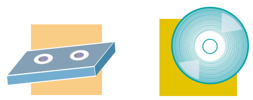
SECTION 11.3

Solving Quadratic Equations by Graphing

11.3 OBJECTIVES

1. Find an axis of symmetry
2. Find a vertex
3. Graph a parabola
4. Solve quadratic equations by graphing
5. Solve an application involving a quadratic equation



In Section 3.3, we learned to graph a linear equation. We discovered that the graph of every linear equation was a straight line. In this section, we will consider the graph of a quadratic equation.

Consider an equation of the form

$$y = ax^2 + bx + c \qquad a \neq 0$$

This equation is quadratic in x and linear in y. Its graph will always be the curve called the parabola.

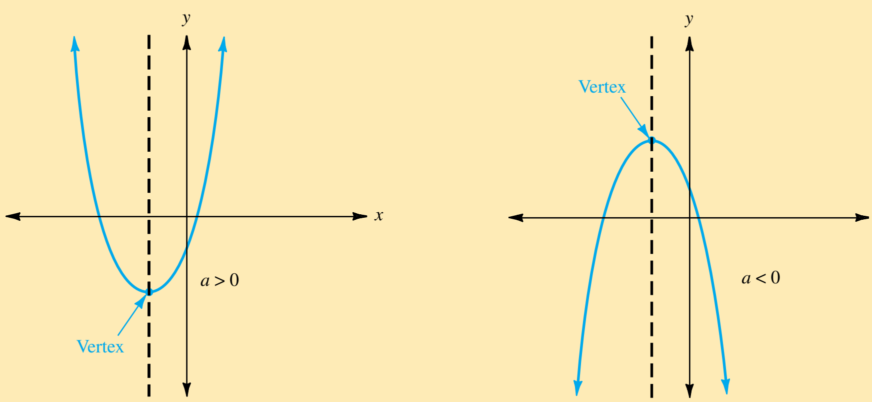
In an equation of the form

$$y = ax^2 + bx + c \qquad a \neq 0$$

the parabola opens upward or downward, as follows:

1. If $a > 0$, the parabola opens *upward.*
2. If $a < 0$, the parabola opens *downward.*

$$y = ax^2 + bx + c$$

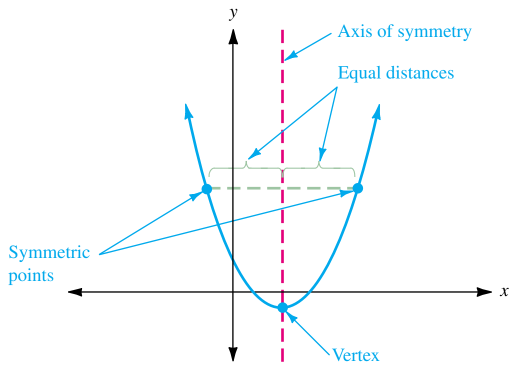


Two concepts regarding the parabola can be made by observation. Consider the above illustrations.

1. There is always a **minimum** (or lowest) point on the parabola if it opens upward. There is always a **maximum** (or highest) point on the parabola if it opens downward. In either case, that maximum or minimum value occurs at the **vertex** of the parabola.

2. Every parabola has an **axis of symmetry.** In the case of parabolas that open upward or downward, that axis of symmetry is a vertical line midway between any pair of symmetric points on the parabola. Also, the point where this axis of symmetry intersects the parabola is the vertex of the parabola.

The following figure summarizes these observations.



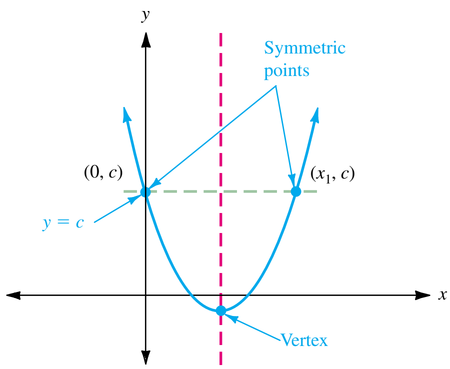
Our objective is to be able to quickly sketch a parabola. This can be done with *as few as three points* if those points are carefully chosen. For this purpose you will want to find the vertex and two symmetric points.

First, let's see how the coordinates of the vertex can be determined from the standard equation

$$y = ax^2 + bx + c \tag{1}$$

In equation (1), if $x = 0$, then $y = c$, and so $(0, c)$ gives the point where the parabola intersects the y axis (the y intercept).

Look at the sketch. To determine the coordinates of the symmetric point (x_1, c), note that it lies along the horizontal line $y = c$.



Therefore, let $y = c$ in equation (1):

$$c = ax^2 + bx + c$$
$$0 = ax^2 + bx$$
$$0 = x(ax + b)$$

and

$$x = 0 \qquad \text{or} \qquad x = -\frac{b}{a}$$

We now know that

$$(0, c) \qquad \text{and} \qquad \left(-\frac{b}{a}, c\right)$$

are the coordinates of the symmetric points shown. Since the axis of symmetry must be midway between these points, the x value along that axis is given by

$$x = \frac{0 + (-b/a)}{2} = -\frac{b}{2a} \tag{2}$$

Since the vertex for any parabola lies on the axis of symmetry, we can now state the following general result.

Vertex of a Parabola

If

$$y = ax^2 + bx + c \qquad a \neq 0$$

then the x coordinate of the vertex of the corresponding parabola is

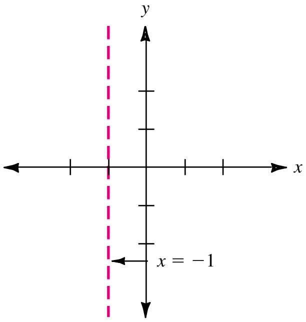
$$x = -\frac{b}{2a}$$

Note: The y coordinate of the vertex can be found most easily by substituting the value found for x into the original equation.

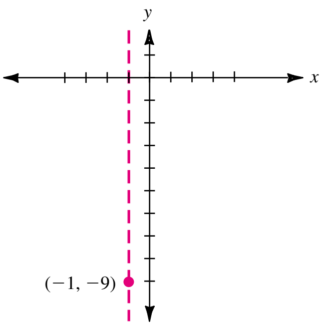
We now know how to find the vertex of a parabola, and if two symmetric points can be determined, we are well on our way to the desired graph. Perhaps the simplest case is when the quadratic member of the given equation is factorable. In most cases, the two x intercepts will then give two symmetric points that are very easily found. Example 1 illustrates such a case.

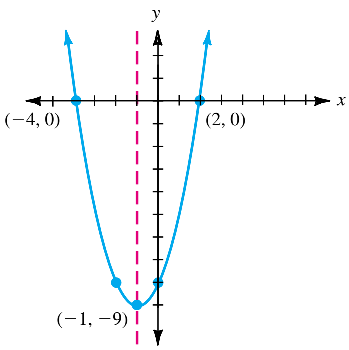
Example 1

Sketch the information to help you solve the problem. Begin by drawing—as a dashed line—the axis of symmetry.



At this point you can plot the vertex along the axis of symmetry.





Graphing a Parabola

Graph the equation

$$y = x^2 + 2x - 8$$

First, find the axis of symmetry. In this equation, $a = 1$, $b = 2$, and $c = -8$. We then have

$$x = -\frac{b}{2a} = -\frac{2}{2 \cdot 1} = -\frac{2}{2} = -1$$

Thus, $x = -1$ is the axis of symmetry.

Second, find the vertex. Since the vertex of the parabola lies on the axis of symmetry, let $x = -1$ in the original equation. If $x = -1$,

$$y = (-1)^2 + 2(-1) - 8 = -9$$

and $(-1, -9)$ is the vertex of the parabola.

Third, find two symmetric points. Note that the quadratic member in this case is factorable, and so setting $y = 0$ in the original equation will quickly give two symmetric points (the x intercepts):

$$\begin{aligned} 0 &= x^2 + 2x - 8 \\ &= (x + 4)(x - 2) \end{aligned}$$

So when $y = 0$,

$$\begin{aligned} x + 4 &= 0 \qquad \text{or} \qquad & x - 2 &= 0 \\ x &= -4 & x &= 2 \end{aligned}$$

and our x intercepts are $(-4, 0)$ and $(2, 0)$.

Fourth, draw a smooth curve connecting the points found above, to form the parabola.

Note: You can choose to find additional pairs of symmetric points at this time if necessary. For instance, the symmetric points $(0, -8)$ and $(-2, -8)$ are easily located.

✓ CHECK YOURSELF 1

Graph the equation

$$y = -x^2 - 2x + 3$$

(*Hint*: Since the coefficient of x^2 is negative, the parabola opens downward.)

A similar process will work if the quadratic member of the given equation is *not* factorable. In that case, one of two things happens:

1. The x intercepts are irrational and therefore not particularly helpful in the graphing process.
2. The x intercepts do not exist.

Consider Example 2.

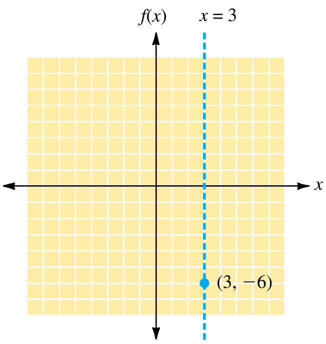
Example 2 Graphing a Parabola

Graph the function

$$f(x) = x^2 - 6x + 3$$

First, find the axis of symmetry. Here $a = 1$, $b = -6$, and $c = 3$. So

$$x = -\frac{b}{2a} = \frac{-(-6)}{2 \cdot 1} = \frac{6}{2} = 3$$



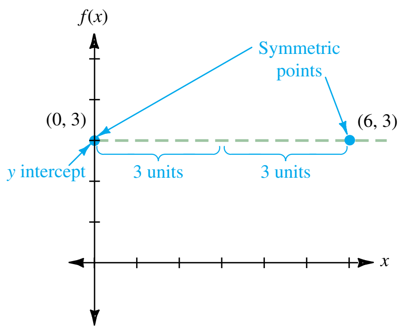
Thus, $x = 3$ is the axis of symmetry.

Second, find the vertex. If $x = 3$,

$$f(3) = 3^2 - 6 \cdot 3 + 3 = -6$$

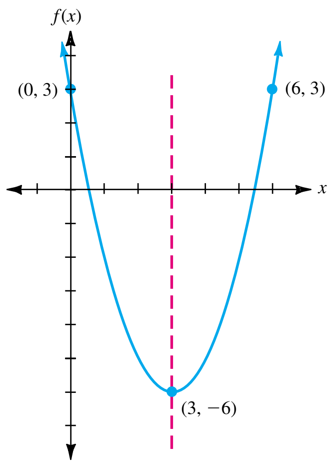
and $(3, -6)$ is the vertex of the desired parabola.

Third, find two symmetric points. Here the quadratic member is not factorable, so we need to find another pair of symmetric points.



Note that $(0, 3)$ is the y intercept of the parabola. We found the axis of symmetry at $x = 3$ in step 1. Note that the symmetric point to $(0, 3)$ lies along the horizontal line through the y intercept at the same distance (3 units) from the axis of symmetry. Hence, $(6, 3)$ is our symmetric point.

Fourth, draw a smooth curve connecting the points found above to form the parabola.



Note: An alternate method is available in step 3. Observing that 3 is the y intercept and that the symmetric point lies along the line $y = 3$, set $f(x) = 3$ in the original equation:

$$3 = x^2 - 6x + 3$$
$$0 = x^2 - 6x$$
$$0 = x(x - 6)$$

so

$$x = 0 \qquad \text{or} \qquad x - 6 = 0$$
$$x = 6$$

and (0, 3) and (6, 3) are the desired symmetric points.

✓ CHECK YOURSELF 2

Graph the function

$$f(x) = x^2 + 4x + 5$$

Thus far the coefficient of x^2 has been 1 or −1. The following example shows the effect of different coefficients on the shape of the graph of a quadratic equation.

Example 3 Graphing a Parabola

Graph the equation

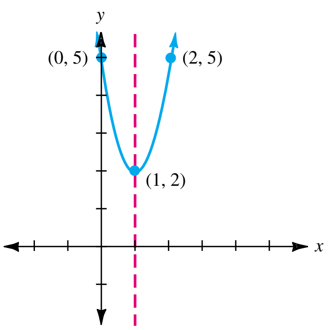
$$y = 3x^2 - 6x + 5$$

First, find the axis of symmetry.

$$x = -\frac{b}{2a} = \frac{-(-6)}{2 \cdot 3} = \frac{6}{6} = 1$$

Second, find the vertex. If $x = 1$,

$$y = 3(1)^2 - 6 \cdot 1 + 5 = 2$$



So (1, 2) is the vertex.

Third, find symmetric points. Again the quadratic member is not factorable, and we use the y intercept (0, 5) and its symmetric point (2, 5).

Fourth, connect the points with a smooth curve to form the parabola. Compare this curve to those in previous examples. Note that the parabola is "tighter" about the axis of symmetry. That is the effect of the larger x^2 coefficient.

✓ *CHECK YOURSELF 3*

Graph the equation

$$y = \frac{1}{2}x^2 - 3x - 1$$

The following algorithm summarizes our work thus far in this section.

To Graph a Parabola

Step 1 Find the axis of symmetry.
Step 2 Find the vertex.
Step 3 Determine two symmetric points.
Note: You can use the x intercepts if the quadratic member of the given equation is factorable. Otherwise use the y intercept and its symmetric point.
Step 4 Draw a smooth curve connecting the points found above to form the parabola. You may choose to find additional pairs of symmetric points at this time.

We have seen that quadratic equations can be solved in three different ways: by factoring (Section 6.5), by completing the square (Section 11.1), or by using the quadratic formula (Section 11.2). We will now look at a fourth technique for solving quadratic equations, a graphical method. Unlike the other methods, the graphical technique may yield only an approximation of the solution(s). On the other hand, it can be a very useful method for checking the reasonableness of exact answers. This is particularly true when technology is used to produce the graph.

We can easily use a graph to identify the number of positive and negative solutions to a quadratic equation. Recall that solutions to the equation

$$0 = ax^2 + bx + c$$

are values for x that make the statement true.

If we have the graph of the parabola

$$y = ax^2 + bx + c$$

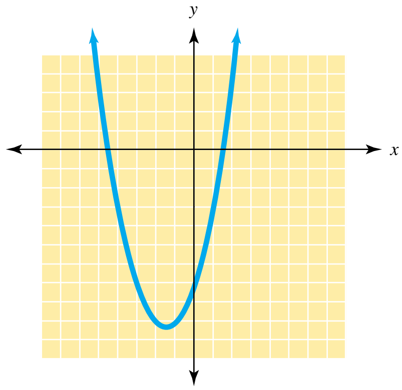
then values for which y is equal to zero are solutions to the original equation. We know that $y = 0$ is the equation for the x axis. Solutions for the equation exist wherever the graph crosses the x axis.

Example 4 Identifying the Number of Solutions to a Quadratic Equation

Find the number of positive solutions and the number of negative solutions for each quadratic equation.

(a) $0 = x^2 + 3x - 7$

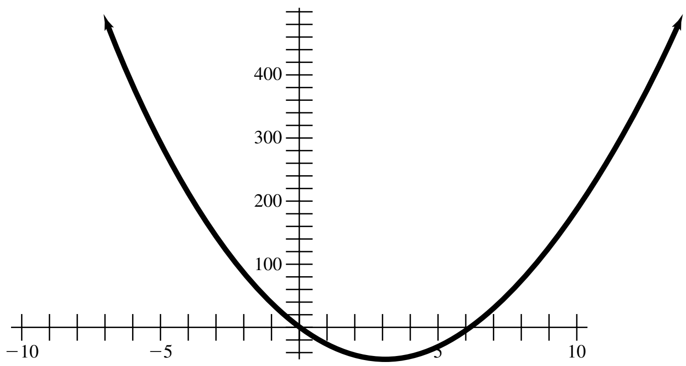
Using either the methods of this section, or a graphing device, we find the graph of the parabola $y = x^2 + 3x - 7$ first. The axis of symmetry is $x = -\frac{3}{2}$. Two points on the graph could be $(0, -7)$ and $(-3, -7)$.



Looking at the graph, we see that the parabola crosses the x axis twice, once to the left of zero and once to the right. There are two real solutions. One is negative and one is positive. Note that due to the basic shape of a parabola, it is not possible for this graph to cross the x axis more than two times.

(b) $0 = 5x^2 - 32x$

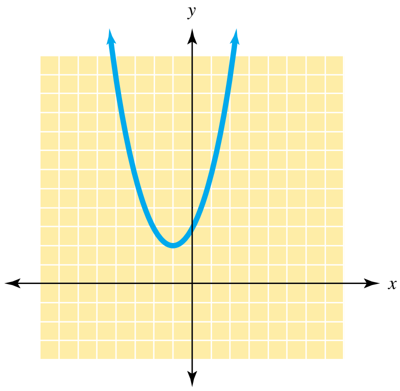
Again, we graph the parabola $y = 5x^2 - 32x$ first. The line of symmetry is $x = \frac{16}{5}$. Two points on the parabola are $(0, 0)$ and $\left(\frac{32}{5}, 0\right)$.



There are two solutions. One is positive and the other appears to be zero. A quick check of the equation confirms that zero is a solution.

(c) $0 = x^2 + 2x + 3$

The axis of symmetry is $x = -1$. Two points on the parabola would be (0, 3) and (−2, 3). Here is the graph of the related parabola.



Notice that the graph never touches the x axis. There are no real solutions to the equation $0 = x^2 + 2x + 3$.

✓ *CHECK YOURSELF 4*

Use the graph of the related parabola to find the number of real solutions to each equation. Identify the number of solutions that are positive and the number that are negative.

(a) $0 = 3x^2 - 16x$ (b) $0 = x^2 + x + 4$ (c) $0 = x^2 - 3x - 9$

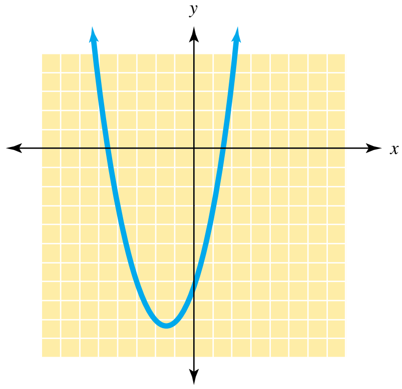
We used the x intercepts of the graph to determine the number of solutions for each equation in Example 4. The value of the x coordinate for each intercept is the exact solution. In the next example, we'll use the graphical method to estimate the solutions of a quadratic equation.

Example 5 Solving a Quadratic Equation

Solve each equation.

(a) $0 = x^2 + 3x - 7$

As we did in Example 4, we find the graph of the parabola $y = x^2 + 3x - 7$ first.

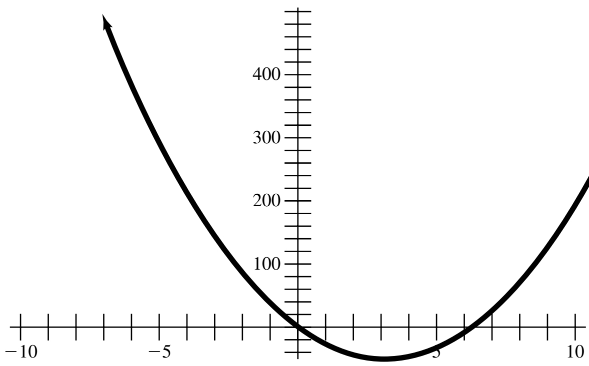


In Example 4, we pointed out that there are two solutions. One solution is between -5 and -4. The other is between 1 and 2. The precision with which we estimate the values depends on the quality of the graph we are examining. From this graph, -4.5 and 1.5 would be reasonable estimates of the solutions.

Using the quadratic formula, we find that the actual solutions are $\frac{(-3 - \sqrt{37})}{2}$ and $\frac{(-3 + \sqrt{37})}{2}$. Using a calculator, we can round these answers to the nearest hundredth. We get 1.54 and -4.54. Our estimates were quite reasonable.

(b) $0 = 5x^2 - 32x$

Again, we graph the parabola $y = 5x^2 - 32x$ first.



There are two solutions. One is a little more than six and the other appears to be zero. Using the quadratic formula (or factoring) we find that the solutions to this equation are 0 and 6.4.

✓ *CHECK YOURSELF 5*

Use the graph of the related parabola to estimate the solutions to each equation.

(a) $0 = 3x^2 - 16x$ (b) $0 = x^2 - 3x - 9$

From graphs of equations of the form $y = ax^2 + bx + c$, we know that if $a > 0$, then the vertex is the lowest point on the graph (the minimum value). Also, if $a < 0$, then the vertex is the highest point on the graph (the maximum value). We can use this result to solve a variety of problems in which we want to find the maximum or minimum value of a variable. The following are just two of many typical examples.

Example 6 An Application Involving a Quadratic Function

A software company sells a word processing program for personal computers. They have found that their monthly profit in dollars, P, from selling x copies of the program is approximated by

$$P(x) = -0.3x^2 + 90x - 1500$$

Find the number of copies of the program that should be sold in order to maximize the profit.

Since the relating equation is quadratic, the graph must be a parabola. Also since the coefficient of x^2 is negative, the parabola must open downward, and thus the vertex will give the maximum value for the profit, P. To find the vertex,

$$x = -\frac{b}{2a} = \frac{-90}{2(-0.3)} = \frac{-90}{-0.6} = 150$$

The maximum profit must then occur when $x = 150$, and we substitute that value into the original equation:

$$\begin{aligned} P(x) &= -0.3(150)^2 + (90)(150) - 1500 \\ &= \$5250 \end{aligned}$$

The maximum profit will occur when 150 copies are sold per month, and that profit will be $5250.

✓ CHECK YOURSELF 6

A company that sells portable radios finds that its weekly profit in dollars, P, and the number of radios sold, x, are related by

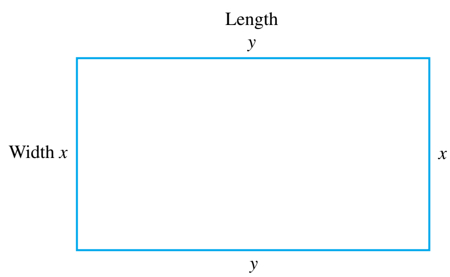
$$P(x) = -0.2x^2 + 40x - 100$$

Find the number of radios that should be sold to have the largest weekly profit. Also find the amount of that profit.

Example 7 An Application Involving a Quadratic Function

A farmer has 3600 ft of fence and wishes to enclose the largest possible rectangular area with that fencing. Find the largest possible area that can be enclosed.

As usual, when dealing with geometric figures, we start by drawing a sketch of the problem.



First, we can write the area, A, as

Area = length × width

$$A = xy \tag{3}$$

Also, since 3600 ft of fence is to be used, we know that

The perimeter of the region is $2x + 2y$

$$\begin{aligned} 2x + 2y &= 3600 \\ 2y &= 3600 - 2x \\ y &= 1800 - x \end{aligned} \tag{4}$$

Substituting for y in equation (3), we have

$$\begin{aligned} A(x) &= x(1800 - x) \\ &= 1800x - x^2 \\ &= -x^2 + 1800x \end{aligned} \tag{5}$$

Again, the graph for A is a parabola opening downward, and the largest possible area will occur at the vertex. As before,

The width x is 900 ft. Since from equation (2)

$$y = 1800 - 900$$
$$= 900 \text{ ft}$$

$$x = \frac{-1800}{2(-1)} = \frac{-1800}{-2} = 900$$

and the largest possible area is

The length is also 900 ft. The desired region is a square.

$$A(x) = -(900)^2 + 1800(900) = 810{,}000 \text{ ft}^2$$

✓ CHECK YOURSELF 7

We want to enclose three sides of the largest possible rectangular area by using 900 ft of fence. Assume that an existing wall makes the fourth side. What will be the dimensions of the rectangle?

✓ CHECK YOURSELF ANSWERS

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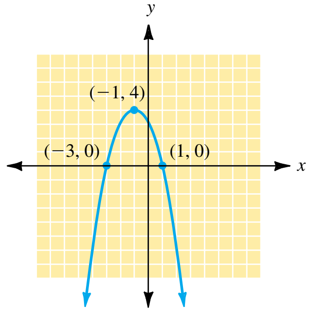

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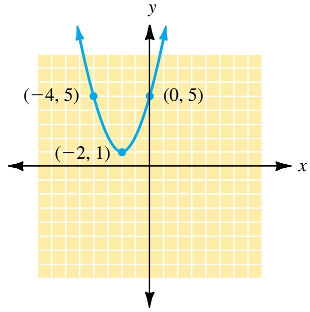

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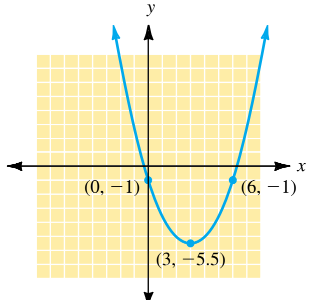


4. (a) Two solutions; one positive, one zero; (b) no real solutions; (c) two solutions; one negative, one positive. **5.** (a) 0, 5.3; (b) −1.9, 4.9. **6.** 100 radios, $1900. **7.** Width 225 ft, length 450 ft.

Electric Power Costs. You and a partner in a small engineering firm have been asked to help calculate the costs involved in running electric power a distance of 3 miles from a public utility company to a nearby community. You have decided to consider three tower heights: 75 feet, 100 feet, and 120 feet. The 75-ft towers cost \$850 each to build and install; the 100-ft towers cost \$1110 each; and the 120-ft towers cost \$1305 each. Spans between towers typically run 300 to 1200 ft. The conducting wires weigh 1900 pounds per 1000 ft, and the tension on the wires is 6000 pounds. To be safe, the conducting wires should never be closer than 45 ft to the ground.

Work with a partner to complete the following.

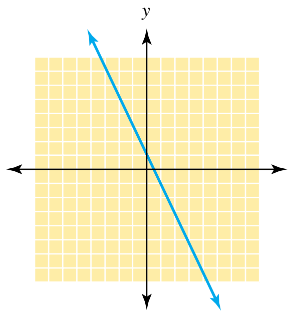
1. Develop a plan for the community showing all three scenarios and their costs. Remember that hot weather will expand the wires and cause them to sag more than normally. Therefore, allow for about a 10% margin of error when you calculate the sag amount.
2. Write an accompanying letter to the town council explaining your recommendation. Be sure to include any equations you used to help you make your decision. Remember to look at the formula at the beginning of the chapter, which gives the amount of sag for wires given a certain weight per foot, span length, and tension on the conductors.

Exercises 11.3

In Exercises 1 to 8, match each graph with one of the equations at the left.

(a) $y = x^2 + 2$
(b) $y = 2x^2 - 1$
(c) $y = 2x + 1$
(d) $y = x^2 - 3x$
(e) $y = -x^2 - 4x$
(f) $y = -2x + 1$
(g) $y = x^2 + 2x - 3$
(h) $y = -x^2 + 6x - 8$

1.



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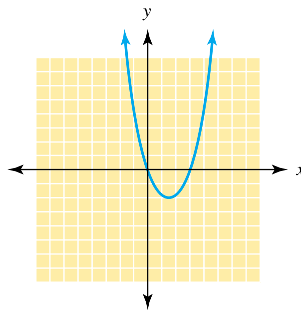

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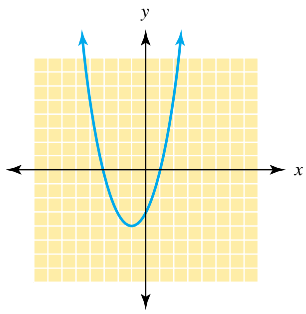

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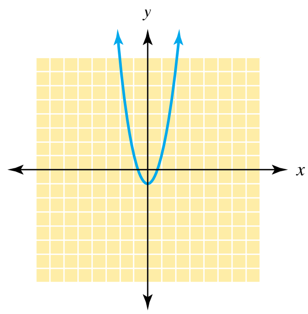

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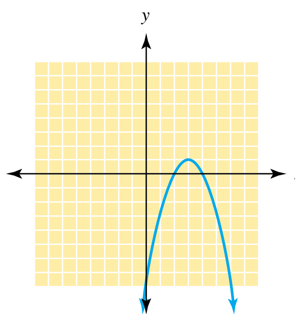

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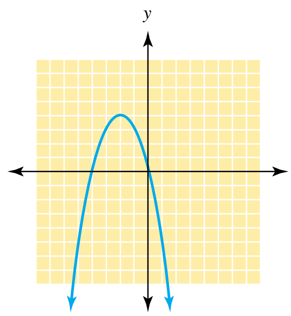

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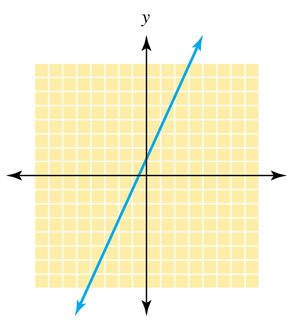

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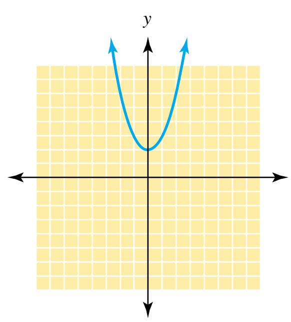

In Exercises 9 to 14, which of the given conditions apply to the graphs of the following equations? Note that more than one condition may apply.

(a) The parabola opens upward.

(b) The parabola opens downward.

(c) The parabola has two x intercepts.

(d) The parabola has one x intercept.

(e) The parabola has no x intercept.

9. $y = x^2 - 3$ **10.** $y = -x^2 + 4x$ **11.** $y = x^2 - 3x - 4$

12. $y = x^2 - 2x + 2$ **13.** $y = -x^2 - 3x + 10$ **14.** $y = x^2 - 8x + 16$

In Exercises 15 to 28, find the equation of the axis of symmetry, the coordinates of the vertex, and the x intercepts. Sketch the graph of each equation.

15. $y = x^2 - 4x$ **16.** $y = x^2 - 1$

17. $y = -x^2 + 4$ **18.** $y = x^2 + 2x$

19. $y = -x^2 - 2x$ **20.** $y = -x^2 - 3x$

21. $y = x^2 - 6x + 5$ **22.** $y = x^2 + x - 6$

23. $y = x^2 - 5x + 6$ **24.** $y = x^2 + 6x + 5$

25. $y = x^2 - 6x + 8$ **26.** $y = -x^2 - 3x + 4$

27. $y = -x^2 - 6x - 5$ **28.** $y = -x^2 + 6x - 8$

In Exercises 29 to 40, find the equation of the axis of symmetry, the coordinates of the vertex, and at least two symmetric points. Sketch the graph of each equation.

29. $y = x^2 - 2x - 1$

30. $y = x^2 + 4x + 6$

31. $y = x^2 - 4x - 1$

32. $y = -x^2 + 6x - 5$

33. $y = -x^2 + 3x - 3$

34. $y = x^2 + 5x + 3$

35. $y = 2x^2 + 4x - 1$

36. $y = \frac{1}{2}x^2 - x - 1$

37. $y = -\frac{1}{3}x^2 + x - 3$

38. $y = -2x^2 - 4x - 1$

39. $y = 3x^2 + 12x + 5$

40. $y = -3x^2 + 6x + 1$

In Exercises 41 to 50, use the graph of the related parabola to find the number of real solutions to each equation. Identify the number of solutions that are positive and the number that are negative.

41. $0 = x^2 + x - 12$

42. $0 = x^2 + 3x + 2$

43. $0 = 6x^2 - 19x$

44. $0 = 7x^2 - 15x$

45. $0 = x^2 - 2x + 5$

46. $0 = x^2 + 6x + 4$

47. $0 = 9x^2 + 12x - 7$

48. $0 = 3x^2 + 9x + 5$

49. $0 = x^2 + 2x - 7$

50. $0 = x^2 - 8x + 11$

In Exercises 51 to 58, use the graph of the related parabola to estimate the solutions to each equation. Round answers to the nearest tenth.

51. $0 = x^2 + x - 12$

52. $0 = x^2 + 3x + 2$

53. $0 = 6x^2 - 19x$

54. $0 = 7x^2 - 15x$

55. $0 = 9x^2 + 12x - 7$

56. $0 = 3x^2 + 9x + 5$

57. $0 = x^2 + 2x - 7$

58. $0 = x^2 - 8x + 11$

59. **Profit.** A company's weekly profit, P, is related to the number of items sold by $P(x) = -0.3x^2 + 60x - 400$. Find the number of items that should be sold each week in order to maximize the profit. Then find the amount of that weekly profit.

60. **Profit.** A company's monthly profit, P, is related to the number of items sold by $P(x) = -0.2x^2 + 50x - 800$. How many items should be sold each month to obtain the largest possible profit? What is the amount of that profit?

61. **Area.** A builder wants to enclose the largest possible rectangular area with 2000 ft of fencing. What should be the dimensions of the rectangle, and what will the area of the rectangle be?

62. **Area.** A farmer wants to enclose a rectangular area along a river on three sides. If 1600 ft of fencing is to be used, what dimensions will give the maximum enclosed area? Find that maximum area.

63. **Motion.** A ball is thrown upward into the air with an initial velocity of 96 ft/s. If h gives the height of the ball at time t, then the equation relating h and t is

$$h(t) = -16t^2 + 96t$$

Find the maximum height the ball will attain.

64. **Motion.** A ball is thrown upward into the air with an initial velocity of 64 ft/s. If h gives the height of the ball at time t, then the equation relating h and t is

$$h(t) = -16t^2 + 64t$$

Find the maximum height the ball will attain.

65. **Motion.** A ball is thrown upward with an initial velocity v. After 3 s, it attains a height of 120 ft. Find the initial velocity using the equation

$$h(t) = -16t^2 + v \cdot t$$

For each of the following quadratic functions, use your graphing calculator to determine (a) the vertex of the parabola and (b) the range of the function.

66. $f(x) = 2(x - 3)^2 + 1$

67. $g(x) = 3(x + 4)^2 + 2$

68. $f(x) = -(x - 1)^2 + 2$

69. $g(x) = -(x + 2)^2 - 1$

70. $f(x) = 3(x + 1)^2 - 2$

71. $g(x) = -2(x - 4)^2$

72. Explain how to determine the domain and range of the function $f(x) = a(x - h)^2 + k$.

In Exercises 73 to 76, describe a viewing window that would include the vertex and all intercepts for the graph of each function.

73. $f(x) = 3x^2 - 25$

74. $f(x) = 9x^2 - 5x - 7$

75. $f(x) = -2x^2 + 5x - 7$

76. $f(x) = -5x^2 + 2x + 7$