OBJECTIVES

1. Use the notation of inequalities
2. Graph the solution set of an inequality
3. Solve an inequality and graph the solution set

As pointed out in the introduction to this chapter, an equation is just a statement that two expressions are equal. In algebra, an inequality is a statement that one expression is less than or greater than another. Four new symbols are used in writing inequalities. The use of two of them is illustrated in Example 1.

Example 1

Reading the Inequality Symbol

5 < 8 is an inequality read “5 is less than 8.”
9 > 6 is an inequality read “9 is greater than 6.”

CHECK YOURSELF 1

Fill in the blanks, using the symbols < and >.

(a) 12 _____ 8
(b) 20 _____ 25

Like an equation, an inequality can be represented by a balance scale. Note that, in each case, the inequality arrow points to the side that is “lighter.”

2x < 4x – 3

The 2x side is less than the 4x – 3 side, so it is “lighter.”

5x – 6 > 9
Just as was the case with equations, inequalities that involve variables may be either true or false depending on the value that we give to the variable. For instance, consider the inequality 

\[ x < 6 \]

Therefore 3, 5, and \(-10\) are some solutions for the inequality \(x < 6\); they make the inequality a true statement. You should see that 8 is not a solution. We call the set of all solutions the solution set for the inequality. Of course, there are many possible solutions.

Because there are so many solutions (an infinite number, in fact), we certainly do not want to try to list them all! A convenient way to show the solution set of an inequality is with the use of a number line.

### Example 2
**Solving Inequalities**

To graph the solution set for the inequality \(x < 6\), we want to include all real numbers that are “less than” 6. This means all numbers to the left of 6 on the number line. We then start at 6 and draw an arrow extending left, as shown:

![Number line with arrow extending left from 6](image)

**Note:** The open circle at 6 means that we do not include 6 in the solution set (6 is not less than itself). The colored arrow shows all the numbers in the solution set, with the arrowhead indicating that the solution set continues indefinitely to the left.

### CHECK YOURSELF 2

**Graph the solution set of** \(x < -2\).

Two other symbols are used in writing inequalities. They are used with inequalities such as

\[ x \geq 5 \quad \text{and} \quad x \leq 2 \]

Here \(x \geq 5\) is really a combination of the two statements \(x > 5\) and \(x = 5\). It is read “\(x\) is greater than or equal to 5.” The solution set includes 5 in this case.

The inequality \(x \leq 2\) combines the statements \(x < 2\) and \(x = 2\). It is read “\(x\) is less than or equal to 2.”

### Example 3
**Graphing Inequalities**

The solution set for \(x \geq 5\) is graphed as follows:

![Number line with filled circle at 5](image)

**Note:** Here the filled-in circle means that we want to include 5 in the solution set. This is often called a closed circle.
Graph the solution sets.

(a) \( x \leq -4 \)  
(b) \( x \geq 3 \)

You have learned how to graph the solution sets of some simple inequalities, such as \( x < 8 \) or \( x \geq 10 \). Now we will look at more complicated inequalities, such as \( 2x - 3 < x + 4 \). This is called a **linear inequality in one variable**. Only one variable is involved in the inequality, and it appears only to the first power. Fortunately, the methods used to solve this type of inequality are very similar to those we used earlier in this chapter to solve linear equations in one variable. Here is our first property for inequalities.

**Rules and Properties: The Addition Property of Inequality**

If \( a < b \) then \( a + c < b + c \)

In words, adding the same quantity to both sides of an inequality gives an equivalent inequality.

Again, we can use the idea of a balance scale to see the significance of this property. If we add the same weight to both sides of an unbalanced scale, it stays unbalanced.

**Example 4**

**Solving Inequalities**

Solve and graph the solution set for \( x - 8 < 7 \).

To solve \( x - 8 < 7 \), add 8 to both sides of the inequality by the addition property.

\[
x - 8 < 7
\]
\[
x + 8 + 8
\]
\[
x < 15 \quad \text{(The inequality is solved)}
\]
The graph of the solution set is

\[ 0 \quad 7 \]

**Example 5**

**Solving Inequalities**

Solve and graph the solution set for \(4x - 2 \geq 3x + 5\).

First, we add \(-3x\) to both sides of the inequality.

\[
\begin{align*}
4x - 2 & \geq 3x + 5 \\
-3x & \quad -3x \\
x - 2 & \geq 5 \\
+2 & \quad +2 \\
x & \geq 7
\end{align*}
\]

Now we add 2 to both sides.

The graph of the solution set is

\[ 0 \quad 7 \]

**CHECK YOURSELF 4**

Solve and graph the solution set for \(x - 9 > -3\).

**CHECK YOURSELF 5**

Solve and graph the solution set.

\(7x - 8 \leq 6x + 2\)

You will also need a rule for multiplying on both sides of an inequality. Here you’ll have to be a bit careful. There is a difference between the multiplication property for inequalities and that for equations. Look at the following:

\[ 2 < 7 \quad \text{(A true inequality)} \]

Let’s multiply both sides by 3.

\[
\begin{align*}
2 & < 7 \\
3 \cdot 2 & < 3 \cdot 7 \\
6 & < 21 \quad \text{(A true inequality)}
\end{align*}
\]

Now we multiply both sides by \(-3\).

\[
\begin{align*}
2 & < 7 \\
(-3)(2) & < (-3)(7) \\
-6 & < -21 \quad \text{(Not a true inequality)}
\end{align*}
\]

Let’s try something different.

\[
\begin{align*}
2 & < 7 \\
(-3)(2) & > (-3)(7) \\
-6 & > -21
\end{align*}
\]

Change the “sense” of the inequality:

\(< \text{ becomes } >\)

(This is now a true inequality.)
This suggests that multiplying both sides of an inequality by a negative number changes the “sense” of the inequality.

We can state the following general property.

**Rules and Properties:** The Multiplication Property of Inequality

If \( a < b \) then \( ac < bc \) when \( c > 0 \)

and \( ac > bc \) when \( c < 0 \)

In words, multiplying both sides of an inequality by the same positive number gives an equivalent inequality.

When both sides of an inequality are multiplied by the same negative number, it is necessary to reverse the sense of the inequality to give an equivalent inequality.

**Example 6**

Solving and Graphing Inequalities

(a) Solve and graph the solution set for \( 5x < 30 \).

Multiplying both sides of the inequality by \( \frac{1}{5} \) gives

\[
\frac{1}{5}(5x) < \frac{1}{5}(30)
\]

Simplifying, we have

\( x < 6 \)

The graph of the solution set is

![Graph of solution set](image)

(b) Solve and graph the solution set for \( -4x \geq 28 \).

In this case we want to multiply both sides of the inequality by \( -\frac{1}{4} \) to leave \( x \) alone on the left.

\[
\left(-\frac{1}{4}\right)(-4x) \leq \left(-\frac{1}{4}\right)(28)
\]

or \( x \leq -7 \)

The graph of the solution set is

![Graph of solution set](image)

**CHECK YOURSELF 6**

Solve and graph the solution sets:

(a) \( 7x > 35 \)  
(b) \( -8x \leq 48 \)
Example 7 illustrates the use of the multiplication property when fractions are involved in an inequality.

**Example 7**

**Solving and Graphing Inequalities**

(a) Solve and graph the solution set for

\[ \frac{x}{4} > 3 \]

Here we multiply both sides of the inequality by 4. This will isolate \( x \) on the left.

\[ 4 \left( \frac{x}{4} \right) > 4(3) \]

\[ x > 12 \]

The graph of the solution set is [Graph showing the interval (12, \( \infty \))]

(b) Solve and graph the solution set for

\[ \frac{-x}{6} \geq -3 \]

In this case, we multiply both sides of the inequality by \(-6\):

\[ (-6) \left( \frac{-x}{6} \right) \leq (-6)(-3) \]

\[ x \leq 18 \]

The graph of the solution set is [Graph showing the interval (-\( \infty \), 18]]

**NOTE** Note that we reverse the sense of the inequality because we are multiplying by a negative number.

---

**CHECK YOURSELF 7**

Solve and graph the solution sets for the following inequalities.

(a) \[ \frac{x}{5} \leq 4 \]

(b) \[ \frac{-x}{3} < -7 \]

---

**Example 8**

**Solving and Graphing Inequalities**

(a) Solve and graph the solution set for \( 5x - 3 < 2x \).

First, add 3 to both sides to undo the subtraction on the left.

\[ 5x - 3 < 2x \]

\[ +3 \quad +3 \]

\[ 5x < 2x + 3 \quad \text{Add 3 to both sides to undo the subtraction.} \]

Now add \(-2x\), so that only the number remains on the right.

\[ 5x \quad < \quad 2x + 3 \]

\[ +(-2x) \quad +(-2x) \]

\[ 3x \quad < \quad 3 \quad \text{Add} -2x \text{ to isolate the number on the right.} \]
Next divide both sides by 3.

\[
\begin{align*}
3x & \leq 3 \\
\frac{3x}{3} & \leq \frac{3}{3} \\
x & \leq 1
\end{align*}
\]

The graph of the solution set is

(b) Solve and graph the solution set for \(2 - 5x < 7\).

\[
\begin{align*}
2 - 5x & < 7 \\
-2 & \quad \text{Add} \ -2. \\
-5x & < 5 \\
\frac{-5x}{-5} & > \frac{5}{-5} \\
x & > -1
\end{align*}
\]

The graph is

CHECK YOURSELF 8

Solve and graph the solution sets.

(a) \(4x + 9 \geq x\)  
(b) \(5 - 6x < 41\)

As with equations, we will collect all variable terms on one side and all constant terms on the other.

Example 9

Solving and Graphing Inequalities

Solve and graph the solution set for \(5x - 5 \geq 3x + 4\).

\[
\begin{align*}
5x - 5 & \geq 3x + 4 \\
+5 & \quad \text{Add} \ 5. \\
5x & \geq 3x + 9 \\
-3x & \quad \text{Add} \ -3x. \\
2x & \geq 9 \\
\frac{2x}{2} & \geq \frac{9}{2} \\
x & \geq \frac{9}{2}
\end{align*}
\]

The graph of the solution set is

\[
\begin{array}{c}
\text{0} \\
\frac{9}{2}
\end{array}
\]
Example 10

Solving and Graphing Inequalities

Solve and graph the solution set for \( 2x + 4 < 5x - 2 \).

\[
\begin{align*}
2x + 4 &< 5x - 2 \\
-4 &< 5x - 6 \\
-5x &< -5x \\
-3x &< -6 \\
\div -3 &> -3 \\
x &> 2
\end{align*}
\]

The graph of the solution set is

\[\text{CHECK YOURSELF 10} \]

Solve and graph the solution set.

\( 5x + 12 \geq 10x - 8 \)

The solution of inequalities may also require the use of the distributive property.

Example 11

Solving and Graphing Inequalities

Solve and graph the solution set for

\( 5(x - 2) \geq -8 \)

Applying the distributive property on the left yields

\( 5x - 10 \geq -8 \)

Solving as before yields

\[
\begin{align*}
5x - 10 &\geq -8 \\
5x &\geq 2 \\
x &\geq \frac{2}{5}
\end{align*}
\]

or \( x \geq \frac{2}{5} \)
The graph of the solution set is

**CHECK YOURSELF 11**

Solve and graph the solution set.

\[ 4(x + 3) < 9 \]

Some applications are solved by using an inequality instead of an equation. Example 12 illustrates such an application.

**Example 12**

**Solving an Application with Inequalities**

Mohammed needs a mean score of 92 or higher on four tests to get an A. So far his scores are 94, 89, and 88. What score on the fourth test will get him an A?

**Step 1** We are looking for the score that will, when combined with the other scores, give Mohammed an A.

**Step 2** Let \( x \) represent a fourth-test score that will get him an A.

**Step 3** The inequality will have the mean on the left side, which must be greater than or equal to the 92 on the right.

\[
\frac{94 + 89 + 88 + x}{4} \geq 92
\]

**Step 4** First, multiply both sides by 4:

\[
94 + 89 + 88 + x \geq 368
\]

Then add the test scores:

\[
183 + 88 + x \geq 368
\]

\[
271 + x \geq 368
\]

Subtracting 271 from both sides,

\[
x \geq 97
\]
Step 5  To check the solution, we find the mean of the four test scores, 94, 89, 88, and 97.

\[
\frac{94 + 89 + 88 + 97}{4} = \frac{368}{4} = 92
\]

CHECK YOURSELF 12

Felicia needs a mean score of at least 75 on five tests to get a passing grade in her health class. On her first four tests she has scores of 68, 79, 71, and 70. What score on the fifth test will give her a passing grade?

The following outline (or algorithm) summarizes our work in this section.

Step by Step:  Solving Linear Inequalities

Step 1  Remove any grouping symbols and combine any like terms appearing on either side of the inequality.

Step 2  Apply the addition property to write an equivalent inequality with the variable term on one side of the inequality and the number on the other.

Step 3  Apply the multiplication property to write an equivalent inequality with the variable isolated on one side of the inequality. Be sure to reverse the sense of the inequality if you multiply or divide by a negative number. The set of solutions derived in step 3 can then be graphed on a number line.

CHECK YOURSELF ANSWERS

1. (a) >; (b) <

2. -2

3. (a) -4

(b) 0

4. x > 6

5. x \leq 10

6. (a) x > 5

(b) x \geq -6

7. (a) x \leq 20

(b) x > 21

8. (a) x \geq -3

(b) x > -6

9. x < -4

10. x \leq 4

11. x < -\frac{3}{4}

12. 87 or greater
2.7 Exercises

Complete the statements, using the symbol < or >.

1. $5 \underline{\quad} 10$
2. $9 \underline{\quad} 8$

3. $7 \underline{\quad} -2$
4. $0 \underline{\quad} -5$

5. $0 \underline{\quad} 4$
6. $-10 \underline{\quad} -5$

7. $-2 \underline{\quad} -5$
8. $-4 \underline{\quad} -11$

Write each inequality in words.

9. $x < 3$

10. $x \leq -5$

11. $x \geq -4$

12. $x < -2$

13. $-5 \leq x$

14. $2 < x$

Graph the solution set of each of the following inequalities.

15. $x > 2$

16. $x < -3$

17. $x < 9$

18. $x > 4$

19. $x > 1$

20. $x < -2$

21. $x < 8$

22. $x > 3$

23. $x > -5$

24. $x < -4$

ANSWERS

1. 

2. 

3. 

4. 

5. 

6. 

7. 

8. 

9. 

10. 

11. 

12. 

13. 

14. 

15. 

16. 

17. 

18. 

19. 

20. 

21. 

22. 

23. 

24. 

231
Solve and graph the solution set of each of the following inequalities.

25. $x \geq 9$

26. $x \geq 0$

27. $x < 0$

28. $x \leq -3$

29. $x - 7 < 6$

30. $x + 5 \leq 4$

31. $x + 8 \geq 10$

32. $x - 11 > -14$

33. $5x < 4x + 7$

34. $3x \geq 2x - 4$

35. $6x - 8 \leq 5x$

36. $3x + 2 > 2x$

37. $4x - 3 \geq 3x + 5$

38. $5x + 2 \leq 4x - 6$

39. $7x + 5 < 6x - 4$

40. $8x - 7 > 7x + 3$

41. $3x \leq 9$

42. $5x > 20$

43. $5x > -35$

44. $7x \leq -21$

45. $-6x \geq 18$

46. $-9x < 45$

47. $-10x < -60$

48. $-12x \geq -48$
49. \( \frac{x}{4} > 5 \)
50. \( \frac{x}{3} \leq -3 \)
51. \( -\frac{x}{2} \geq -3 \)
52. \( -\frac{x}{5} < 4 \)
53. \( \frac{2x}{3} < 6 \)
54. \( \frac{3x}{4} \geq -9 \)
55. \( 5x > 3x + 8 \)
56. \( 4x \leq x - 9 \)
57. \( 5x - 2 > 3x \)
58. \( 7x + 3 \equiv 2x \)
59. \( 3 - 2x > 5 \)
60. \( 5 - 3x \leq 17 \)
61. \( 2x \equiv 5x + 18 \)
62. \( 3x < 7x - 28 \)
63. \( 5x - 3 \leq 3x + 15 \)
64. \( 8x + 7 > 5x + 34 \)
65. \( 9x + 7 > 2x - 28 \)
66. \( 10x - 5 \leq 8x - 25 \)
67. \( 7x - 5 < 3x + 2 \)
68. \( 5x - 2 \equiv 2x - 7 \)
69. \( 5x + 7 > 8x - 17 \)
70. \( 4x - 3 \leq 9x + 27 \)
71. \( 3x - 2 \leq 5x + 3 \)
72. \( 2x + 3 > 8x - 2 \)
Translate the following statements into inequalities. Let $x$ represent the number in each case.

77. 5 more than a number is greater than 3.

78. 3 less than a number is less than or equal to 5.

79. 4 less than twice a number is less than or equal to 7.

80. 10 more than a number is greater than negative 2.

81. 4 times a number, decreased by 15, is greater than that number.

82. 2 times a number, increased by 28, is less than or equal to 6 times that number.

Match each inequality on the right with a statement on the left.

83. $x$ is nonnegative  
   a. $x \geq 0$

84. $x$ is negative  
   b. $x \geq 5$

85. $x$ is no more than 5  
   c. $x \leq 5$

86. $x$ is positive  
   d. $x > 0$

87. $x$ is at least 5  
   e. $x < 5$

88. $x$ is less than 5  
   f. $x < 0$

89. **Panda population.** There are fewer than 1000 wild giant pandas left in the bamboo forests of China. Write an inequality expressing this relationship.
90. **Forestry.** Let \( C \) represent the amount of Canadian forest and \( M \) represent the amount of Mexican forest. Write an inequality showing the relationship of the forests of Mexico and Canada if Canada contains at least 9 times as much forest as Mexico.

\[ C \geq 9M \]

91. **Test scores.** To pass a course with a grade of B or better, Liza must have an average of 80 or more. Her grades on three tests are 72, 81, and 79. Write an inequality representing the score that Liza must get on the fourth test to obtain a B average or better for the course.

\[ \frac{72 + 81 + 79 + x}{4} \geq 80 \]

92. **Test scores.** Sam must have an average of 70 or more in his summer course to obtain a grade of C. His first three test grades were 75, 63, and 68. Write an inequality representing the score that Sam must get in the last test to get a C grade.

\[ \frac{75 + 63 + 68 + x}{4} \geq 70 \]

93. **Commission.** Juanita is a salesperson for a manufacturing company. She may choose to receive $500 or 5 percent commission on her sales as payment for her work. How much does she need to sell to make the 5 percent offer a better deal?

\[ \frac{0.05S}{500} = 0.5 \]

94. **Telephone costs.** The cost for a long distance telephone call is $0.36 for the first minute and $0.21 for each additional minute or portion thereof. The total cost of the call cannot exceed $3. Write an inequality representing the number of minutes a person could talk without exceeding $3.

\[ 0.36 + 0.21(x - 1) \leq 3 \]

95. **Geometry.** The perimeter of a rectangle is to be no greater than 250 centimeters (cm) and the length must be 105 cm. Find the maximum width of the rectangle.

96. **Recreation.** Sarah bowled 136 and 189 in her first two games. What must she bowl in her third game to have an average of at least 170?

\[ \frac{136 + 189 + x}{3} \geq 170 \]

97. You are the office manager for a small company. You need to acquire a new copier for the office. You find a suitable one that leases for $250 a month from the copy machine company. It costs 2.5¢ per copy to run the machine. You purchase paper for $3.50 a ream (500 sheets). If your copying budget is no more than $950 per month, is this machine a good choice? Write a brief recommendation to the Purchasing Department. Use equations and inequalities to explain your recommendation.
98. Your aunt calls to ask your help in making a decision about buying a new refrigerator. She says that she found two that seem to fit her needs, and both are supposed to last at least 14 years, according to Consumer Reports. The initial cost for one refrigerator is $712, but it only uses 88 kilowatt-hours (kWh) per month. The other refrigerator costs $519 and uses an estimated 100 kWh per month. You do not know the price of electricity per kilowatt-hour where your aunt lives, so you will have to decide what in cents per kilowatt-hour will make the first refrigerator cheaper to run for its 14 years of expected usefulness. Write your aunt a letter explaining what you did to calculate this cost, and tell her to make her decision based on how the kilowatt-hour rate she has to pay in her area compares with your estimation.