3.2 Negative Exponents and Scientific Notation

OBJECTIVES

1. Evaluate expressions involving zero or a negative exponent
2. Simplify expressions involving zero or a negative exponent
3. Write a decimal number in scientific notation
4. Solve an application of scientific notation

In Section 3.1, we discussed exponents. We now want to extend our exponent notation to include 0 and negative integers as exponents.

First, what do we do with \( x^0 \)? It will help to look at a problem that gives us \( x^0 \) as a result. What if the numerator and denominator of a fraction have the same base raised to the same power and we extend our division rule? For example,

\[
\frac{a^5}{a^{10}} = a^{5-10} = a^{-5}
\]

But from our experience with fractions we know that

\[
\frac{a^5}{a^7} = \frac{1}{a^2}
\]

By comparing equations (1) and (2), it seems reasonable to make the following definition:

**Definitions: Zero Power**

For any number \( a, a \neq 0 \),

\[ a^0 = 1 \]

In words, any expression, except 0, raised to the 0 power is 1.

Example 1 illustrates the use of this definition.

**Example 1**

Raising Expressions to the Zero Power

Evaluate. Assume all variables are nonzero.

(a) \( 5^0 = 1 \)

(b) \( 27^0 = 1 \)

(c) \( (x^2y)^0 = 1 \) if \( x \neq 0 \) and \( y \neq 0 \)

(d) \( 6x^0 = 6 \cdot 1 = 6 \) if \( x \neq 0 \)

**CHECK YOURSELF 1**

Evaluate. Assume all variables are nonzero.

(a) \( 7^0 \)  
(b) \( (-8)^0 \)  
(c) \( (xy^3)^0 \)  
(d) \( 3x^0 \)
The second property of exponents allows us to define a negative exponent. Suppose that the exponent in the denominator is greater than the exponent in the numerator. Consider the expression \( \frac{x^2}{x^3} \).

Our previous work with fractions tells us that

\[
\frac{x^2}{x^3} = \frac{x \cdot x}{x \cdot x \cdot x} = \frac{1}{x}\normalsize{^1}
\]

(1)

However, if we extend the second property to let \( n \) be greater than \( m \), we have

\[
\frac{x^2}{x^3} = x^{2-3} = x^{-1}
\]

(2)

Now, by comparing equations (1) and (2), it seems reasonable to define \( x^{-3} \) as \( \frac{1}{x^3} \).

In general, we have this result:

**Definitions: Negative Powers**

For any number \( a \), \( a \neq 0 \), and any positive integer \( n \),

\[
a^{-n} = \frac{1}{a^n}
\]

**NOTE** John Wallis (1616–1703), an English mathematician, was the first to fully discuss the meaning of 0 and negative exponents.

### Example 2

**Rewriting Expressions That Contain Negative Exponents**

Rewrite each expression, using only positive exponents.

(a) \( x^{-4} = \frac{1}{x^4} \)

(b) \( m^{-7} = \frac{1}{m^7} \)

(c) \( 3^{-2} = \frac{1}{3^2} \) or \( \frac{1}{9} \)

(d) \( 10^{-3} = \frac{1}{10^3} \) or \( \frac{1}{1000} \)

(e) \( 2x^{-3} = 2 \cdot \frac{1}{x^3} = \frac{2}{x^3} \)

The \( -3 \) exponent applies only to \( x \), because \( x \) is the base.

(f) \( \frac{a^5}{a^9} = a^{5-9} = a^{-4} = \frac{1}{a^4} \)

(g) \( -4x^{-5} = -4 \cdot \frac{1}{x^5} = -\frac{4}{x^5} \)
We will now allow negative integers as exponents in our first property for exponents. Consider Example 3.

Example 3
Simplifying Expressions Containing Exponents

Simplify (write an equivalent expression that uses only positive exponents).

(a) \(a^{-10}\)

(b) \(4^{-3}\)

(c) \(3x^{-2}\)

(d) \(\frac{x^5}{x^2}\)

We will now allow negative integers as exponents in our first property for exponents. Consider Example 3.

Example 3
Simplifying Expressions Containing Exponents

Simplify (write an equivalent expression that uses only positive exponents).

(a) \(x^5x^{-2} = x^{5+(-2)} = x^3\)

Note: An alternative approach would be

\[ x^5x^{-2} = \frac{x^5}{x^2} = \frac{x}{x^2} = x^3 \]

(b) \(a^7a^{-5} = a^{7+(-5)} = a^2\)

(c) \(y^5y^{-9} = y^{5+(-9)} = y^{-4} = \frac{1}{y^4}\)

Example 4 shows that all the properties of exponents introduced in the last section can be extended to expressions with negative exponents.

Example 4
Simplifying Expressions Containing Exponents

Simplify each expression.

(a) \(\frac{m^{-3}}{m^4} = m^{-3-4} = m^{-7}\)

(b) \(\frac{a^{-2}b^6}{a^5b^{-2}} = a^{-2-5}b^{6-(-4)} = a^{-7}b^{10} = \frac{b^{10}}{a^7}\)
Simplify each expression.

(a) \(3a^3\) \(11002\) \(4\) 
(b) \(r^3\) \(11002\) \(2\) 
(c) \(m^3n^{-5}\) \(m^{-2}n^3\) 
(d) \((y^3)^{-2}\) 

\[ (2x^3)^{-3} = \left( \frac{1}{2x^3} \right)^3 \quad \text{Definition of the negative exponent} \]
\[ = \frac{1}{8x^9} \quad \text{Property 3} \]
\[ = \frac{1}{8x^9} \quad \text{Property 4} \]
\[ = y^{-2} = \frac{1}{y^2} \quad \text{Property 2} \]

**CHECK YOURSELF 4**

Simplify each expression.

(\(x^5\) \(x^{-3}\) 
(\(m^3n^{-5}\) \(m^{-2}n^3\) 
(\(3a^3\)^{-4} 
(\((r^3)^{-2}\) 

Let us now take a look at an important use of exponents, scientific notation. We begin the discussion with a calculator exercise. On most calculators, if you multiply 2.3 times 1000, the display will read 2300.

Multiply by 1000 a second time. Now you will see 2300000.

Multiplying by 1000 a third time will result in the display 2 300 000.

This must equal 2,300,000,000.

**NOTE** Consider the following table:

<table>
<thead>
<tr>
<th>Number</th>
<th>Scientific Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3</td>
<td>(2.3 \times 10^0)</td>
</tr>
<tr>
<td>23</td>
<td>(2.3 \times 10^1)</td>
</tr>
<tr>
<td>230</td>
<td>(2.3 \times 10^2)</td>
</tr>
<tr>
<td>2300</td>
<td>(2.3 \times 10^3)</td>
</tr>
<tr>
<td>23,000</td>
<td>(2.3 \times 10^4)</td>
</tr>
<tr>
<td>230,000</td>
<td>(2.3 \times 10^5)</td>
</tr>
</tbody>
</table>

Can you see what is happening? This is the way calculators display very large numbers. The number on the left is always between 1 and 10, and the number on the right indicates the number of places the decimal point must be moved to the right to put the answer in standard (or decimal) form.

This notation is used frequently in science. It is not uncommon in scientific applications of algebra to find yourself working with very large or very small numbers. Even in the time of Archimedes (287–212 B.C.E.), the study of such numbers was not unusual. Archimedes estimated that the universe was 23,000,000,000,000,000 m in diameter, which is the approximate distance light travels in \(2\frac{1}{2}\) years. By comparison, Polaris (the North Star) is actually 680 light-years from the earth. Example 6 will discuss the idea of light-years.
In scientific notation, Archimedes’s estimate for the diameter of the universe would be

\[ 2.3 \times 10^{16} \text{ m} \]

In general, we can define scientific notation as follows.

**Definitions: Scientific Notation**

Any number written in the form

\[ a \times 10^n \]

in which \( 1 \leq a < 10 \) and \( n \) is an integer, is written in scientific notation.

**Example 5**

Using Scientific Notation

Write each of the following numbers in scientific notation.

(a) \( 120,000,000,000,000,000 = 1.2 \times 10^5 \)

(b) \( 88,000,000 = 8.8 \times 10^7 \)

(c) \( 520,000,000 = 5.2 \times 10^8 \)

(d) \( 4,000,000,000 = 4 \times 10^9 \)

(e) \( 0.0005 = 5 \times 10^{-4} \)

(f) \( 0.0000000081 = 8.1 \times 10^{-9} \)

**Check Yourself 5**

Write in scientific notation.

(a) \( 212,000,000,000,000,000 \)

(b) \( 0.00079 \)

(c) \( 5,600,000 \)

(d) \( 0.0000007 \)
Example 6

An Application of Scientific Notation

(a) Light travels at a speed of $3.05 \times 10^8$ meters per second (m/s). There are approximately $3.15 \times 10^7$ s in a year. How far does light travel in a year?

We multiply the distance traveled in 1 s by the number of seconds in a year. This yields

$$(3.05 \times 10^8)(3.15 \times 10^7) = (3.05 \cdot 3.15)(10^8 \cdot 10^7) = 9.6075 \times 10^{15}$$

For our purposes we round the distance light travels in 1 year to $10^{16}$ m. This unit is called a **light-year**, and it is used to measure astronomical distances.

(b) The distance from earth to the star Spica (in Virgo) is $2.2 \times 10^{18}$ m. How many light-years is Spica from earth?

![Diagram of Earth and Spica](image)

For our purposes we round the distance light travels in 1 year to $10^{16}$ m. This unit is called a **light-year**, and it is used to measure astronomical distances.

$$\frac{2.2 \times 10^{18}}{10^{16}} = 2.2 \times 10^{18-16} = 2.2 \times 10^2 = 220 \text{ light-years}$$

NOTE We divide the distance (in meters) by the number of meters in 1 light-year.

CHECK YOURSELF 6

The farthest object that can be seen with the unaided eye is the Andromeda galaxy. This galaxy is $2.3 \times 10^{22}$ m from earth. What is this distance in light-years?

CHECK YOURSELF ANSWERS

1. (a) 1; (b) 1; (c) 1; (d) 3
2. (a) $\frac{1}{a^5}$; (b) $\frac{1}{4^3}$ or $\frac{1}{64}$; (c) $\frac{3}{x^3}$; (d) $\frac{1}{x^3}$
3. (a) $x^5$; (b) $\frac{1}{b^3}$
4. (a) $x^5$; (b) $\frac{m^3}{n^8}$; (c) $\frac{1}{81a^{12}}$; (d) $r^2$
5. (a) $2.12 \times 10^{13}$; (b) $7.9 \times 10^{-4}$; (c) $5.6 \times 10^4$; (d) $7 \times 10^{-7}$
6. 2,300,000 light-years
### Exercises

Evaluate (assume the variables are nonzero).

1. $4^0$
2. $(-7)^0$
3. $(-29)^0$
4. $75^0$
5. $(x^3y^2)^0$
6. $7m^0$
7. $11x^0$
8. $(2a^3b^7)^0$
9. $(-3p^6q^8)^0$
10. $-7x^0$

Write each of the following expressions using positive exponents; simplify when possible.

11. $b^{-8}$
12. $p^{-12}$
13. $3^{-4}$
14. $2^{-5}$
15. $5^{-2}$
16. $4^{-3}$
17. $10^{-4}$
18. $10^{-5}$
19. $5x^{-1}$
20. $3a^{-2}$
21. $(5x)^{-1}$
22. $(3a)^{-2}$
23. $-2x^{-5}$
24. $3x^{-4}$
25. $(-2x)^{-5}$
26. $(3x)^{-4}$

Use Properties 1 and 2 to simplify each of the following expressions. Write your answers with positive exponents only.

27. $a^5a^3$
28. $m^5m^7$
29. $x^8x^{-2}$
30. $a^{12}a^{-8}$
31. $b^7b^{-11}$
32. $y^5y^{-12}$
33. $x^0x^5$
34. $r^{-3}r^0$
35. $\frac{a^8}{a^7}$
Simplify each of the following expressions. Write your answers with positive exponents only.

36. \( \frac{m^9}{m^3} \)  
37. \( \frac{x^7}{x^9} \)  
38. \( \frac{a^3}{a^{10}} \)  
39. \( \frac{r^{-3}}{r} \)  
40. \( \frac{x^3}{x^5} \)  
41. \( \frac{x^{-4}}{x^5} \)  
42. \( \frac{p^{-6}}{p^3} \)

In exercises 55 to 58, express each number in scientific notation.

55. The distance from the earth to the sun: 93,000,000 mi.

56. The diameter of a grain of sand: 0.000021 m.

57. The diameter of the sun: 130,000,000,000 cm.

58. The number of molecules in 22.4 L of a gas: 602,000,000,000,000,000,000,000 (Avogadro’s number).

59. The mass of the sun is approximately \( 1.98 \times 10^{30} \) kg. If this were written in standard or decimal form, how many 0s would follow the digit 8?
60. Archimedes estimated the universe to be $2.3 \times 10^{19}$ millimeters (mm) in diameter. If this number were written in standard or decimal form, how many 0s would follow the digit 3?

In exercises 61 to 64, write each expression in standard notation.

61. $8 \times 10^{-3}$
62. $7.5 \times 10^{-6}$
63. $2.8 \times 10^{-5}$
64. $5.21 \times 10^{-4}$

In exercises 65 to 68, write each of the following in scientific notation.

65. 0.0005
66. 0.000003
67. 0.00037
68. 0.000051

In exercises 69 to 72, compute the expressions using scientific notation, and write your answer in that form.

69. $(4 \times 10^{-3})(2 \times 10^{-5})$
70. $(1.5 \times 10^{-6})(4 \times 10^{2})$
71. $\frac{9 \times 10^{3}}{3 \times 10^{-2}}$
72. $\frac{7.5 \times 10^{-4}}{1.5 \times 10^{3}}$

In exercises 73 to 78, perform the indicated calculations. Write your result in scientific notation.

73. $(2 \times 10^{5})(4 \times 10^{4})$
74. $(2.5 \times 10^{7})(3 \times 10^{5})$
75. $\frac{6 \times 10^{9}}{3 \times 10^{7}}$
76. $\frac{4.5 \times 10^{12}}{1.5 \times 10^{7}}$
77. $\frac{(3.3 \times 10^{15})(6 \times 10^{15})}{(1.1 \times 10^{9})(3 \times 10^{9})}$
78. $\frac{(6 \times 10^{12})(3.2 \times 10^{5})}{(1.6 \times 10^{6})(3 \times 10^{2})}$

In 1975 the population of Earth was approximately 4 billion and doubling every 35 years. The formula for the population $P$ in year $Y$ for this doubling rate is

$$P \text{ (in billions)} = 4 \times 2^{(Y-1975)/35}$$

79. What was the approximate population of Earth in 1960?

80. What will Earth’s population be in 2025?

The United States population in 1990 was approximately 250 million, and the average growth rate for the past 30 years gives a doubling time of 66 years. The above formula for the United States then becomes

$$P \text{ (in millions)} = 250 \times 2^{(Y-1990)/66}$$

81. What was the approximate population of the United States in 1960?

82. What will be the population of the United States in 2025 if this growth rate continues?
83. Megrez, the nearest of the Big Dipper stars, is $6.6 \times 10^{17}$ m from Earth. Approximately how long does it take light, traveling at $10^{16}$ m/year, to travel from Megrez to Earth?

84. Alkaid, the most distant star in the Big Dipper, is $2.1 \times 10^{18}$ m from Earth. Approximately how long does it take light to travel from Alkaid to Earth?

85. The number of liters (L) of water on Earth is $15,500$ followed by 19 zeros. Write this number in scientific notation. Then use the number of liters of water on Earth to find out how much water is available for each person on Earth. The population of Earth is 6 billion.

86. If there are $6 \times 10^9$ people on Earth and there is enough freshwater to provide each person with $8.79 \times 10^5$ L, how much freshwater is on Earth?

87. The United States uses an average of $2.6 \times 10^6$ L of water per person each year. The United States has $3.2 \times 10^8$ people. How many liters of water does the United States use each year?

Getting Ready for Section 3.3 [Section 1.6]

Combine like terms where possible.

(a) $8m + 7m$  
(b) $9x - 5x$

c. $9m^2 - 8m$  
(d) $8x^2 - 7x^2$

e. $5c^3 + 15c^3$  
(f) $9s^3 + 8s^3$

(g) $8c^2 - 6c + 2c^2$  
(h) $8r^3 - 7r^2 + 5r^3$

Answers

1. 1  
3. 1  
5. 1  
7. 11  
9. 1  
11. $\frac{1}{b^8}$  
13. $\frac{1}{81}$  
15. $\frac{1}{25}$  
17. $\frac{1}{10,000}$  
19. $\frac{5}{x}$  
21. $\frac{1}{5x}$  
23. $-\frac{2}{x^2}$

25. $-\frac{1}{32x^7}$  
27. $a^8$  
29. $x^6$  
31. $\frac{1}{b^4}$  
33. $x^5$  
35. $a^3$  
37. $\frac{1}{x^2}$

39. $\frac{1}{r^3}$  
41. $x$  
43. $\frac{m^6}{n^3}$  
45. $\frac{16}{d^{12}}$  
47. $\frac{y^4}{y^6}$  
49. $\frac{1}{r^2}$

51. $\frac{1}{x}$  
53. $\frac{1}{a^{11}}$  
55. $9.3 \times 10^7$ mi  
57. $1.3 \times 10^{11}$ cm  
59. 28  
61. 0.008

63. 0.000028  
65. $5 \times 10^{-4}$  
67. $3.7 \times 10^{-4}$  
69. $8 \times 10^{-8}$

71. $3 \times 10^5$  
73. $8 \times 10^9$  
75. $2 \times 10^2$  
77. $6 \times 10^{16}$

79. 2.97 billion  
81. 182 million  
83. 66 years

85. $1.55 \times 10^{23}$ L; $2.58 \times 10^{13}$ L  
87. $8.32 \times 10^{14}$ L  

a. $15m$  

b. $4x$

c. $9m^2 - 8m$  

d. $x^2$  

e. $20c^3$  

f. $17s^5$  
g. $10c^2 - 6c$  
h. $13r^3 - 7r^2$