4.1 An Introduction to Factoring

OBJECTIVES

1. Remove the greatest common factor (GCF)
2. Remove a binomial GCF

Overcoming Math Anxiety

Hint #5
Working Together
How many of your classmates do you know? Whether you are by nature gregarious or shy, you have much to gain by getting to know your classmates.

1. It is important to have someone to call when you have missed class or if you are unclear on an assignment.
2. Working with another person is almost always beneficial to both people. If you don’t understand something, it helps to have someone to ask about it. If you do understand something, nothing will cement that understanding more than explaining the idea to another person.
3. Sometimes we need to commiserate. If an assignment is particularly frustrating, it is reassuring to find that it is also frustrating for other students.
4. Have you ever thought you had the right answer, but it doesn’t match the answer in the text? Frequently the answers are equivalent, but that’s not always easy to see. A different perspective can help you see that. Occasionally there is an error in a textbook (here we are talking about other textbooks). In such cases it is wonderfully reassuring to find that someone else has the same answer you do.

In Chapter 3 you were given factors and asked to find a product. We are now going to reverse the process. You will be given a polynomial and asked to find its factors. This is called factoring.

Let’s start with an example from arithmetic. To multiply $5 \cdot 7$, you write

$$5 \cdot 7 = 35$$

To factor 35, you would write

$$35 = 5 \cdot 7$$

Factoring is the reverse of multiplication.

Now let’s look at factoring in algebra. You have used the distributive property as

$$a(b + c) = ab + ac$$

For instance,

$$3(x + 5) = 3x + 15$$

To use the distributive property in factoring, we apply that property in the opposite fashion, as

$$ab + ac = a(b + c)$$

The property lets us remove the common monomial factor $a$ from the terms of $ab + ac$. To use this in factoring, the first step is to see whether each term of the polynomial has a common monomial factor. In our earlier example,

$$3x + 15 = 3 \cdot x + 3 \cdot 5$$

Common factor

NOTE 3 and $x + 5$ are the factors of $3x + 15$. 

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So, by the distributive property,
\[ 3x + 15 = 3(x + 5) \]
The original terms are each divided by the greatest common factor to determine the terms in parentheses.

To check this, multiply \(3(x + 5)\).

\[ 3(x + 5) = 3x + 15 \]

The first step in factoring is to identify the greatest common factor (GCF) of a set of terms. This is the monomial with the largest common numerical coefficient and the largest power common to any variables.

### Definitions: Greatest Common Factor

The greatest common factor (GCF) of a polynomial is the monomial with the highest degree and the largest numerical coefficient that is a factor of each term of the polynomial.

### Example 1

**Finding the GCF**

Find the GCF for each set of terms.

(a) 9 and 12 The largest number that is a factor of both is 3.
(b) 10, 25, 150 The GCF is 5.
(c) \(x^4\) and \(x^7\) The largest power that divides both terms is \(x^4\).
(d) \(12a^3\) and \(18a^2\) The GCF is \(6a^2\).

### CHECK YOURSELF 1

Find the GCF for each set of terms.

(a) 14, 24  (b) 9, 27, 81  (c) \(a^9, a^5\)  (d) \(10x^5, 35x^4\)

### Step by Step: To Factor a Monomial from a Polynomial

**Step 1** Find the greatest common factor (GCF) for all the terms.
**Step 2** Factor the GCF from each term, then apply the distributive property.
**Step 3** Mentally check your factoring by multiplication.

### Example 2

**Finding the GCF of a Binomial**

(a) Factor \(8x^2 + 12x\).

NOTE Again, factoring is the reverse of multiplication.

NOTE This diagram relates the idea of multiplication and factoring.

NOTE In fact, we will see that factoring out the GCF is the first method to try in any of the factoring problems we will discuss.

CHECK YOURSELF 1

Find the GCF for each set of terms.

(a) 14, 24  (b) 9, 27, 81  (c) \(a^9, a^5\)  (d) \(10x^5, 35x^4\)

Step by Step: To Factor a Monomial from a Polynomial

**Step 1** Find the greatest common factor (GCF) for all the terms.
**Step 2** Factor the GCF from each term, then apply the distributive property.
**Step 3** Mentally check your factoring by multiplication.
The largest common numerical factor of 8 and 12 is 4, and \( x \) is the variable factor with the largest power. So 4\( x \) is the GCF. Write

\[
8x^2 + 12x = 4x \cdot 2x + 4x \cdot 3 \quad \text{GCF}
\]

Now, by the distributive property, we have

\[
8x^2 + 12x = 4x(2x + 3)
\]

(b) Factor \( 6a^2 - 18a^2 \).

The GCF in this case is \( 6a^2 \). Write

\[
6a^4 + (-18a^2) = 6a^2 \cdot a^2 + 6a^2 \cdot (-3) \quad \text{GCF}
\]

Again, using the distributive property yields

\[
6a^4 - 18a^2 = 6a^2(a^2 - 3)
\]

You should check this by multiplying.

**CHECK YOURSELF 2**

Factor each of the following polynomials.

(a) \( 5x + 20 \)  
(b) \( 6x^2 - 24x \)  
(c) \( 10a^3 - 15a^2 \)

The process is exactly the same for polynomials with more than two terms. Consider Example 3.

**Example 3**

Finding the GCF of a Polynomial

(a) Factor \( 5x^2 - 10x + 15 \).

**NOTE** The GCF is 5.

\[
5x^2 - 10x + 15 = 5 \cdot x^2 - 5 \cdot 2x + 5 \cdot 3 \quad \text{GCF}
\]

\[
= 5(x^2 - 2x + 3)
\]

(b) Factor \( 6ab + 9ab^2 - 15a^3 \).

**NOTE** The GCF is \( 3a \).

\[
6ab + 9ab^2 - 15a^3 = 3a \cdot 2b + 3a \cdot 3b^2 - 3a \cdot 5a \quad \text{GCF}
\]

\[
= 3a(2b + 3b^2 - 5a)
\]

(c) Factor \( 4a^4 + 12a^3 - 20a^2 \).

**NOTE** The GCF is \( 4a^2 \).

\[
4a^4 + 12a^3 - 20a^2 = 4a^2 \cdot a^2 + 4a^2 \cdot 3a - 4a^2 \cdot 5 \quad \text{GCF}
\]

\[
= 4a^2(a^2 + 3a - 5)
\]
Note In each of these examples, you will want to check the result by multiplying the factors.

(d) Factor \(6a^2b + 9ab^2 + 3ab\).

Mentally note that 3, \(a\), and \(b\) are factors of each term, so

\[
6a^2b + 9ab^2 + 3ab = 3ab(2a + 3b + 1)
\]

Check Yourself 3

Factor each of the following polynomials.

(a) \(8b^2 + 16b - 32\)  
(b) \(4xy - 8x^2y + 12x^3\)  
(c) \(7x^4 - 14x^3 + 21x^2\)  
(d) \(5x^2y^2 - 10xy^2 + 15x^3y\)

We can have two or more terms that have a binomial factor in common, as is the case in Example 4.

Example 4

Finding a Common Factor

(a) Factor \(3x(x + y) + 2(x + y)\).

We see that the binomial \(x + y\) is a common factor and can be removed.

\[
3x(x + y) + 2(x + y) = (x + y) \cdot 3x + (x + y) \cdot 2 = (x + y)(3x + 2)
\]

(b) Factor \(3x^2(x - y) + 6x(x - y) + 9(x - y)\).

We note that here the GCF is \(3(x - y)\). Factoring as before, we have

\[
3(x - y)(x^2 + 2x + 3)
\]

Check Yourself 4

Completely factor each of the polynomials.

(a) \(7a(a - 2b) + 3(a - 2b)\)  
(b) \(4x^2(x + y) - 8x(x + y) - 16(x + y)\)

Check Yourself Answers

1. (a) 2; (b) 9; (c) \(a^2\); (d) \(5x^4\)  
2. (a) \(5(x + 4)\); (b) \(6x(x - 4)\); (c) \(5a^2(2a - 3)\)  
3. (a) \(8(b^2 + 2b - 4)\); (b) \(4x(y - 2xy + 3x^2)\); (c) \(7x^2(x^2 - 2x + 3)\); (d) \(5xy(xy - 2y + 3x)\)  
4. (a) \((a - 2b)(7a + 3)\); (b) \(4(x + y)(x^2 - 2x - 4)\)
4.1 Exercises

Find the greatest common factor for each of the following sets of terms.

1. 10, 12
2. 15, 35

3. 16, 32, 88
4. 55, 33, 132

5. $x^2, x^5$
6. $y^7, y^9$

7. $a^1, a^6, a^9$
8. $b^4, b^6, b^8$

9. $5x^4, 10x^5$
10. $8y^9, 24y^3$

11. $8a^4, 6a^6, 10a^{10}$
12. $9b^3, 6b^5, 12b^4$

13. $9x^2y, 12xy^2, 15x^2y^3$
14. $12a^3b^2, 18a^2b^3, 6a^4b^4$

15. $15ab^3, 10a^2bc, 25b^2c^3$
16. $9x^3, 3xy^3, 6y^3$

17. $15a^2bc^2, 9ab^2c^2, 6a^2b^2c^2$
18. $18x^3y^2z^3, 27x^4y^2z^3, 81xy^2z^3$

19. $(x + y)^2, (x + y)^3$
20. $12(a + b)^4, 4(a + b)^3$

Factor each of the following polynomials.

21. $8a + 4$
22. $5x − 15$

23. $24m − 32n$
24. $7p − 21q$

25. $12m^2 + 8m$
26. $24m^2 − 32n$
27. $10s^2 + 5s$

28. $12y^2 - 6y$

29. $12x^2 + 24x$

30. $14b^2 - 28b$

31. $15a^3 - 25a^2$

32. $36b^4 + 24b^2$

33. $6pq + 18p^2q$

34. $8ab - 24ab^2$

35. $7m^3n - 21mn^3$

36. $36p^2q^2 - 9pq$

37. $6x^2 - 18x + 30$

38. $7a^2 + 21a - 42$

39. $3a^3 + 6a^2 - 12a$

40. $5x^3 - 15x^2 + 25x$

41. $6m + 9mn - 15mn^2$

42. $4s + 6st - 14st^2$

43. $10x^2y + 15xy - 5xy^2$

44. $3ab^2 + 6ab - 15a^2b$

45. $10r^3s^2 + 25r^2s^2 - 15r^2s^3$

46. $28x^3y^3 - 35x^2y^2 + 42x^3y$

47. $9a^5 - 15a^4 + 21a^3 - 27a$

48. $8p^5 - 40p^4 + 24p^3 - 16p^2$

49. $15m^3n^3 - 20m^2n + 35mn^3 - 10mn$

50. $14ab^4 + 21a^2b^3 - 35a^3b^2 + 28ab^2$

51. $x(x - 2) + 3(x - 2)$

52. $y(y + 5) - 3(y + 5)$
53. The GCF of \(2x - 6\) is 2. The GCF of \(5x + 10\) is 5. Find the greatest common factor of the product \((2x - 6)(5x + 10)\).

54. The GCF of \(3z + 12\) is 3. The GCF of \(4z + 8\) is 4. Find the GCF of the product \((3z + 12)(4z + 8)\).

55. The GCF of \(2x^3 - 4x\) is 2x. The GCF of \(3x + 6\) is 3. Find the GCF of the product \((2x^3 - 4x)(3x + 6)\).

56. State, in a sentence, the rule that the previous three exercises illustrated.

Find the GCF for each product.

57. \((2a + 8)(3a - 6)\)

58. \((5b - 10)(2b + 4)\)

59. \((2x^2 + 5x)(7x - 14)\)

60. \((6y^2 - 3y)(y + 7)\)

61. **Area of a rectangle.** The area of a rectangle with width \(t\) is given by \(33t - t^2\). Factor the expression and determine the length of the rectangle in terms of \(t\).

62. **Area of a rectangle.** The area of a rectangle of length \(x\) is given by \(3x^2 + 5x\). Find the width of the rectangle.

63. For centuries, mathematicians have found factoring numbers into prime factors a fascinating subject. A prime number is a number that cannot be written as a product of any numbers but 1 and itself. The list of primes begins with 2 because 1 is not considered a prime number and then goes on: 3, 5, 7, 11, . . . What are the first 10 primes? What are the primes less than 100? If you list the numbers from 1 to 100 and then cross out all numbers that are multiples of 2, 3, 5, and 7, what is left? Are all the numbers not crossed out prime? Write a paragraph to explain why this might be so. You might want to investigate the sieve of Eratosthenes, a system from 230 B.C.E. for finding prime numbers.

64. If we made a list of all the prime numbers, what number would be at the end of the list? Because there are an infinite number of prime numbers, there is no “largest prime number.” But is there some formula that will give us all the primes? Here are some formulas proposed over the centuries:

\[ n^2 + n + 17 \quad 2n^2 + 29 \quad n^2 - n + 11 \]

In all these expressions, \(n = +1, 2, 3, 4, \ldots\), that is, a positive integer beginning with 1. Investigate these expressions with a partner. Do the expressions give prime numbers when they are evaluated for these values of \(n\)? Do the expressions give every prime in the range of resulting numbers? Can you put in any positive number for \(n\)?

65. How are primes used in coding messages and for security? Work together to decode the messages. The messages are coded using this code: After the numbers are factored into prime factors, the power of 2 gives the number of the letter in the alphabet. This code would be easy for a code breaker to figure out, but you might make up code that would be more difficult to break.

a. \(1310720, 229376, 1572864, 1760, 460, 2097152, 336\)

b. \(786432, 143, 4608, 278528, 1344, 98304, 1835008, 352, 4718592, 5242880\)

c. Code a message using this rule. Exchange your message with a partner to decode it.
Multiply.

(a) \((a - 1)(a + 4)\)  
(b) \((x - 1)(x + 3)\)  
(c) \((x - 3)(x - 3)\)  
(d) \((y - 11)(y + 3)\)  
(e) \((x + 5)(x + 7)\)  
(f) \((y + 1)(y - 13)\)

Answers

1. 2  
3. 8  
5. \(x^2\)  
7. \(a^3\)  
9. \(5x^4\)  
11. \(2a^4\)  
13. \(3xy\)  
15. \(5b\)  
17. \(3abc^2\)  
19. \((x + y)^2\)  
21. \(4(2a + 1)\)  
23. \(8(3m - 4n)\)  
25. \(4m(3m + 2)\)  
27. \(5s(2s + 1)\)  
29. \(12x(x + 2)\)  
31. \(5a^2(3a - 5)\)  
33. \(6pq(1 + 3p)\)  
35. \(7mn(m^2 - 3n^2)\)  
37. \(6(x^2 - 3x + 5)\)  
39. \(3a(a^2 + 2a - 4)\)  
41. \(3m(2 + 3n - 5m^2)\)  
43. \(5xy(2x + 3 - y)\)  
45. \(5r^2s^2(2r + 5 - 3s)\)  
47. \(3a(3a^4 - 5a^2 + 7a^3 - 9)\)  
49. \(5mn(3m^2n - 4m + 7n^3 - 2)\)  
51. \((x - 2)(x + 3)\)  
53. 10  
55. \(6x\)  
57. 6  
59. \(7x\)  
61. \(t(33 - t); 33 - t\)  
63.  
65.  

a. \(a^2 + 3a - 4\)  
b. \(x^2 + 2x - 3\)  
c. \(x^2 - 6x + 9\)  
d. \(y^2 - 8y - 33\)  
e. \(x^2 + 12x + 35\)  
f. \(y^2 - 12y - 13\)