# 4.3 Factoring Trinomials of the Form $ax^2 + bx + c$

## OBJECTIVES

1. Factor a trinomial of the form $ax^2 + bx + c$
2. Completely factor a trinomial

Factoring trinomials is more time-consuming when the coefficient of the first term is not 1. Look at the following multiplication.

$$(5x + 2)(2x + 3) = 10x^2 + 19x + 6$$

Do you see the additional problem? We must consider all possible factors of the first coefficient (10 in the example) as well as those of the third term (6 in our example).

There is no easy way out! You need to form all possible combinations of factors and then check the middle term until the proper pair is found. If this seems a bit like guesswork, you’re almost right. In fact some call this process factoring by *trial and error*.

We can simplify the work a bit by reviewing the sign patterns found in Section 4.2.

## Rules and Properties: Sign Patterns for Factoring Trinomials

1. If all terms of a trinomial are positive, the signs between the terms in the binomial factors are both plus signs.
2. If the third term of the trinomial is positive and the middle term is negative, the signs between the terms in the binomial factors are both minus signs.
3. If the third term of the trinomial is negative, the signs between the terms in the binomial factors are opposite (one is $+$ and one is $-$).

### Example 1

**Factoring a Trinomial**

Factor $3x^2 + 14x + 15$.

First, list the possible factors of 3, the coefficient of the first term.

$$3 = 1 \cdot 3$$

Now list the factors of 15, the last term.

$$15 = 1 \cdot 15 = 3 \cdot 5$$

Because the signs of the trinomial are all positive, we know any factors will have the form

$$(_x + _)(_x + _)$$

The product of the last terms must be 15.

The product of the numbers in the first blanks must be 3.
So the following are the possible factors and the corresponding middle terms:

<table>
<thead>
<tr>
<th>Possible Factors</th>
<th>Middle Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x + 1)(3x + 15))</td>
<td>18x</td>
</tr>
<tr>
<td>((x + 15)(3x + 1))</td>
<td>46x</td>
</tr>
<tr>
<td>((3x + 3)(x + 5))</td>
<td>18x</td>
</tr>
<tr>
<td>((3x + 5)(x + 3))</td>
<td>14x</td>
</tr>
</tbody>
</table>

The correct middle term

So

\[3x^2 + 14x + 15 = (3x + 5)(x + 3)\]

**CHECK YOURSELF 1**

Factor.

(a) \(5x^2 + 14x + 8\)  
(b) \(3x^2 + 20x + 12\)

**Example 2**

**Factoring a Trinomial**

Factor \(4x^2 - 11x + 6\).

Because only the middle term is negative, we know the factors have the form

\[(_x - _)(_x - _)\]

Both signs are negative.

Now look at the factors of the first coefficient and the last term.

\[4 = 1 \cdot 4 \quad 6 = 1 \cdot 6\]

\[= 2 \cdot 2 \quad = 2 \cdot 3\]

This gives us the possible factors:

<table>
<thead>
<tr>
<th>Possible Factors</th>
<th>Middle Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x - 1)(4x - 6))</td>
<td>(-10x)</td>
</tr>
<tr>
<td>((x - 6)(4x - 1))</td>
<td>(-25x)</td>
</tr>
<tr>
<td>((x - 2)(4x - 3))</td>
<td>(-11x)</td>
</tr>
</tbody>
</table>

The correct middle term

Note that, in this example, we stopped as soon as the correct pair of factors was found. So

\[4x^2 - 11x + 6 = (x - 2)(4x - 3)\]

**CHECK YOURSELF 2**

Factor.

(a) \(2x^2 - 9x + 9\)  
(b) \(6x^2 - 17x + 10\)

Let’s factor a trinomial whose last term is negative.
Example 3

Factoring a Trinomial

Factor $5x^2 + 6x - 8$.

Because the last term is negative, the factors have the form

$(_x + _)(_x - _)$

Consider the factors of the first coefficient and the last term.

$5 = 1 \cdot 5$ \hspace{1cm} $8 = 1 \cdot 8$

$= 2 \cdot 4$

The possible factors are then

<table>
<thead>
<tr>
<th>Possible Factors</th>
<th>Middle Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x + 1)(5x - 8)$</td>
<td>$-3x$</td>
</tr>
<tr>
<td>$(x + 8)(5x - 1)$</td>
<td>$39x$</td>
</tr>
<tr>
<td>$(5x + 1)(x - 8)$</td>
<td>$-39x$</td>
</tr>
<tr>
<td>$(5x + 8)(x - 1)$</td>
<td>$3x$</td>
</tr>
<tr>
<td>$(x + 2)(5x - 4)$</td>
<td>$6x$</td>
</tr>
</tbody>
</table>

Again we stop as soon as the correct pair of factors is found.

$5x^2 + 6x - 8 = (x + 2)(5x - 4)$

CHECK YOURSELF 3

Factor $4x^2 + 5x - 6$.

The same process is used to factor a trinomial with more than one variable.

Example 4

Factoring a Trinomial

Factor $6x^2 + 7xy - 10y^2$.

The form of the factors must be

$(_x + _y)(_x - _y)$

The product of the first terms is an $x^2$ term. The product of the second terms is a $y^2$ term.

The signs are opposite because the last term is negative.

Again look at the factors of the first and last coefficients.

$6 = 1 \cdot 6$ \hspace{1cm} $10 = 1 \cdot 10$

$= 2 \cdot 3$ \hspace{1cm} $= 2 \cdot 5$
Once more, we stop as soon as the correct factors are found.

\[ 6x^2 + 7xy - 10y^2 = (x + 2y)(6x - 5y) \]

**CHECK YOURSELF 4**

Factor \( 15x^2 - 4xy - 4y^2 \).

The next example illustrates a special kind of trinomial called a **perfect square trinomial**.

**Example 5**

**Factoring a Trinomial**

Factor \( 9x^2 + 12xy + 4y^2 \).

Because all terms are positive, the form of the factors must be

\[ (\_x + \_y)(\_x + \_y) \]

Consider the factors of the first and last coefficients.

\[
egin{align*}
9 &= 9 \cdot 1 & 4 &= 4 \cdot 1 \\
   &= 3 \cdot 3 & &= 2 \cdot 2
\end{align*}
\]

**Possible Factors**  |  **Middle Terms**
---|---
\((x + y)(9x + 4y)\)  |  \(13xy\)
\((x + 4y)(9x + y)\)  |  \(37xy\)
\((3x + 2y)(3x + 2y)\)  |  \(12xy\)

So

\[ 9x^2 + 12xy + 4y^2 = (3x + 2y)(3x + 2y) = (3x + 2y)^2 \]

This trinomial is the result of squaring a binomial, thus the special name of perfect square trinomial.

**CHECK YOURSELF 5**

**Factor**.

(a) \( 4x^2 + 28x + 49 \)  
(b) \( 16x^2 - 40xy + 25y^2 \)
Before we look at our next example, let's review one important point from Section 4.2. Recall that when you factor trinomials, you should not forget to look for a common factor as the first step. If there is a common factor, remove it and factor the remaining trinomial as before.

### Example 6

**Factoring a Trinomial**

Factor $18x^2 - 18x + 4$.

First look for a common factor in all three terms. Here that factor is 2, so write

$$18x^2 - 18x + 4 = 2(9x^2 - 9x + 2)$$

By our earlier methods, we can factor the remaining trinomial as

$$9x^2 - 9x + 2 = (3x - 1)(3x - 2)$$

So

$$18x^2 - 18x + 4 = 2(3x - 1)(3x - 2)$$

Don't forget the 2 that was factored out!

### CHECK YOURSELF 6

*Factor* $16x^2 + 44x - 12$.

Let's look at an example in which the common factor includes a variable.

### Example 7

**Factoring a Trinomial**

Factor

$$6x^3 + 10x^2 - 4x$$

The common factor is $2x$.

So

$$6x^3 + 10x^2 - 4x = 2x(3x^2 + 5x - 2)$$

Because

$$3x^2 + 5x - 2 = (3x - 1)(x + 2)$$

we have

$$6x^3 + 10x^2 - 4x = 2x(3x - 1)(x + 2)$$

**NOTE** Remember to include the monomial factor.

### CHECK YOURSELF 7

*Factor* $6x^3 - 27x^2 + 30x$. 
You have now had a chance to work with a variety of factoring techniques. Your success in factoring polynomials depends on your ability to recognize when to use which technique. Here are some guidelines to help you apply the factoring methods you have studied in this chapter.

**Step by Step: Factoring Polynomials**

**Step 1** Look for a greatest common factor other than 1. If such a factor exists, factor out the GCF.

**Step 2** If the polynomial that remains is a **trinomial**, try to factor the trinomial by the trial-and-error methods of Sections 4.2 and 4.3.

The following example illustrates the use of this strategy.

**Example 8**

**Factoring a Trinomial**

(a) Factor $5m^2n + 20n$.
First, we see that the GCF is $5n$. Removing that factor gives

$$5m^2n + 20n = 5n(m^2 + 4)$$

(b) Factor $3x^3 - 24x^2 + 48x$.
First, we see that the GCF is $3x$. Factoring out $3x$ yields

$$3x^3 - 24x^2 + 48x = 3x(x^2 - 8x + 16) = 3x(x - 4)(x - 4)$$

(c) Factor $8r^2s + 20rs^2 - 12s^3$.
First, the GCF is $4s$, and we can write the original polynomial as

$$8r^2s + 20rs^2 - 12s^3 = 4s(2r^2 + 5rs - 3s^2)$$

Because the remaining polynomial is a trinomial, we can use the trial-and-error method to complete the factoring as

$$8r^2s + 20rs^2 - 12s^3 = 4s(2r - s)(r + 3s)$$

**CHECK YOURSELF 8**

Factor the following polynomials.

(a) $8a^3 + 32a^2b + 32ab^2$
(b) $7x^3 + 7x^2y - 42xy^2$
(c) $5m^4 + 15m^3 + 5m^2$

**CHECK YOURSELF ANSWERS**

1. (a) $(5x + 4)(x + 2)$; (b) $(3x + 2)(x + 6)$
2. (a) $(2x - 3)(x - 3)$;
(b) $(6x - 5)(x - 2)$
3. $(4x - 3)(x + 2)$
4. $(3x - 2y)(5x + 2y)$
5. (a) $(2x + 7)^2$; (b) $(4x - 5y)^2$
6. $(4x - 1)(x + 3)$
7. $3x(2x - 5)(x - 2)$
8. (a) $8a(a + 2b)(a + 2b)$; (b) $7x(x + 3y)(x - 2y)$; (c) $5m^2(m^2 + 3m + 1)$
4.3 Exercises

Complete each of the following statements.

1. \(4x^2 - 4x - 3 = (2x + 1)(\quad)\)

2. \(3w^2 + 11w - 4 = (w + 4)(\quad)\)

3. \(6a^2 + 13a + 6 = (2a + 3)(\quad)\)

4. \(25y^2 - 10y + 1 = (5y - 1)(\quad)\)

5. \(15x^2 - 16x + 4 = (3x - 2)(\quad)\)

6. \(6m^2 + 5m - 4 = (3m + 4)(\quad)\)

7. \(16a^2 + 8ab + b^2 = (4a + b)(\quad)\)

8. \(6x^2 + 5xy - 4y^2 = (3x + 4y)(\quad)\)

9. \(4m^2 + 5mn - 6n^2 = (m + 2n)(\quad)\)

10. \(10p^2 - pq - 3q^2 = (5p - 3q)(\quad)\)

Factor each of the following polynomials.

11. \(3x^2 + 7x + 2\)

12. \(5y^2 + 8y + 3\)

13. \(2w^2 + 13w + 15\)

14. \(3x^2 - 16x + 21\)

15. \(5x^2 - 16x + 3\)

16. \(2a^2 + 7a + 5\)

17. \(4x^2 - 12x + 5\)

18. \(2x^2 + 11x + 12\)

19. \(3x^2 - 5x - 2\)

20. \(4m^2 - 23m + 15\)

21. \(4p^2 + 19p - 5\)

22. \(5x^2 - 36x + 7\)

23. \(6x^2 + 19x + 10\)

24. \(6x^2 - 7x - 3\)

ANSWERS
25. \(15x^2 + x - 6\)  

26. \(12w^2 + 19w + 4\)  

27. \(6m^2 + 25m - 25\)  

28. \(8x^2 - 6x - 9\)  

29. \(9x^2 - 12x + 4\)  

30. \(20x^2 - 23x + 6\)  

31. \(12x^2 - 8x - 15\)  

32. \(16a^2 + 40a + 25\)  

33. \(3y^2 + 7y - 6\)  

34. \(12x^2 + 11x - 15\)  

35. \(8x^2 - 27x - 20\)  

36. \(24v^2 + 5v - 36\)  

37. \(2x^2 + 3xy + y^2\)  

38. \(3x^2 - 5xy + 2y^2\)  

39. \(5a^2 - 8ab - 4b^2\)  

40. \(5x^2 + 7xy - 6y^2\)  

41. \(9x^2 + 4xy - 5y^2\)  

42. \(16x^2 + 32xy + 15y^2\)  

43. \(6m^2 - 17mn + 12n^2\)  

44. \(15x^2 - xy - 6y^2\)  

45. \(36a^2 - 3ab - 5b^2\)  

46. \(3q^2 - 17qr - 6r^2\)  

47. \(x^2 + 4xy + 4y^2\)  

48. \(25b^2 - 80bc + 64c^2\)
Factor each of the following polynomials completely.

49. \(20x^2 - 20x - 15\)  
50. \(24x^2 - 18x - 6\)

51. \(8m^2 + 12m + 4\)  
52. \(14x^2 - 20x + 6\)

53. \(15r^2 - 21rs + 6s^2\)  
54. \(10x^2 + 5xy - 30y^2\)

55. \(2x^3 - 2x^2 - 4x\)  
56. \(2y^3 + y^2 - 3y\)

57. \(2y^4 + 5y^3 + 3y^2\)  
58. \(4z^3 - 18z^2 - 10z\)

59. \(36a^3 - 66a^2 + 18a\)  
60. \(20n^4 - 22n^3 - 12n^2\)

61. \(9p^2 + 30pq + 21q^2\)  
62. \(12x^2 + 2xy - 24y^2\)

Factor each of the following polynomials completely.

63. \(10(x + y)^2 - 11(x + y) - 6\)  
64. \(8(a - b)^2 + 14(a - b) - 15\)

65. \(5(x - 1)^2 - 15(x - 1) - 350\)  
66. \(3(x + 1)^2 - 6(x + 1) - 45\)

67. \(15 + 29x - 48x^2\)  
68. \(12 + 4a - 21a^2\)

69. \(-6x^2 + 19x - 15\)  
70. \(-3s^2 - 10s + 8\)
Getting Ready for Section 4.4 [Section 3.5]

Multiply.

(a) \((x - 1)(x + 1)\)  
(b) \((a + 7)(a - 7)\)  
(c) \((x - y)(x + y)\)  
(d) \((2x - 5)(2x + 5)\)  
(e) \((3a - b)(3a + b)\)  
(f) \((5a - 4b)(5a + 4b)\)

Answers

1. \(2x - 3\)  
3. \(3a + 2\)  
5. \(5x - 2\)  
7. \(4a + b\)  
9. \(4m - 3n\)  
11. \((3x + 1)(x + 2)\)  
13. \((2w + 3)(w + 5)\)  
15. \((5x - 1)(x - 3)\)  
17. \((2x - 5)(2x - 1)\)  
19. \((3x + 1)(x - 2)\)  
21. \((4p - 1)(p + 5)\)  
23. \((3x + 2)(2x + 5)\)  
25. \((5x - 3)(3x + 2)\)  
27. \((6m - 5)(m + 5)\)  
29. \((3x - 2)(3x - 2)\)  
31. \((6x + 5)(2x - 3)\)  
33. \((3y - 2)(y + 3)\)  
35. \((8x + 5)(x - 4)\)  
37. \((2x + y)(x + y)\)  
39. \((5a + 2b)(a - 2b)\)  
41. \((9x - 5y)(x + y)\)  
43. \((3m - 4n)(2m - 3n)\)  
45. \((12a - 5b)(3a + b)\)  
47. \((x + 2y)^2\)  
49. \((2x - 3)(2x + 1)\)  
51. \((2m + 1)(m + 1)\)  
53. \(3(5r - 2s)(r - s)\)  
55. \(2x(x - 2)(x + 1)\)  
57. \(y^2(2y + 3)(y + 1)\)  
59. \(6a(3a - 1)(2a - 3)\)  
61. \(3(p + q)(3p + 7q)\)  
63. \((5x + 5y + 2)(2x + 2y - 3)\)  
65. \(5(x - 11)(x + 6)\)  
67. \((1 + 3x)(15 - 16x)\)  
69. \((3x - 5)(-2x + 3)\)

\(a. \quad x^2 - 1\)  
\(b. \quad a^2 - 49\)  
\(c. \quad x^2 - y^2\)  
\(d. \quad 4x^2 - 25\)  
\(e. \quad 9a^2 - b^2\)  
\(f. \quad 25a^2 - 16b^2\)