OBJECTIVE

1. Apply the Pythagorean theorem in solving problems

Perhaps the most famous theorem in all of mathematics is the **Pythagorean theorem**. The theorem was named for the Greek mathematician Pythagoras, born in 572 B.C. Pythagoras was the founder of the Greek society the Pythagoreans. Although the theorem bears Pythagoras’ name, his own work on this theorem is uncertain because the Pythagoreans credited new discoveries to their founder.

**Rules and Properties: The Pythagorean Theorem**

For every right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

\[ c^2 = a^2 + b^2 \]

**Example 1**

**Verifying the Pythagorean Theorem**

Verify the Pythagorean theorem for the given triangles.

(a) For the right triangle below,

\[
5^2 = 3^2 + 4^2 \\
25 = 9 + 16 \\
25 = 25
\]

(b) For the right triangle below,

\[
13^2 = 12^2 + 5^2 \\
169 = 144 + 25 \\
169 = 169
\]
**CHECK YOURSELF 1**

Verify the Pythagorean theorem for the right triangle shown.

The Pythagorean theorem can be used to find the length of one side of a right triangle when the lengths of the two other sides are known.

**Example 2**

Solving for the Length of the Hypotenuse

Find length $x$.

**NOTE** Notice $x$ will be longer than the given sides because it is the hypotenuse.

$$x^2 = 9^2 + 12^2$$
$$= 81 + 144$$
$$= 225$$

so

$$x = 15 \quad \text{or} \quad x = -15$$

We reject this solution because a length must be positive.

**CHECK YOURSELF 2**

Find length $x$.

Sometimes, one or more of the lengths of the sides may be represented by an irrational number.

**Example 3**

Solving for the Length of the Leg

Find length $x$. Then use your calculator to give an approximation to the nearest tenth.

**NOTE** You can approximate $3\sqrt{3}$ (or $\sqrt{27}$) with the use of a calculator.

$$6^2 + 3^2 = x^2$$
$$36 + 9 = x^2$$
$$x^2 = 45$$

$$x = \pm \sqrt{45}$$
The Pythagorean theorem can be applied to solve a variety of geometric problems.

**Example 4**

**Solving for the Length of the Diagonal**

Find, to the nearest tenth, the length of the diagonal of a rectangle that is 8 centimeters (cm) long and 5 cm wide. Let $x$ be the unknown length of the diagonal:

\[
x^2 = 5^2 + 8^2 = 25 + 64 = 89 \]

\[
x = \sqrt{89} \approx 9.4 \text{ cm}
\]

**NOTE** Again, distance cannot be negative, so we eliminate $x = -\sqrt{89}$.

**CHECK YOURSELF 4**

The diagonal of a rectangle is 12 inches (in.) and its width is 6 in. Find its length to the nearest tenth.

The next application also makes use of the Pythagorean theorem.
Example 5

Solving an Application

How long must a guywire be to reach from the top of a 30-foot (ft) pole to a point on the ground 20 ft from the base of the pole?

Again be sure to draw a sketch of the problem.

\[ x^2 = 20^2 + 30^2 \]
\[ = 400 + 900 \]
\[ = 1300 \]
\[ x = \sqrt{1300} \]
\[ \approx 36 \text{ ft} \]

CHECK YOURSELF 5

A 16-ft ladder leans against a wall with its base 4 ft from the wall. How far off the floor is the top of the ladder?

To find the distance between any two points in the plane, we use a formula derived from the Pythagorean theorem.

Definitions: Pythagorean Theorem

Given a right triangle in which \( c \) is the length of the hypotenuse, we have the equation

\[ c^2 = a^2 + b^2 \]

We can rewrite the formula as

\[ c = \sqrt{a^2 + b^2} \]

We use this form of the Pythagorean theorem in Example 6.

Example 6

Finding the Distance Between Two Points

Find the distance from (2, 3) to (5, 7).

The distance can be seen as the hypotenuse of a right triangle.

The lengths of the two legs can be found by finding the difference of the two \( x \) coordinates and the difference of the two \( y \) coordinates. So

\[ a = 5 - 2 = 3 \quad \text{and} \quad b = 7 - 3 = 4 \]
The distance, \( c \), can then be found using the formula

\[
c = \sqrt{a^2 + b^2}
\]

or, in this case

\[
c = \sqrt{3^2 + 4^2}
\]

\[
c = \sqrt{9 + 16}
\]

\[
= \sqrt{25}
\]

\[
= 5
\]

The distance is 5 units.

CHECK YOURSELF 6

Find the distance between \((0, 2)\) and \((5, 14)\).

If we call our points \((x_1, y_1)\) and \((x_2, y_2)\), we can state the **distance formula**.

**Definitions: Distance Formula**

The distance between points \((x_1, y_1)\) and \((x_2, y_2)\) can be found using the formula

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

**Example 7**

**Finding the Distance Between Two Points**

Find the distance between \((-2, 5)\) and \((2, -3)\). Simplify the radical answer.

Using the formula,

\[
d = \sqrt{[2 - (-2)]^2 + [(-3) - 5]^2}
\]

\[
= \sqrt{(4)^2 + (-8)^2}
\]

\[
= \sqrt{16 + 64}
\]

\[
= \sqrt{80}
\]

\[
= 4\sqrt{5}
\]

**NOTE** \(
\sqrt{80} = \sqrt{16 \cdot 5} = 4\sqrt{5}
\)

CHECK YOURSELF 7

Find the distance between \((2, 5)\) and \((-5, 2)\). Simplify the radical answer.

In Example 7, you were asked to find the distance between \((-2, 5)\) and \((2, -3)\).
To form a right triangle, we include the point \((-2, -3)\).

Note that the lengths of the two sides of the right triangle are 4 and 8. By the Pythagorean theorem, the hypotenuse must have length \(\sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}\). The distance formula is an application of the Pythagorean theorem.

Using the square root key on a calculator, it is easy to approximate the length of a diagonal line. This is particularly useful in checking to see if an object is square or rectangular.

**Example 8**

**Approximating Length with a Calculator**

Approximate the length of the diagonal of a rectangle. The diagonal forms the hypotenuse of a triangle with legs 12.2 in. and 15.7 in. The length of the diagonal would be \(\sqrt{12.2^2 + 15.7^2} = \sqrt{395.33} \approx 19.88\) in. Use your calculator to confirm the approximation.

**CHECK YOURSELF 8**

Approximate the length of the diagonal of the rectangle to the nearest tenth.

**CHECK YOURSELF ANSWERS**

1. \(10^2 \approx 8^2 + 6^2; 100 \approx 64 + 36; 100 = 100\)
2. 13
3. \(2\sqrt{7}\); or approximately 5.3
4. Length is approximately 10.4 in.
5. The height is approximately 15.5 ft.
6. 13
7. \(\sqrt{58}\)
8. \(\approx 24.0\) in.
Find the length $x$ in each triangle. Express your answer in simplified radical form.

1. 

2. 

3. 

4. 

5. 

6. 

7. Length of a diagonal. Find the length of the diagonal of a rectangle with a length of 10 centimeters (cm) and a width of 7 cm.

8. Length of a diagonal. Find the length of the diagonal of a rectangle with 5 inches (in.) width and 7 in. length.

9. Width of a rectangle. Find the width of a rectangle whose diagonal is 12 feet (ft) and whose length is 10 ft.

10. Length of a rectangle. Find the length of a rectangle whose diagonal is 9 in. and whose width is 6 in.

11. Length of a wire. How long must a guywire be to run from the top of a 20-ft pole to a point on the ground 8 ft from the base of the pole?
12. **Height of a ladder.** The base of a 15-ft ladder is 5 ft away from a wall. How high from the floor is the top of the ladder?

Find the altitude of each triangle.

13.  

14.  

15. **Length of insulation.** A homeowner wishes to insulate her attic with fiberglass insulation to conserve energy. The insulation comes in 40-cm wide rolls that are cut to fit between the rafters in the attic. If the roof is 6 meters (m) from peak to eave and the attic space is 2 m high at the peak, how long does each of the pieces of insulation need to be? Round to the nearest tenth.

16. **Length of insulation.** For the home described in exercise 15, if the roof is 7 m from peak to eave and the attic space is 3 m high at the peak, how long does each of the pieces of insulation need to be? Round to the nearest tenth.

17. **Base of a triangle.** A solar collector and its stand are in the shape of a right triangle. The collector is 5.00 m long, the upright leg is 3.00 m long, and the base leg is 4.00 m long. Because of inefficiencies in the collector’s position, it needs to be raised by 0.50 m on the upright leg. How long will the new base leg be? Round to the nearest tenth.

18. **Base of a triangle.** A solar collector and its stand are in the shape of a right triangle. The collector is 5.00 m long, the upright leg is 2.00 m long, and the base leg is 4.58 m long. Because of inefficiencies in the collector’s position, it needs to be lowered by 0.50 m on the upright leg. How long will the new base leg be? Round to the nearest tenth.
Find the distance between each pair of points.

19. (2, 0) and (−4, 0)  
20. (−3, 0) and (4, 0)

21. (0, −2) and (0, −9)  
22. (0, 8) and (0, −4)

23. (2, 5) and (5, 2)  
24. (3, 3) and (5, 7)

25. (5, 1) and (3, 8)  
26. (2, 9) and (7, 4)

27. (−2, 8) and (1, 5)  
28. (2, 6) and (−3, 4)

29. (6, −1) and (2, 2)  
30. (2, −8) and (1, 0)

31. (−1, −1) and (2, 5)  
32. (−2, −2) and (3, 3)

33. (−2, 9) and (−3, 3)  
34. (4, −1) and (0, −5)

35. (−1, −4) and (−3, 5)  
36. (−2, 3) and (−7, −1)

37. (−2, −4) and (−4, 1)  
38. (−1, −1) and (4, −2)

39. (−4, −2) and (−1, −5)  
40. (−2, −2) and (−4, −4)

41. (−2, 0) and (−4, −1)  
42. (−5, −2) and (−7, −1)

Use the distance formula to show that each set of points describes an isosceles triangle (a triangle with two sides of equal length).

43. (−3, 0), (2, 3), and (1, −1)

44. (−2, 4), (2, 7), and (5, 3)
45. **Dimensions of a triangle.** The length of one leg of a right triangle is 3 in. more than the other. If the length of the hypotenuse is 15 in., what are the lengths of the two legs?

46. **Dimensions of a rectangle.** The length of a rectangle is 1 cm longer than its width. If the diagonal of the rectangle is 5 cm, what are the dimensions (the length and width) of the rectangle?

47. Use the Pythagorean theorem to determine the length of each line segment. Where appropriate, round to the nearest hundredth.

48. 

49. 

50. 

51. 

For each figure, use the slope concept and the Pythagorean theorem to show that the figure is a square. (Recall that a square must have four right angles and four equal sides.) Then give the area of the square to the nearest hundredth.
52.

53. Your architecture firm just received this memo.

To: Algebra Expert Architecture, Inc.
From: Microbeans Coffee Company, Inc.
Re: Design for On-Site Day Care Facility
Date: Aug. 10, 2000

We are requesting that you submit a design for a nursery for preschool children. We are planning to provide free on-site day care for the workers at our corporate headquarters.

The nursery should be large enough to serve the needs of 20 preschoolers. There will be three child care workers in this facility. We want the nursery to be 3000 square feet in area. It needs a playroom, a small kitchen and eating space, and bathroom facilities. There should be some space to store toys and books, many of which should be accessible to children. The company plans to put this facility on the first floor on an outside wall so the children can go outside to play without disturbing workers. You are free to add to this design as you see fit.

Please send us your design drawn to a scale of 1 ft to 0.25 in., with precise measurements and descriptions. We would like to receive this design in 1 week from today. Please give us some estimate of the cost of this renovation to our building.

Submit a design, keeping in mind that the design has to conform to strict design specifications for buildings designated as nurseries, including:

1. Number of exits: Two exits for the first 7 people and one exit for every additional 7 people.
2. Width of exits: The total width of exits in inches shall not be less than the total occupant load served by an exit multiplied by 0.3 for stairs and 0.2 for other exits. No exit shall be less than 3 ft wide and 6 ft 8 in. high.
3. Arrangements of exits: If two exits are required, they shall be placed a distance apart equal to but not less than one-half the length of the maximum overall diagonal dimension of the building or area to be served measured in a straight line between exits. Where three or more exits are required, two shall be placed as above and the additional exits arranged a reasonable distance apart.
4. Distance to exits: Maximum distance to travel from any point to an exterior door shall not exceed 100 ft.
Answers
1. 15  3. 15  5. $2\sqrt{6}$  7. $\approx 12.207$ cm  9. $\approx 6.633$ ft
11. $\approx 21.541$ ft  13. 4  15. $4\sqrt{2} = 5.7$ m  17. $\approx 3.6$ m  19. 6
21. 7  23. $3\sqrt{2}$  25. $\sqrt{53}$  27. $3\sqrt{2}$  29. 5  31. $3\sqrt{5}$
33. $\sqrt{37}$  35. $\sqrt{85}$  37. $\sqrt{29}$  39. $3\sqrt{3}$  41. $\sqrt{5}$
43. Sides have length $\sqrt{34}$, $\sqrt{17}$, and $\sqrt{17}$  45. 9 in., 12 in.  47. 4.12
49. 5  51. 13  53.