text{or} \quad x &= -1
\end{align*}
\]

What happened to the 5? Explain.

---

**Collaborative Activities**

**Topic:** Two students per group

**Grouping:** Practice with exponent rules, multiplying polynomials

**Sadar:** Now stop teasing me. I know you only have one x-term left, so subtract it.

**Jim:** Ha, ha, I could give you a hint about the coefficient, but that wouldn’t be fair, would it?

**Sadar:** Well you could, and then I could tell you your number, or you could just tell me the number you have left after subtracting.

**Jim:** Okay, the number I had left at the end was 25. Let’s see if you can tell me what the coefficient of the x-term I subtracted is.

**Sadar:** Aha, then the number you chose at the beginning was 5, and the coefficient was 10!

**Jim:** Hey, you’re right! How did you do that?
Definitions

Definition of negative integral exponents
If \( a \) is a nonzero real number and \( n \) is a positive integer, then
\[
a^{-n} = \frac{1}{a^n}.
\]

Definition of zero exponent
If \( a \) is any nonzero real number, then \( a^0 = 1 \). The expression \( 0^0 \) is undefined.

Rules of Exponents

If \( a \) and \( b \) are nonzero real numbers and \( m \) and \( n \) are integers, then the following rules hold.

Negative exponent rules
\[
a^{-n} = \left(\frac{1}{a}\right)^n, \quad a^{-1} = \frac{1}{a}, \quad \text{and} \quad \frac{1}{a^{-n}} = a^n
\]

Product rule
\[
a^m \cdot a^n = a^{m+n}
\]

Quotient rule
\[
\frac{a^m}{a^n} = a^{m-n}
\]

Power of a power rule
\[
(a^m)^n = a^{mn}
\]

Power of a product rule
\[
(ab)^n = a^n b^n
\]

Power of a quotient rule
\[
\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}
\]

Scientific Notation

Converting from scientific notation
1. Determine the number of places to move the decimal point by examining the exponent on the 10.
2. Move to the right for a positive exponent and to the left for a negative exponent.

Examples

<table>
<thead>
<tr>
<th>Definition</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{-3} = \frac{1}{2^3} = 8</td>
<td>3^0 = 1</td>
</tr>
</tbody>
</table>

| Negative exponent rules | 5^{-1} = \frac{1}{5^1} = \frac{1}{5} = 5^3 |
| Product rule | (2\cdot3)^{-2} = \left(\frac{3}{2}\right)^2 |
| Quotient rule | \frac{x^8}{x^5} = x^3 \cdot \frac{5^4}{5^3} = 5^1 |
| Power of a power rule | (5^2)^3 = 5^6 |
| Power of a product rule | (2x)^3 = 8x^3 |
| Power of a quotient rule | \left(\frac{x}{3}\right)^2 = \frac{x^2}{9} |

| Converting from scientific notation | 4 \times 10^3 = 4000 |
| Scientific Notation Example | 3 \times 10^{-4} = 0.0003 |
### Converting to Scientific Notation

1. Count the number of places \( (n) \) that the decimal point must be moved so that it will follow the first nonzero digit of the number.
2. If the original number was larger than 10, use \( 10^n \).
3. If the original number was smaller than 1, use \( 10^{-n} \).

\[
\begin{align*}
67,000 &= 6.7 \times 10^4 \\
0.009 &= 9 \times 10^{-3}
\end{align*}
\]

### Polynomials

**Term of a polynomial**
The product of a number (coefficient) and one or more variables raised to whole number powers.

**Polynomial**
A single term or a finite sum of terms.

**Adding or subtracting polynomials**
Add or subtract the like terms.

\[
(x + 3) + (x - 7) = 2x - 4 \\
(x^2 - 2x) - (3x^2 - x) = -2x^2 - x
\]

**Multiplying two polynomials**
Multiply each term of the first polynomial by each term of the second polynomial, then combine like terms.

\[
(x^2 + 2x + 3)(x + 1) \\
= (x^2 + 2x + 3)x + (x^2 + 2x + 3)1 \\
= x^3 + 2x^2 + 3x + x^2 + 2x + 3 \\
= x^3 + 3x^2 + 5x + 3
\]

**Dividing polynomials**
Ordinary division or long division

\[
\begin{align*}
\text{dividend} &= \text{(quotient)}(\text{divisor}) + \text{remainder} \\
\text{dividend} \div \text{divisor} &= \text{quotient} + \frac{\text{remainder}}{\text{divisor}} \\
\frac{x - 7}{x^2 + 2x - 7x - 14} &= \frac{x - 7}{x^2 + 2x - 7x - 14} \\
&= \frac{x - 7}{x^2 + 2x - 7x - 14} \\
&= \frac{x - 7}{x^2 + 2x - 7x - 14} \\
&= \frac{x - 7}{x^2 + 2x - 7x - 14} \\
&= \frac{x - 7}{x^2 + 2x - 7x - 14}
\end{align*}
\]

**Synthetic division**
A condensed version of long division, used only for dividing by a polynomial of the form \( x - c \).

If the remainder is 0, then the dividend factors as

\[
\text{dividend} = \text{(quotient)}(\text{divisor}).
\]

\[
\begin{array}{c|ccc}
-2 & 1 & -5 & -14 \\
\hline
& -2 & 14 & \\
1 & -7 & 0
\end{array}
\]

\[
x^2 - 5x - 14 = (x - 7)(x + 2)
\]

### Shortcuts for Multiplying Two Binomials

**FOIL**
The product of two binomials can be found quickly by multiplying their First, Outer, Inner, and Last terms.

\[
(x + 2)(x + 3) = x^2 + 5x + 6
\]

**Square of a sum**
\[
(a + b)^2 = a^2 + 2ab + b^2
\]

**Square of a difference**
\[
(a - b)^2 = a^2 - 2ab + b^2
\]

**Product of a sum and a difference**
\[
(a + b)(a - b) = a^2 - b^2
\]

\[
(x + 3)(x - 3) = x^2 - 9
\]
### Factoring

<table>
<thead>
<tr>
<th>Factoring a polynomial</th>
<th>Write a polynomial as a product of two or more polynomials. A polynomial is factored completely if it is a product of prime polynomials.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common factors</td>
<td>Factor out the greatest common factor (GCF).</td>
</tr>
<tr>
<td>(a^2 - b^2 = (a + b)(a - b))</td>
<td>(The sum of two squares (a^2 + b^2) is prime.)</td>
</tr>
<tr>
<td>Perfect square trinomials</td>
<td>(a^2 + 2ab + b^2 = (a + b)^2) (a^2 - 2ab + b^2 = (a - b)^2)</td>
</tr>
<tr>
<td>Difference of two cubes</td>
<td>(a^3 - b^3 = (a - b)(a^2 + ab + b^2))</td>
</tr>
<tr>
<td>Sum of two cubes</td>
<td>(a^3 + b^3 = (a + b)(a^2 - ab + b^2))</td>
</tr>
<tr>
<td>Grouping</td>
<td>Factor out common factors from groups of terms.</td>
</tr>
</tbody>
</table>

#### Factoring \(ax^2 + bx + c\)

By the \(ac\) method:
1. Find two numbers that have a product equal to \(ac\) and a sum equal to \(b\).
2. Replace \(bx\) by two terms using the two new numbers as coefficients.
3. Factor the resulting four-term polynomial by grouping.

By trial and error:
Try possibilities by considering factors of the first term and factors of the last term. Check them by FOIL.

#### Substitution

Use substitution on higher-degree polynomials to reduce the degree to 2 or 3.

#### Solving Equations by Factoring

<table>
<thead>
<tr>
<th>Strategy</th>
<th>1. Write the equation with 0 on the right-hand side.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2. Factor the left-hand side.</td>
</tr>
<tr>
<td></td>
<td>3. Set each factor equal to 0.</td>
</tr>
<tr>
<td></td>
<td>4. Solve the simpler equations.</td>
</tr>
<tr>
<td></td>
<td>5. Check the answers in the original equation.</td>
</tr>
</tbody>
</table>

#### Examples

<table>
<thead>
<tr>
<th>Example</th>
<th>(3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2x^3 - 6x = 2x(x^2 - 3))</td>
<td>(m^2 - 25 = (m + 5)(m - 5))</td>
</tr>
<tr>
<td>(m^2 + 25) is prime.</td>
<td>(m^2 - 25 = (m + 5)(m - 5))</td>
</tr>
<tr>
<td>(x^2 + 10x + 25 = (x + 5)^2)</td>
<td>(x^2 - 6x + 9 = (x - 3)^2)</td>
</tr>
<tr>
<td>(x^3 - 8 = (x - 2)(x^2 + 2x + 4))</td>
<td>(x^3 + 27 = (x + 3)(x^2 - 3x + 9))</td>
</tr>
<tr>
<td>(3x + 3w + bw = 3(x + w) + b(x + w) = (3 + b)(x + w))</td>
<td>(12x^2 + 19x - 18 = (3x - 2)(4x + 9))</td>
</tr>
<tr>
<td>(x^4 - 3x^2 - 18)</td>
<td>(a^2 - 3a - 18)</td>
</tr>
<tr>
<td>Let (a = x^2).</td>
<td>(x^2 - 3x - 18 = 0)</td>
</tr>
<tr>
<td>((x - 6)(x + 3) = 0)</td>
<td>(x - 6 = 0) or (x + 3 = 0)</td>
</tr>
<tr>
<td>(x = 6) or (x = -3)</td>
<td>(6^2 - 3(6) - 18 = 0)</td>
</tr>
<tr>
<td>((-3)^2 - 3(-3) - 18 = 0)</td>
<td>(6^2 - 3(6) - 18 = 0)</td>
</tr>
</tbody>
</table>
For each mathematical term, choose the correct meaning.

1. polynomial
   a. four or more terms
   b. many numbers
   c. a sum of four or more numbers
   d. a single term or a finite sum of terms

2. degree of a polynomial
   a. the number of terms in a polynomial
   b. the highest degree of any of the terms of a polynomial
   c. the value of a polynomial when \( x = 0 \)
   d. the largest coefficient of any of the terms of a polynomial

3. leading coefficient
   a. the first coefficient
   b. the largest coefficient
   c. the coefficient of the first term when a polynomial is written with decreasing exponents
   d. the most important coefficient

4. monomial
   a. a single polynomial
   b. one number
   c. an equation that has only one solution
   d. a polynomial that has one term

5. FOIL
   a. a method for adding polynomials
   b. first, outer, inner, last
   c. an equation with no solution
   d. a polynomial with five terms

6. binomial
   a. a polynomial with two terms
   b. any two numbers
   c. the two coordinates in an ordered pair
   d. an equation with two variables

7. scientific notation
   a. the notation of rational exponents
   b. the notation of algebra
   c. a notation for expressing large or small numbers with powers of 10
   d. radical notation

8. trinomial
   a. a polynomial with three terms
   b. an ordered triple of real numbers
   c. a sum of three numbers
   d. a product of three numbers

9. synthetic division
   a. division of nonreal numbers
   b. division by zero
   c. multiplication that looks like division
   d. a quick method for dividing by \( x - c \)

10. factor
    a. to write an expression as a product
    b. to multiply
    c. what two numbers have in common
    d. to FOIL

11. prime number
    a. a polynomial that cannot be factored
    b. a number with no divisors
    c. an integer between 1 and 10
    d. an integer larger than 1 that has no integral factors other than itself and 1

12. greatest common factor
    a. the least common multiple
    b. the least common denominator
    c. the largest integer that is a factor of two or more integers
    d. the largest number in a product

13. prime polynomial
    a. a polynomial that has no factors
    b. a product of prime numbers
    c. a first-degree polynomial
    d. a monomial

14. factor completely
    a. to factor by grouping
    b. to factor out a prime number
    c. to write as a product of primes
    d. to factor by trial-and-error

15. sum of two cubes
    a. \((a + b)^3\)
    b. \(a^3 + b^3\)
    c. \(a^3 - b^3\)
    d. \(a^3b^3\)

16. quadratic equation
    a. \(ax + b = 0\), where \(a \neq 0\)
    b. \(ax + b = cx + d\)
    c. \(ax^2 + bx + c = 0\), where \(a \neq 0\)
    d. any equation with four terms

17. zero factor property
    a. If \(ab = 0\), then \(a = 0\) or \(b = 0\)
    b. \(a \cdot 0 = 0\) for any \(a\)
    c. \(a = a + 0\) for any real number \(a\)
    d. \(a + (-a) = 0\) for any real number \(a\)

18. difference of two squares
    a. \(a^3 - b^3\)
    b. \(2a - 2b\)
    c. \(a^2 - b^2\)
    d. \((a - b)^2\)
5.1 Simplify each expression. Assume all variables represent nonzero real numbers. Write your answers with positive exponents.

1. \(2 \cdot 2 \cdot 2^{-1}\)  
2. \(5^{-1} \cdot 5\)  
3. \(2^3 \cdot 3^2\)  
4. \(3^2 \cdot 5^3\)  
5. \((-3)^{-3}\)  
6. \((-2)^{-2}\)  
7. \(-(-1)^{-3}\)  
8. \(3^4 \cdot 3^2\)  
9. \(2x^3 \cdot 4x^{-6}\)  
10. \(-3a^{-3} \cdot 4a^{-4}\)  
11. \(\frac{y^{-5}}{y^{-3}}\)  
12. \(\frac{w^3}{w^{-3}}\)  
13. \(\frac{a^5 \cdot a^{-2}}{a^{-4}}\)  
14. \(\frac{2m^3 \cdot m^5}{2m^{-2}}\)  
15. \(\frac{6x^2}{3x^2}\)  
16. \(\frac{-5y^3}{5y^{-2}x^{-3}}\)

Write each number in standard notation.

17. \(8.36 \times 10^6\)  
18. \(3.4 \times 10^7\)  
19. \(5.7 \times 10^{-4}\)  
20. \(4 \times 10^{-3}\)

Write each number in scientific notation.

21. \(8,070,000\)  
22. \(90,000\)  
23. \(0.000709\)  
24. \(0.0000005\)

Perform each computation without a calculator. Write the answer in scientific notation.

25. \(\frac{(4,000,000,000)(0.00000006)}{(0.000012)(2,000,000)}\)
26. \(\frac{(1.2 \times 10^{12})(2 \times 10^{-5})}{4 \times 10^{-7}}\)

5.2 Simplify each expression. Assume all variables represent nonzero real numbers. Write your answers with positive exponents.

27. \((a^{-3})^{-2} \cdot a^{-7}\)  
28. \((-3x^{-2}y^3)^{-4}\)
29. \((m^2n^3)^{-2}(m^{-3}n^4)^3\)  
30. \((w^{-3}xy)^{-1}(wx^{-3}y)^2\)
31. \(\left(\frac{2}{3}\right)^{-4}\)  
32. \(\left(\frac{a^4}{3}\right)^{-2}\)
33. \(\left(\frac{1}{2} + \frac{1}{3}\right)^2\)  
34. \(\left(\frac{1}{2} - \frac{1}{3}\right)^{-2}\)
35. \(\left(-\frac{3a}{4b^{-1}}\right)^{-1}\)  
36. \(\left(-\frac{4x^3}{5y^{-3}}\right)^{-1}\)
37. \(\left(\frac{(a^{-3}b)^2}{(ab^{-2})^{-5}}\right)\)  
38. \(\left(\frac{2x^3y^2}{3x^2}\right)^{-2}\)

Simplify each expression. Assume that the variables represent integers.

39. \(5^{2w} \cdot 5^{4w} \cdot 5^{-1}\)  
40. \(3(3^2y)^3\)
41. \(\left(\frac{7b^3}{p^8}\right)^5\)  
42. \(\left(\frac{2^{6-k}}{2^{2-8k}}\right)^3\)

5.3 Perform the indicated operations.

43. \((2w - 3) + (6w + 5)\)  
44. \((3a - 2xy) + (5xy - 7a)\)
45. \((x^2 - 3x - 4) - (x^2 + 3x - 7)\)
46. \((7 - 2x - x^2) - (x^2 - 5x + 6)\)
47. \((x^2 - 2x + 4)(x - 2)\)
48. \((x + 5)(x^2 - 2x + 10)\)
49. \(xy + 7z - 5(xy - 3z)\)
50. \(7 - 4(x - 3)\)
51. \(m^3(5m^3 - m + 2)\)
52. \((a + 2)^3\)

5.4 Perform the following computations mentally. Write down only the answers.

53. \((x - 3)(x + 7)\)  
54. \((k - 5)(k + 4)\)
55. \((z - 5y)(z + 5y)\)  
56. \((m - 3)(m + 3)\)
57. \((m + 8)^2\)  
58. \((b + 2a)^2\)
59. \((w - 6x)(w - 4x)\)  
60. \((2w - 3)(w + 6)\)
61. \((k - 3)^2\)  
62. \((n - 5)^2\)
63. \((m^2 - 5)(m^2 + 5)\)  
64. \((3k^2 - 5t)(2k^2 + 6t)\)

5.5 Find the quotient and remainder.

65. \((x^3 + x^2 - 11x + 10) \div (x - 2)\)  
66. \((2x^3 + 5x^2 + 9) \div (x + 3)\)
67. \((m^4 - 1) \div (m + 1)\)
68. \((x^4 - 1) \div (x - 1)\)
69. \((a^9 - 8) \div (a^3 - 2)\)
70. \((a^2 - b^2) \div (a - b)\)
71. \((3m^3 + 6m^2 - 18m) \div (3m)\)
72. \((w - 3) \div (3 - w)\)

Rewrite each expression in the form \(\text{quotient} + \frac{\text{remainder}}{\text{divisor}}\).

Use synthetic division.

73. \(\frac{x^2 - 5}{x - 1}\)
74. \(\frac{x^2 + 3x + 2}{x + 3}\)
75. \(\frac{3x}{x - 2}\)
76. \(\frac{4x}{x - 5}\)
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5.6 Complete the factoring by filling in the parentheses.

85. $3y - 6 = 3( \quad )$
86. $7x^2 - x = x( \quad )$
87. $4a - 20 = -4( \quad )$
88. $w^2 - w = -w( \quad )$
89. $3w - w^2 = -w( \quad )$
90. $3x - 6 = ( \quad )(2 - x)$

Factor each polynomial.

91. $y^2 - 81$
92. $x^2 - 9a^2$
93. $4x^2 + 28x + 49$
94. $y^2 - 20y + 100$
95. $t^2 - 18t + 81$
96. $4w^2 + 4ws + s^2$
97. $t^2 - 125$
98. $8y^3 + 1$

5.7 Factor each polynomial.

99. $x^3 - 7x - 30$
100. $x^3 + 4y - 32$
101. $w^3 - 3w^2 - 28$
102. $6t^2 - 5t + 1$
103. $2m^2 + 5m - 7$
104. $12x^2 - 17x + 6$
105. $m^3 - 3m^4 - 10m$
106. $6w^3 - 7w^3 - 5w$

5.8 Factor each polynomial completely.

107. $5x^3 + 40$
108. $w^3 - 6w^2 + 9w$
109. $9x^2 + 9x + 2$
110. $ax^3 + a$
111. $x^3 + x^2 - x - 1$
112. $16x^2 - 4x - 2$
113. $-x^2y + 16y$
114. $-5m^2 + 5$
115. $-a^2b^2 + 2ab^2 - ab^2$
116. $-2w^2 - 16w - 32$
117. $x^3 - x^2 + 9x - 9$
118. $w^4 + 2w^2 - 3$

119. $x^4 - x^2 - 12$
120. $8x^3 - 1$
121. $a^6 - a^3$
122. $a^2 - ab + 2a - 2b$
123. $-8m^2 - 24m - 18$
124. $-3x^2 - 9x + 30$
125. $(2x - 3)^2 - 16$
126. $(m - 6)^3 - (m - 6) - 12$
127. $x^6 + 7x^3 - 8$
128. $32a^2 - 2a$
129. $(a^2 - 9)^2 - 5(a^2 - 9) + 6$
130. $x^3 - 9x + x^2 - 9$

Factor each polynomial completely. Variables used as exponents represent positive integers.

131. $x^{2k} - 49$
132. $x^{6k} - 1$
133. $m^{2a} - 2m^a - 3$
134. $2y^{2n} - 7y^n + 6$
135. $9z^{2k} - 12z^6 + 4$
136. $25a^{6m} + 20z^{3m} + 4$
137. $y^{2a} - by^a + cy^a - bc$
138. $x^3y^b - xy^a + 2x^3 - 2x$

5.9 Solve each equation.

139. $x^3 - 5x^2 = 0$
140. $2m^2 + 10m + 12 = 0$
141. $(a - 2)(a - 3) = 6$
142. $(w - 2)(w + 3) = 50$
143. $2m^2 - 9m - 5 = 0$
144. $m^3 + 4m^2 - 9m - 36 = 0$
145. $w^3 + 5w^2 - w - 5 = 0$
146. $12x^2 + 5x - 3 = 0$
147. $|x^2 - 5| = 4$
148. $|x^2 - 3x - 7| = 3$

MISCELLANEOUS

Solve each problem.

149. Roadrunner and the coyote. The roadrunner has just taken a position atop a giant saguaro cactus. While positioning a 10-foot Acme ladder against the cactus, Wile E. Coyote notices a warning label on the ladder. For safety, Acme recommends that the distance from the ground to the top of the ladder, measured vertically along the cactus, must be 2 feet longer than the distance between the bottom of the ladder and the cactus. How far from the cactus should he place the bottom of this ladder?
150. **Three consecutive integers.** Find three consecutive integers such that the sum of their squares is 50.

151. **Perimeter of a square.** If the area of a square field is $9a^2 + 6a + 1$ square kilometers, then what is its perimeter?

152. **Landscape design.** Rico planted red tulips in a square flower bed with an area of $x^2$ square feet (ft²). He plans to surround the tulips with daffodils in a uniform border with a width of 3 feet. Write a polynomial that represents the area planted in daffodils.

153. **Life expectancy of black males.** The age at which people die is precisely measured and provides an indication of the health of the population as a whole. The formula $L = 64.3(1.0033)^a$ can be used to model life expectancy $L$ for U.S. black males with present age $a$ (National Center for Health Statistics, www.cdc.gov/nchswww).

   a) To what age can a 20-year-old black male expect to live?
   b) How many more years is a 20-year-old white male expected to live than a 20-year-old black male? (See Section 5.2 Exercise 95.)

154. **Life expectancy of black females.** The formula $L = 72.9(1.002)^a$ can be used to model life expectancy for U.S. black females with present age $a$. How long can a 20-year-old black female expect to live?

155. **Golden years.** A person earning $80,000 per year should expect to receive 21% of her retirement income from Social Security and the rest from personal savings. To calculate the amount of regular savings, we use the formula

$$S = R \cdot \frac{(1 + i)^n - 1}{i},$$

where $S$ is the amount at the end of $n$ years of $n$ investments of $R$ dollars each year earning interest rate $i$ compounded annually.

   a) Use the accompanying graph to estimate the interest rate needed to get an investment of $1 per year for 20 years to amount to $100.
   b) Use the formula to determine the annual savings for 20 years that would amount to $500,000 at 7% compounded annually.

![Figure for Exercise 155](image)

**FIGURE FOR EXERCISE 155**

156. **Costly education.** The average cost for tuition, room, and board for one year at a private college was $16,222 for 1994–1995 (National Center for Educational Statistics, www.nces.ed.gov). Use the formula in the previous exercise to find the annual savings for 18 years that would amount to $16,222 with an annual return of 8%.

---

**CHAPTER 5 TEST**

Simplify each expression. Assume all variables represent nonzero real numbers. Exponents in your answers should be positive exponents.

1. $3^{-2}$
2. $\frac{1}{6^{-2}}$
3. $(\frac{1}{2})^{-3}$
4. $3x^4 \cdot 4x^3$
5. $\frac{8y^9}{2y^{-3}}$
6. $(4a^2b)^3$
7. $(\frac{2k^3}{3})^{-3}$
8. $\frac{(2^{-1}a^2b)^{-3}}{4a^{-9}}$

Convert to standard notation.

9. $3.24 \times 10^9$
10. $8.673 \times 10^{-4}$

Perform each computation by converting each number to scientific notation. Give the answer in scientific notation.

11. $\frac{(80,000)(0.0006)}{2,000,000}$
12. $(0.00006)(500)$

Perform the indicated operations.

13. $(3x^3 - x^2 + 6) + (4x^2 - 2x - 3)$
14. $(x^2 - 6x - 7) - (3x^2 + 2x - 4)$
15. $(x^3 - 3x + 7)(x - 2)$
16. $(x^3 + 7x^2 + 7x - 15) \div (x + 3)$
17. $(x - 2)^3$
18. $(x - 3) \div (3 - x)$
Find the products.
19. \((x - 7)(2x + 3)\) \hspace{1cm} 20. \((x - 6)^2\)
21. \((2x + 5)^2\) \hspace{1cm} 22. \((3y^3 - 5)(3y^2 + 5)\)

Rewrite each expression in the form \(\frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}\).

Use synthetic division.
23. \(\frac{5x}{x + 3}\) \hspace{1cm} 24. \(\frac{x^2 + 3x - 6}{x - 2}\)

Factor completely.
25. \(a^2 - 2a - 24\) \hspace{1cm} 26. \(4x^2 + 28x + 49\) \hspace{1cm} 27. \(3m^3 - 24\) \hspace{1cm} 28. \(2x^3y - 32y\)

Solve each equation.
29. \(2x + 3a - 10x - 15\) \hspace{1cm} 30. \(x^4 + 3x^3 - 4\)

Solve each equation.
31. \(2m^2 + 7m - 15 = 0\) \hspace{1cm} 32. \(x^3 - 4x = 0\) \hspace{1cm} 33. \(|x^2 + x - 9| = 3\)

Write a complete solution for each problem.
34. A portable television is advertised as having a 10-inch diagonal measure screen. If the width of the screen is 2 inches more than the height, then what are the dimensions of the screen?

35. The infant mortality rate for the United States, the number of deaths per 100,000 live births, has decreased dramatically since 1950. The formula \(d = (1.8 \times 10^{28})(1.032)^{-y}\) gives the infant mortality rate \(d\) as a function of the year \(y\) (National Center for Health Statistics, www.cdc.gov/nchswww). Find the infant mortality rates in 1950, 1990, and 2000.
Simplify each expression.

1. \(4^2\)
2. \(4(-2)\)
3. \(4^{-2}\)
4. \(2^3 \cdot 4^{-1}\)
5. \(2^{-1} + 2^{-1}\)
6. \(2^{-1} \cdot 3^{-1}\)
7. \(3^{-1} - 2^{-2}\)
8. \(3^2 - 4(5)(-2)\)
9. \(2^7 - 2^6\)
10. \(0.08(32) + 0.08(68)\)
11. \(3 - 2|5 - 7 \cdot 3|\)
12. \(5^{-1} + 6^{-1}\)

Solve each equation.

13. \(0.05a - 0.04(a - 50) = 4\)
14. \(15b - 27 = 0\)
15. \(2c^2 + 15c - 27 = 0\)
16. \(2r^2 + 15r = 0\)
17. \(|15u - 27| = 3\)
18. \(|15v - 27| = 0\)
19. \(|15x - 27| = -\)
20. \(|x^2 + x - 4| = 2\)
21. \((2x - 1)(x + 5) = 0\)
22. \(|3x - 1| + 6 = 9\)

Solve each problem.

23. \((1.5 \times 10^{-4})w - 5 \times 10^3 = 7 \times 10^6\)
24. \((3 \times 10^7)(y - 5 \times 10^3) = 6 \times 10^{12}\)

25. **Negative income tax.** In a negative income tax proposal, the function

\[D = 0.75E + 500\]

is used to determine the disposable income \(D\) (the amount available for spending) for an earned income \(E\) (the amount earned). If \(E > D\), then the difference is paid in federal taxes. If \(D > E\), then the difference is paid to the wage earner by Uncle Sam.

a) Find the amount of tax paid by a person who earns $100,000.
b) Find the amount received from Uncle Sam by a person who earns $10,000.
c) The accompanying graph shows the lines \(D = 0.75E + 5000\) and \(D = E\). Find the intersection of these lines.
d) How much tax does a person pay whose earned income is at the intersection found in part (c)?