4.2 Factoring Trinomials of the Form $x^2 + bx + c$

4.2 OBJECTIVES

1. Factor a trinomial of the form $x^2 + bx + c$
2. Factor a trinomial containing a common factor

NOTE The process used to factor here is frequently called the *trial-and-error method.* You'll see the reason for the name as you work through this section.

You learned how to find the product of any two binomials by using the FOIL method in Section 3.4. Because factoring is the reverse of multiplication, we now want to use that pattern to find the factors of certain trinomials.

Recall that to multiply two binomials, we have

$$(x + 2)(x + 3) = x^2 + 5x + 6$$

The product of the first terms ($x \cdot x$). The sum of the products of the outer and inner terms ($3x$ and $2x$). The product of the last terms ($2 \cdot 3$).

CAUTION

Not every trinomial can be written as the product of two binomials.

Suppose now that you are given $x^2 + 5x + 6$ and want to find its factors. First, you know that the factors of a trinomial may be two binomials. So write

$$x^2 + 5x + 6 = (\quad)(\quad)$$

Because the first term of the trinomial is x^2, the first terms of the binomial factors must be x and x. We now have

$$x^2 + 5x + 6 = (x \quad)(x \quad)$$

The product of the last terms must be 6. Because 6 is positive, the factors must have *like* signs. Here are the possibilities:

$$6 = 1 \cdot 6$$
$$= 2 \cdot 3$$
$$= (-1)(-6)$$
$$= (-2)(-3)$$

This means that the possible factors of the trinomial are

$$(x + 1)(x + 6)$$
$$(x + 2)(x + 3)$$
$$(x - 1)(x - 6)$$
$$(x - 2)(x - 3)$$

How do we tell which is the correct pair? From the FOIL pattern we know that the sum of the outer and inner products must equal the middle term of the trinomial, in this case $5x$. This is the crucial step!

Possible Factors	Middle Terms
$(x + 1)(x + 6)$	$7x$
$(x + 2)(x + 3)$	$5x$
$(x - 1)(x - 6)$	$-7x$
$(x - 2)(x - 3)$	$-5x$

The correct middle term!

So we know that the correct factorization is

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Are there any clues so far that will make this process quicker? Yes, there is an important one that you may have spotted. We started with a trinomial that had a positive middle term and a positive last term. The negative pairs of factors for 6 led to negative middle terms. So you don't need to bother with the negative factors if the middle term and the last term of the trinomial are both positive.

Example 1

Factoring a Trinomial

(a) Factor $x^2 + 9x + 8$.

Because the middle term and the last term of the trinomial are both positive, consider only the positive factors of 8, that is, $8 = 1 \cdot 8$ or $8 = 2 \cdot 4$.

Possible Factors	Middle Terms
$(x + 1)(x + 8)$	$9x$
$(x + 2)(x + 4)$	$6x$

NOTE If you are wondering why we didn't list $(x + 8)(x + 1)$ as a possibility, remember that multiplication is commutative. The order doesn't matter!

Because the first pair gives the correct middle term,

$$x^2 + 9x + 8 = (x + 1)(x + 8)$$

(b) Factor $x^2 + 12x + 20$.

Possible Factors	Middle Terms
$(x + 1)(x + 20)$	$21x$
$(x + 2)(x + 10)$	$12x$
$(x + 4)(x + 5)$	$9x$

NOTE The factors for 20 are

$20 = 1 \cdot 20$
$= 2 \cdot 10$
$= 4 \cdot 5$

So

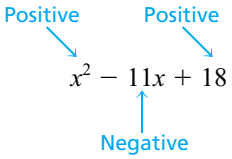
$$x^2 + 12x + 20 = (x + 2)(x + 10)$$

CHECK YOURSELF 1

Factor.

(a) $x^2 + 6x + 5$ **(b)** $x^2 + 10x + 16$

Let's look at some examples in which the middle term of the trinomial is negative but the first and last terms are still positive. Consider



Because we want a negative middle term $(-11x)$, we use *two negative factors* for 18. Recall that the product of two negative numbers is positive.

Example 2

Factoring a Trinomial

(a) Factor $x^2 - 11x + 18$.

Possible Factors	Middle Terms
$(x - 1)(x - 18)$	$-19x$
$(x - 2)(x - 9)$	$-11x$
$(x - 3)(x - 6)$	$-9x$

NOTE The negative factors of 18 are

$18 = (-1)(-18)$
$= (-2)(-9)$
$= (-3)(-6)$

So

$$x^2 - 11x + 18 = (x - 2)(x - 9)$$

(b) Factor $x^2 - 13x + 12$.

Possible Factors	Middle Terms
$(x - 1)(x - 12)$	$-13x$
$(x - 2)(x - 6)$	$-8x$
$(x - 3)(x - 4)$	$-7x$

NOTE The negative factors of 12 are

$12 = (-1)(-12)$
$= (-2)(-6)$
$= (-3)(-4)$

So

$$x^2 - 13x + 12 = (x - 1)(x - 12)$$

A few more clues: We have listed all the possible factors in the above examples. It really isn't necessary. Just work until you find the right pair. Also, with practice much of this work can be done mentally.

CHECK YOURSELF 2

Factor.

(a) $x^2 - 10x + 9$

(b) $x^2 - 10x + 21$

Let's look now at the process of factoring a trinomial whose last term is negative. For instance, to factor $x^2 + 2x - 15$, we can start as before:

$$x^2 + 2x - 15 = (x \quad ?)(x \quad ?)$$

Note that the product of the last terms must be negative (-15 here). So we must choose factors that have different signs.

What are our choices for the factors of -15?

$$\begin{aligned} -15 &= (1)(-15) \\ &= (-1)(15) \\ &= (3)(-5) \\ &= (-3)(5) \end{aligned}$$

This means that the possible factors and the resulting middle terms are

Possible Factors	Middle Terms
$(x + 1)(x - 15)$	$-14x$
$(x - 1)(x + 15)$	$14x$
$(x + 3)(x - 5)$	$-2x$
$(x - 3)(x + 5)$	$2x$

NOTE Another clue: Some students prefer to look at the list of numerical factors rather than looking at the actual algebraic factors. Here you want the pair whose sum is 2, the coefficient of the middle term of the trinomial. That pair is -3 and 5, which leads us to the correct factors.

So $x^2 + 2x - 15 = (x - 3)(x + 5)$.

Let's work through some examples in which the constant term is negative.

Example 3

Factoring a Trinomial

(a) Factor $x^2 - 5x - 6$.

First, list the factors of -6. Of course, one factor will be positive, and one will be negative.

NOTE You may be able to pick the factors directly from this list. You want the pair whose sum is -5 (the coefficient of the middle term).

$$\begin{aligned} -6 &= (1)(-6) \\ &= (-1)(6) \\ &= (2)(-3) \\ &= (-2)(3) \end{aligned}$$

For the trinomial, then, we have

Possible Factors	Middle Terms
$(x + 1)(x - 6)$	$-5x$
$(x - 1)(x + 6)$	$5x$
$(x + 2)(x - 3)$	$-x$
$(x - 2)(x + 3)$	x

So $x^2 - 5x - 6 = (x + 1)(x - 6)$.

(b) Factor $x^2 + 8xy - 9y^2$.

The process is similar if two variables are involved in the trinomial you are to factor. Start with

$x^2 + 8xy - 9y^2 = (x \quad ?)(x \quad ?).$

The product of the last terms must be $-9y^2$.

$-9y^2 = (-y)(9y)$

$= (y)(-9y)$

$= (3y)(-3y)$

Possible Factors	Middle Terms
$(x - y)(x + 9y)$	$8xy$
$(x + y)(x - 9y)$	$-8xy$
$(x + 3y)(x - 3y)$	0

So $x^2 + 8xy - 9y^2 = (x - y)(x + 9y)$.

CHECK YOURSELF 3

Factor.

(a) $x^2 + 7x - 30$

(b) $x^2 - 3xy - 10y^2$

As was pointed out in the last section, any time that we have a common factor, that factor should be removed *before* we try any other factoring technique. Consider the following example.

Example 4

Factoring a Trinomial

(a) Factor $3x^2 - 21x + 18$.

$3x^2 - 21x + 18 = 3(x^2 - 7x + 6)$ Remove the common factor of 3.

We now factor the remaining trinomial. For $x^2 - 7x + 6$:

Possible Factors	Middle Terms
$(x - 2)(x - 3)$	$-5x$
$(x - 1)(x - 6)$	$-7x$

The correct middle term

So $3x^2 - 21x + 18 = 3(x - 1)(x - 6)$.

(b) Factor $2x^3 + 16x^2 - 40x$.

$2x^3 + 16x^2 - 40x = 2x(x^2 + 8x - 20)$ Remove the common factor of $2x$.

CAUTION

A common mistake is to forget to write the 3 that was factored out as the first step.

To factor the remaining trinomial, which is $x^2 + 8x - 20$, we have

Possible Factors	Middle Terms	
$(x - 4)(x + 5)$	x	
$(x - 5)(x + 4)$	$-x$	
$(x - 10)(x + 2)$	$-8x$	
$(x - 2)(x + 10)$	$8x$	The correct middle term

NOTE Once we have found the desired middle term, there is no need to continue.

So $2x^3 + 16x^2 - 40x = 2x(x - 2)(x + 10)$.

CHECK YOURSELF 4

Factor.

(a) $3x^2 - 3x - 36$ **(b)** $4x^3 + 24x^2 + 32x$

One further comment: Have you wondered if all trinomials are factorable? Look at the trinomial

$x^2 + 2x + 6$

The only possible factors are $(x + 1)(x + 6)$ and $(x + 2)(x + 3)$. Neither pair is correct (you should check the middle terms), and so this trinomial does not have factors with integer coefficients. Of course, there are many others.

CHECK YOURSELF ANSWERS

1. (a) $(x + 1)(x + 5)$; **(b)** $(x + 2)(x + 8)$ **2. (a)** $(x - 9)(x - 1)$; **(b)** $(x - 3)(x - 7)$
3. (a) $(x + 10)(x - 3)$; **(b)** $(x + 2y)(x - 5y)$
4. (a) $3(x - 4)(x + 3)$; **(b)** $4x(x + 2)(x + 4)$

4.2 Exercises

Name _____

Section _____ Date _____

Complete each of the following statements.

1. $x^2 - 8x + 15 = (x - 3)($ $)$

2. $y^2 - 3y - 18 = (y - 6)($ $)$

3. $m^2 + 8m + 12 = (m + 2)($ $)$

4. $x^2 - 10x + 24 = (x - 6)($ $)$

5. $p^2 - 8p - 20 = (p + 2)($ $)$

6. $a^2 + 9a - 36 = (a + 12)($ $)$

7. $x^2 - 16x + 64 = (x - 8)($ $)$

8. $w^2 - 12w - 45 = (w + 3)($ $)$

9. $x^2 - 7xy + 10y^2 = (x - 2y)($ $)$

10. $a^2 + 18ab + 81b^2 = (a + 9b)($ $)$

Factor each of the following trinomials.

11. $x^2 + 8x + 15$

12. $x^2 - 11x + 24$

13. $x^2 - 11x + 28$

14. $y^2 - y - 20$

15. $s^2 + 13s + 30$

16. $b^2 + 14b + 33$

17. $a^2 - 2a - 48$

18. $x^2 - 17x + 60$

19. $x^2 - 8x + 7$

20. $x^2 + 7x - 18$

21. $m^2 + 3m - 28$

22. $a^2 + 10a + 25$

ANSWERS

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23. $x^2 - 6x - 40$

24. $x^2 - 11x + 10$

25. $x^2 - 14x + 49$

26. $s^2 - 4s - 32$

27. $p^2 - 10p - 24$

28. $x^2 - 11x - 60$

29. $x^2 + 5x - 66$

30. $a^2 + 2a - 80$

31. $c^2 + 19c + 60$

32. $t^2 - 4t - 60$

33. $n^2 + 5n - 50$

34. $x^2 - 16x + 63$

35. $x^2 + 7xy + 10y^2$

36. $x^2 - 8xy + 12y^2$

37. $a^2 - ab - 42b^2$

38. $m^2 - 8mn + 16n^2$

39. $x^2 - 13xy + 40y^2$

40. $r^2 - 9rs - 36s^2$

41. $b^2 + 6ab + 9a^2$

42. $x^2 + 3xy - 10y^2$

43. $x^2 - 2xy - 8y^2$

44. $u^2 + 6uv - 55v^2$

45. $25m^2 + 10mn + n^2$

46. $64m^2 - 16mn + n^2$

Factor each of the following trinomials completely. Factor out the greatest common factor first.

47. $3a^2 - 3a - 126$

48. $2c^2 + 2c - 60$

49. $r^3 + 7r^2 - 18r$

50. $m^3 + 5m^2 - 14m$

51. $2x^3 - 20x^2 - 48x$

52. $3p^3 + 48p^2 - 108p$

53. $x^2y - 9xy^2 - 36y^3$

54. $4s^4 - 20s^3t - 96s^2t^2$

55. $m^3 - 29m^2n + 120mn^2$

56. $2a^3 - 52a^2b + 96ab^2$

Find a positive value for k for which each of the following can be factored.

57. $x^2 + kx + 8$

58. $x^2 + kx + 9$

59. $x^2 - kx + 16$

60. $x^2 - kx + 17$

61. $x^2 - kx - 5$

62. $x^2 - kx - 7$

63. $x^2 + 3x + k$

64. $x^2 + 5x + k$

65. $x^2 + 2x - k$

66. $x^2 + x - k$

Getting Ready for Section 4.3 [Section 3.3]

Multiply.

(a) $(2x - 1)(2x + 3)$
(b) $(3a - 1)(a + 4)$
(c) $(x - 4)(2x - 3)$
(d) $(2w - 11)(w + 2)$
(e) $(y + 5)(2y + 9)$
(f) $(2x + 1)(x - 12)$
(g) $(p + 9)(2p + 5)$
(h) $(3a - 5)(2a + 4)$

ANSWERS

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48. _____
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e. _____
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g. _____
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Answers

1. $x - 5$ **3.** $m + 6$ **5.** $p - 10$ **7.** $x - 8$ **9.** $x - 5y$
11. $(x + 3)(x + 5)$ **13.** $(x - 4)(x - 7)$ **15.** $(s + 10)(s + 3)$
17. $(a - 8)(a + 6)$ **19.** $(x - 1)(x - 7)$ **21.** $(m + 7)(m - 4)$
23. $(x - 10)(x + 4)$ **25.** $(x - 7)(x - 7)$ **27.** $(p - 12)(p + 2)$
29. $(x + 11)(x - 6)$ **31.** $(c + 4)(c + 15)$ **33.** $(n + 10)(n - 5)$
35. $(x + 2y)(x + 5y)$ **37.** $(a - 7b)(a + 6b)$ **39.** $(x - 5y)(x - 8y)$
41. $(b + 3a)(b + 3a)$ **43.** $(x - 4y)(x + 2y)$ **45.** $(5m + n)(5m + n)$
47. $3(a - 7)(a + 6)$ **49.** $r(r + 9)(r - 2)$ **51.** $2x(x - 12)(x + 2)$
53. $y(x - 12y)(x + 3y)$ **55.** $m(m - 5n)(m - 24n)$ **57.** 6 or 9
59. 8, 10, or 17 **61.** 4 **63.** 2 **65.** 3, 8, 15, 24, . . . **a.** $4x^2 + 4x - 3$
b. $3a^2 + 11a - 4$ **c.** $2x^2 - 11x + 12$ **d.** $2w^2 - 7w - 22$
e. $2y^2 + 19y + 45$ **f.** $2x^2 - 23x - 12$ **g.** $2p^2 + 23p + 45$
h. $6a^2 + 2a - 20$