GETTING MORE INVOLVED

63. Cooperative learning. Write a step-by-step strategy for simplifying complex fractions with negative exponents. Have a classmate use your strategy to simplify some complex fractions from Exercises 35-54.

64. Discussion. a) Find the exact value of each expression.

\[
\begin{align*}
\text{i) } \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}} & \\
\text{ii) } \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}}} & \\
\end{align*}
\]

b) Explain why in each case the exact value must be less than 1.

65. Cooperative Learning. Work with a group to simplify the complex fraction. For what values of \(x\) is the complex fraction undefined?

SOLVING EQUATIONS INVOLVING RATIONAL EXPRESSIONS

In this section you will learn how to solve equations that have rational expressions, and in Section 6.6 we will solve problems using these equations.

Multiplying by the LCD

To solve equations having rational expressions, we multiply each side of the equation by the LCD of the rational expressions.

An equation with rational expressions

Solve \(\frac{1}{x} + \frac{1}{4} = \frac{1}{6}\).

Solution

The LCD for the denominators 4, 6, and \(x\) is 12x:

\[
12x \left( \frac{1}{x} + \frac{1}{4} \right) = 12x \left( \frac{1}{6} \right) \quad \text{Multiply each side by 12x.}
\]
\[
12x \cdot \frac{1}{x} + 12x \cdot \frac{3}{4} = 12x \cdot \frac{1}{6} \quad \text{Distributive property}
\]
\[
12 + 3x = 2x \quad \text{Divide out the common factors.}
\]
\[
12 + x = 0
\]
\[
x = -12
\]

Check \(-12\) in the original equation. The solution set is \{-12\}.
EXAMPLE 2
An equation with rational expressions

Solve \( \frac{1}{x} + \frac{2}{3x} = \frac{1}{5} \).

Solution

Multiply each side by 15x, the LCD for x, 3x, and 5:

\[
15x \left( \frac{1}{x} + \frac{2}{3x} \right) = 15x \left( \frac{1}{5} \right)
\]

\[
15 \cdot \frac{1}{x} + 15 \cdot \frac{2}{3x} = 3x
\]

\[
15 + 10 = 3x
\]

\[
25 = 3x
\]

\[
\frac{25}{3} = x
\]

Check \( \frac{25}{3} \) in the original equation. The solution set is \( \{\frac{25}{3}\} \).

CAUTION
To solve an equation with rational expressions, we do not convert the rational expressions to ones with a common denominator. Instead, we multiply each side by the LCD to eliminate the denominators.

EXAMPLE 3
An equation with two solutions

Solve \( \frac{200}{x} + \frac{300}{x+20} = 10 \).

Solution

\[
x(x+20) \left( \frac{200}{x} + \frac{300}{x+20} \right) = x(x+20)10
\]

\[
x(x+20) \frac{200}{x} + x(x+20) \frac{300}{x+20} = x(x+20)10
\]

\[
(x+20)200 + x(300) = (x^2 + 20x)10
\]

\[
200x + 4000 + 300x = 10x^2 + 200x
\]

\[
400 + 300x = x^2
\]

\[
0 = x^2 - 30x - 400
\]

\[
0 = (x - 40)(x + 10)
\]

\[
x = 40 \quad \text{or} \quad x = -10
\]

Check these values in the original equation. The solution set is \( \{-10, 40\} \).

Extraneous Roots

Because equations involving rational expressions have variables in denominators, a root to the equation might cause a 0 to appear in a denominator. In this case the root does not satisfy the original equation, and so it is called an extraneous root.
**Example 4**  
**An equation with an extraneous root**  
Solve \( \frac{3}{x} + \frac{6}{x - 2} = \frac{12}{x^2 - 2x} \).

**Solution**  
Because \( x^2 - 2x = x(x - 2) \), the LCD for \( x, x - 2, \) and \( x^2 - 2x = x(x - 2) \) is \( x(x - 2) \).  
Multiply each side by \( x(x - 2) \):  
\[
3(x - 2) + 6x = 12  
3x - 6 + 6x = 12  
9x - 6 = 12  
9x = 18  
x = 2
\]
Neither 0 nor 2 could be a solution because replacing \( x \) by either 0 or 2 would cause 0 to appear in a denominator in the original equation. So 2 is an extraneous root and the solution set is the empty set, \( \emptyset \).

**Example 5**  
**An equation with an extraneous root**  
Solve \( \frac{x}{x - 2} + \frac{x}{x - 2} = \frac{2}{x - 2} \).

**Solution**  
Because the LCD is \( x - 2 \), we multiply each side by \( x - 2 \):  
\[
(x - 2)(x + 2) + (x - 2) \cdot \frac{x}{x - 2} = (x - 2) \cdot \frac{2}{x - 2}  
x^2 - 4 + x = 2  
x^2 + x - 6 = 0  
(x + 3)(x - 2) = 0  
x + 3 = 0 \text{ or } x - 2 = 0  
x = -3 \text{ or } x = 2
\]
Replacing \( x \) by 2 in the original equation would cause 0 to appear in a denominator. So 2 is an extraneous root. Check that the original equation is satisfied if \( x = -3 \). The solution set is \( \{-3\} \).

**Proportions**  
An equation that expresses the equality of two rational expressions is called a **proportion**. The equation  
\[
\frac{a}{b} = \frac{c}{d}
\]
is a proportion. The terms in the position of \( b \) and \( c \) are called the **means**. The terms in the position of \( a \) and \( d \) are called the **extremes**. If we multiply this proportion by the LCD, \( bd \), we get  
\[
bd \cdot \frac{a}{b} = bd \cdot \frac{c}{d}
\]
or  
\[
ad = bc.
\]
The equation \( ad = bc \) says that the product of the extremes is equal to the product of the means. When solving a proportion, we can omit multiplication by the LCD and just remember the result, \( ad = bc \), as the **extremes-means property**.

### Extremes-Means Property

If \( \frac{a}{b} = \frac{c}{d} \), then \( ad = bc \).

The extremes-means property makes it easier to solve proportions.

**Example 6**

**A proportion with one solution**

Solve \( \frac{20}{x} = \frac{30}{x + 20} \).

**Solution**

Rather than multiplying by the LCD, we use the extremes-means property to eliminate the denominators:

\[
\frac{20}{x} = \frac{30}{x + 20} \\
20(x + 20) = 30x \\
20x + 400 = 30x \\
400 = 10x \\
40 = x
\]

Check 40 in the original equation. The solution set is \( \{40\} \).

**Example 7**

**A proportion with two solutions**

Solve \( \frac{2}{x} = \frac{x + 3}{5} \).

**Solution**

Use the extremes-means property to write an equivalent equation:

\[
x(x + 3) = 2 \cdot 5 \\
x^2 + 3x = 10 \\
x^2 + 3x - 10 = 0 \\
(x + 5)(x - 2) = 0 \\
x + 5 = 0 \quad \text{or} \quad x - 2 = 0 \\
x = -5 \quad \text{or} \quad x = 2
\]

Both \(-5\) and \(2\) satisfy the original equation. The solution set is \( \{-5, 2\} \).

**Caution**

Use the extremes-means property only when solving a proportion. It cannot be used on an equation such as

\[
\frac{3}{x} = \frac{2}{x + 1} + 5.
\]
EXAMPLE 8

Ratios and proportions

The ratio of men to women at a football game was 4 to 3. If there were 12,000 more men than women in attendance, then how many men and how many women were in attendance?

Solution

Let \( x \) represent the number of men in attendance and \( x - 12,000 \) represent the number of women in attendance. Because the ratio of men to women was 4 to 3, we can write the following proportion:

\[
\frac{4}{3} = \frac{x}{x - 12,000}
\]

\[4x - 48,000 = 3x\]

\[x = 48,000\]

So there were 48,000 men and 36,000 women at the game.

WARM-UPS

True or false? Explain.

1. In solving an equation involving rational expressions, multiply each side by the LCD for all of the denominators.
2. To solve \( \frac{1}{x} + \frac{1}{2x} = \frac{1}{3} \) first change each rational expression to an equivalent rational expression with a denominator of 6x.
3. Extraneous roots are not real numbers.
4. To solve \( \frac{1}{x - 2} + 3 = \frac{1}{x + 2} \), multiply each side by \( x^2 - 4 \).
5. The solution set to \( \frac{x}{3x + 4} - \frac{6}{2x + 1} = \frac{7}{5} \) is \( \left\{ \frac{-4}{3}, -\frac{1}{2} \right\} \).
Find the solution set to each equation. See Examples 1–5.

6. The solution set to \( \frac{3}{x} = \frac{2}{5} \) is \( \left\{ \frac{15}{2} \right\} \).

7. We should use the extremes-means property to solve \( \frac{x - 2}{x + 3} + 1 = \frac{1}{x} \).

8. The equation \( x^2 = x \) is equivalent to the equation \( x = 0 \).

9. The solution set to \((2x - 3)(3x + 4) = 0\) is \( \left\{ \frac{3}{2}, \frac{4}{3} \right\} \).

10. The equation \( \frac{2}{x + 1} = \frac{x - 4}{4} \) is equivalent to \( x^2 - 1 = 8 \).

6.5 Exercises

Reading and Writing  After reading this section, write out the answers to these questions. Use complete sentences.

1. What is the usual first step in solving an equation involving rational expressions?

2. How can an equation involving rational expressions have an extraneous root?

3. What is a proportion?

4. What are the means?

5. What are the extremes?

6. What is the extremes-means property?

Find the solution set to each equation. See Examples 1–5.

7. \( \frac{1}{x} + \frac{1}{6} = \frac{1}{8} \)

8. \( \frac{3}{x} + \frac{1}{5} = \frac{1}{2} \)

9. \( \frac{2}{3x} + \frac{1}{15x} = \frac{1}{2} \)

10. \( \frac{5}{6x} - \frac{1}{8x} = \frac{17}{24} \)

11. \( \frac{3}{x - 2} + \frac{5}{x} = \frac{10}{x} \)

12. \( \frac{5}{x - 1} + \frac{1}{2x} = \frac{1}{x} \)

13. \( \frac{x}{x - 2} + \frac{3}{x} = 2 \)

14. \( \frac{x}{x - 5} + \frac{5}{x} = \frac{11}{6} \)

15. \( \frac{100}{x} = \frac{150}{x + 5} - 1 \)

16. \( \frac{30}{x} = \frac{50}{x + 10} + \frac{1}{2} \)

17. \( \frac{3x - 5}{x - 1} = 2 - \frac{2x}{x - 1} \)

18. \( \frac{x - 3}{x + 2} = 3 - \frac{1 - 2x}{x + 2} \)

19. \( x + 1 + \frac{2x - 5}{x - 5} = \frac{x}{x - 5} \)

20. \( \frac{x - 3}{2} - \frac{1}{x - 3} = \frac{8 - 3x}{x - 3} \)

21. \( \frac{x}{x + 2} + \frac{x}{x - 3} + \frac{1}{x^2 - x - 6} = 0 \)

22. \( \frac{x - 4}{x^2 + 2x - 15} = \frac{2}{x - 3} \)

Find the solution set to each equation. See Examples 6 and 7.

23. \( \frac{2}{x} = \frac{3}{4} \)

24. \( \frac{5}{x} = \frac{7}{9} \)

25. \( \frac{a}{3} = \frac{-1}{4} \)

26. \( \frac{b}{5} = \frac{-3}{7} \)

27. \( \frac{-5}{7} = \frac{2}{x} \)

28. \( \frac{-3}{8} = \frac{5}{x} \)

29. \( \frac{10}{x} = \frac{20}{x + 20} \)

30. \( \frac{x}{5} = \frac{x + 2}{3} \)

31. \( \frac{2}{x + 1} = \frac{x - 1}{4} \)

32. \( \frac{3}{x - 2} = \frac{x + 2}{7} \)

33. \( \frac{x}{6} = \frac{5}{x - 1} \)

34. \( \frac{x + 5}{2} = \frac{3}{x} \)

35. \( \frac{x}{x - 3} = \frac{x + 2}{x} \)

36. \( \frac{x + 1}{x - 5} = \frac{x + 2}{x - 4} \)

37. \( \frac{x - 2}{x - 3} = \frac{x + 5}{x + 2} \)

38. \( \frac{x}{x + 5} = \frac{x}{x - 2} \)

Solve each equation.

39. \( \frac{a}{9} = \frac{4}{a} \)

40. \( \frac{y}{3} = \frac{27}{y} \)

41. \( \frac{1}{2x - 4} + \frac{1}{x - 2} = \frac{1}{4} \)

42. \( \frac{7}{3x - 9} - \frac{1}{x - 3} = \frac{4}{9} \)

43. \( \frac{x - 2}{4} = \frac{x - 2}{x} \)

44. \( \frac{y + 5}{2} = \frac{y + 5}{y} \)

45. \( \frac{5}{2x + 4} - \frac{1}{x - 1} = \frac{3}{x + 2} \)
65. Cleaning up the river. Pollution in the Tickfaw River has been blamed primarily on pesticide runoff from area farms. The formula
\[ C = \frac{4,000,000p}{100 - p} \]
has been used to model the cost in dollars for removing \( p\% \) of the pollution in the river. If the state gets a $1 million federal grant for cleaning up the river, then what percentage of the pollution can be removed? Use the bar graph to estimate the percentage that can be cleaned up with a $100 million grant.

![Graph showing percentage of pollution removed](image)

66. Campaigning for governor. A campaign manager for a gubernatorial candidate estimates that the cost in dollars for an advertising campaign that will get his candidate \( p\% \) of the votes is given by
\[ C = \frac{1,000,000 + 2,000,000p}{100 - p} \]
If the candidate can spend only $2 million for advertising, then what percentage of the votes can she expect to receive? Use the bar graph to estimate the percentage of votes expected if $4 million is spent.

![Graph showing cost in millions of dollars](image)
67. **Wealth-building portfolio.** Misty decided to invest her annual bonus in a wealth-building portfolio as shown in the figure (Fidelity Investments, Boston).

a) If the amount that she invested in stocks was $20,000 greater than her investment in bonds, then how much did she invest in bonds?

b) What was the amount of her annual bonus?

68. **Estimating weapons.** When intelligence agents obtain enemy weapons marked with serial numbers, they use the formula $N = (1 + 1/C)B - 1$ to estimate the total number of such weapons $N$ that the enemy has produced (New Scientist, May 1998). $B$ is the biggest serial number obtained and $C$ is the number of weapons obtained. It is assumed the weapons are numbered 1 through $N$.

a) Find $N$ if agents obtain five nerve gas containers numbered 45, 143, 258, 301, and 465.

b) Find $C$ if agents estimate that the enemy has 255 tanks from a group of captured tanks on which the biggest serial number is 224.

69. **Writing.** In this chapter the LCD is used to add rational expressions and to solve equations. Explain the difference between using the LCD to solve the equation

$$\frac{3}{x - 2} + \frac{7}{x + 2} = 2$$

and using the LCD to find the sum

$$\frac{3}{x - 2} + \frac{7}{x + 2}.$$