Absolute Value Equations and Inequalities

**OBJECTIVES**

1. Solve an absolute value equation in one variable
2. Solve an absolute value inequality in one variable

Equations and inequalities may involve the absolute value notation in their statements. In this section we build on the tools developed in Sections 2.1 and 2.4 and on our earlier work with absolute value for the necessary solution techniques.

Recall from Section 1.3 that the absolute value of \(x\), written \(|x|\), is the distance between \(x\) and 0 on the number line. Consider, for example, the absolute value equation

\[ |x| = 4 \]

This means that the distance between \(x\) and 0 is 4, as is pictured below.

As the sketch illustrates, \(x = 4\) and \(x = -4\) are the two solutions for the equation. This observation suggests the more general statement.

**NOTE** Technically we mean the distance between the point corresponding to \(x\) and the point corresponding to 0, the origin.

**CAUTION**

\(p\) must be positive because an equation such as \(|x - 2| = -3\) has no solution. The absolute value of a quantity must always be equal to a nonnegative number.

**Rules and Properties:** Absolute Value Equations—Property 1

For any positive number \(p\), if

\[ |x| = p \]

then

\[ x = p \quad \text{or} \quad x = -p \]

This property allows us to “translate” an equation involving absolute value to two linear equations that we can then solve separately. The following example illustrates.

**Example 1**

Solving an Absolute Value Equation

Solve for \(x\):

\[ |3x - 2| = 4 \]

From Property 1 we know that \(|3x - 2| = 4\) is equivalent to the equations

\[ 3x - 2 = 4 \quad \text{or} \quad 3x - 2 = -4 \]

**NOTE** Add 2.

\[ 3x = 6 \quad \text{or} \quad 3x = -2 \]

\[ x = 2 \quad \text{or} \quad x = -\frac{2}{3} \]

**NOTE** Divide by 3.
The solutions are $\frac{2}{3}$ and 2. These solutions are easily checked by replacing $x$ with $-\frac{2}{3}$ and 2 in the original absolute value equation.

**CHECK YOURSELF 1**

**Solve for** $x$.

$$|4x + 1| = 9$$

An equation involving absolute value may have to be rewritten before you can apply Property 1. Consider the following example.

**Example 2**

**Solving an Absolute Value Equation**

Solve for $x$:

$$|2 - 3x| + 5 = 10$$

To use Property 1, we must first isolate the absolute value on the left side of the equation. This is easily done by subtracting 5 from both sides for the result:

$$|2 - 3x| = 5$$

We can now proceed as before by using Property 1.

$$2 - 3x = 5 \quad \text{or} \quad 2 - 3x = -5$$

$$-3x = 3 \quad \quad -3x = -7$$

$$x = -1 \quad \quad x = \frac{7}{3}$$

The solution set is $\left\{-1, \frac{7}{3}\right\}$.

**CHECK YOURSELF 2**

**Solve for** $x$.

$$|5 - 2x| - 4 = 7$$

In some applications more than one absolute value is involved in an equation. Consider an equation of the form

$$|x| = |y|$$

Because the absolute values of $x$ and $y$ are equal, $x$ and $y$ are the same distance from 0. This means they are either equal or opposite in sign. This leads to a second general property of absolute value equations.
Example 3

Solving an Absolute Value Equation

Solve for $x$:

$$|3x - 4| = |x + 2|$$

By Property 2, we can write

$$3x - 4 = x + 2 \quad \text{or} \quad 3x - 4 = -(x + 2)$$

$$3x = x + 6 \quad \quad \quad \quad \quad 3x = -x + 2$$

$$2x = 6 \quad \quad \quad \quad \quad 4x = 2$$

$$x = 3 \quad \quad \quad \quad \quad x = \frac{1}{2}$$

The solution set is $\left\{ \frac{1}{2}, 3 \right\}$.

CHECK YOURSELF 3

Solve for $x$.

$$|4x - 1| = |x + 5|$$

We started this section by noting that the solution set for the equation

$$|x| = 4$$

consists of those numbers whose distance from the origin is equal to 4. Similarly, the solution set for the absolute value inequality

$$|x| < 4$$

consists of those numbers whose distance from the origin is less than 4, that is, all numbers between $-4$ and 4. The solution set is pictured below.
The solution set can be described by the compound inequality

\[-4 < x < 4\]

and this suggests the following general statement.

**Rules and Properties:** Absolute Value Inequalities—Property 1

For any positive number \(p\), if

\[|x| < p\]

then

\[-p < x < p\]

Let’s look at an application of Property 1 in solving an absolute value inequality.

**Example 4**

Solving an Absolute Value Inequality

Solve and graph the solution set of

\[|2x - 3| < 5\]

From Property 1, we know that the given absolute value inequality is equivalent to the compound inequality

\[-5 < 2x - 3 < 5\]

Solving as before, we isolate the variable in the center term.

\[-2 < 2x < 8 \quad \text{Add 3 to all three parts.}\]

\[-1 < x < 4 \quad \text{Divide by 2.}\]

The solution set is

\[\{x \mid -1 < x < 4\}\]

The graph is shown below.

**NOTE** Notice that the solution is an open interval on the number line.

**CHECK YOURSELF 4**

Solve and graph the solution set.

\[|3x - 4| \leq 8\]

We know that the solution set for the absolute value inequality

\[|x| < 4\]

consists of those numbers whose distance from the origin is *less than* 4. Now consider the solution set for

\[|x| > 4\]
It must consist of those numbers whose distance from the origin is greater than 4. The solution set is pictured below.

The solution set can be described by the compound inequality

\[ x < -4 \quad \text{or} \quad x > 4 \]

and this suggests the following general statement.

**Rules and Properties:** Absolute Value Inequalities—Property 2

For any positive number \( p \), if

\[ |x| > p \]

then

\[ x < -p \quad \text{or} \quad x > p \]

Let’s apply Property 2 to the solution of an absolute value inequality.

**Example 5**

**Solving an Absolute Value Inequality**

Solve and graph the solution set of

\[ |5x - 2| > 8 \]

From Property 2, we know that the given absolute value inequality is equivalent to the compound inequality

\[ 5x - 2 < -8 \quad \text{or} \quad 5x - 2 > 8 \]

Solving as before, we have

\[ 5x < -6 \quad \text{or} \quad 5x > 10 \]

\[ x < -\frac{6}{5} \quad \text{or} \quad x > 2 \]

The solution set is \( \left\{ x | x < -\frac{6}{5} \text{ or } x > 2 \right\} \) and the graph is shown below.

**CHECK YOURSELF 5**

Solve and graph the solution set.

\[ |3 - 2x| \geq 9 \]
The following chart summarizes our discussion of absolute value inequalities.

<table>
<thead>
<tr>
<th>Type of Inequality</th>
<th>Equivalent Inequality</th>
<th>Graph of Solution Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>ax + b</td>
<td>&lt; p )</td>
</tr>
<tr>
<td>(</td>
<td>ax + b</td>
<td>&gt; p )</td>
</tr>
</tbody>
</table>

**NOTE** As before, \( p \) must be a positive number.

**CHECK YOURSELF ANSWERS**

1. \( \left\{-\frac{5}{2}, 2\right\} \)
2. \( \{-3, 8\} \)
3. \( \left\{2, -\frac{4}{5}\right\} \)
4. \( \left\{-\frac{4}{3} \leq x \leq 4\right\} \)
5. \( \{x \leq -3 \text{ or } x \geq 6\} \)
Exercises

Solve each of the following absolute value equations.

1. \( |x| = 5 \)
2. \( |x| = 7 \)
3. \( |x - 2| = 3 \)
4. \( |x + 5| = 6 \)
5. \( |x + 6| = 0 \)
6. \( |x - 3| = 0 \)
7. \( |3 - x| = 7 \)
8. \( |5 - x| = 4 \)
9. \( |2x - 3| = 9 \)
10. \( |3x + 5| = 11 \)
11. \( |5 - 4x| = 1 \)
12. \( |3 - 6x| = 9 \)
13. \( \left| \frac{1}{2}x + 5 \right| = 7 \)
14. \( \left| \frac{2}{3}x - 4 \right| = 6 \)
15. \( \left| 4 - \frac{3}{4}x \right| = 8 \)
16. \( \left| 3 - \frac{2}{5}x \right| = 9 \)
17. \( |3x + 1| = -2 \)
18. \( |5x - 2| = -3 \)

Rewrite each of the following absolute value equations, and then solve the equations.

19. \( |x| - 3 = 2 \)
20. \( |x| + 4 = 6 \)
21. \( |x - 2| + 3 = 5 \)
22. \( |x + 5| - 2 = 5 \)
23. \( |2x - 3| - 1 = 6 \)
24. \( |3x + 5| + 2 = 4 \)
## ANSWERS

25. \[ \left| \frac{1}{2}x + 2 \right| - 3 = 5 \]

26. \[ \left| \frac{1}{3}x - 4 \right| + 3 = 9 \]

27. \[ 8 - |x - 4| = 5 \]

28. \[ 10 - |2x + 1| = 3 \]

29. \[ |3x - 2| + 4 = 3 \]

30. \[ |5x - 3| + 5 = 3 \]

Solve each of the following absolute value equations.

31. \[ |2x - 1| = |x + 3| \]

32. \[ |3x + 1| = |2x - 3| \]

33. \[ |5x - 2| = |2x + 4| \]

34. \[ |7x - 3| = |2x + 7| \]

35. \[ |x - 2| = |x + 1| \]

36. \[ |x + 3| = |x - 2| \]

37. \[ |2x - 5| = |2x - 3| \]

38. \[ |3x + 1| = |3x - 1| \]

39. \[ |x - 2| = |2 - x| \]

40. \[ |x - 4| = |4 - x| \]

Find and graph the solution set for each of the following absolute value inequalities.

41. \[ |x| < 5 \]

42. \[ |x| > 3 \]

43. \[ |x| \geq 7 \]

44. \[ |x| \leq 4 \]

45. \[ |x - 4| > 2 \]

46. \[ |x + 5| < 3 \]

47. \[ |x + 6| \leq 4 \]

48. \[ |x - 7| \geq 5 \]
49. $|3 - x| > 5$  
50. $|5 - x| < 3$

51. $|x - 7| < 0$  
52. $|x + 5| \geq 0$

53. $|2x - 5| < 3$  
54. $|3x - 1| > 8$

55. $|3x + 4| \geq 5$  
56. $|2x + 3| \leq 9$

57. $|5x - 3| > 7$  
58. $|6x - 5| < 13$

59. $|2 - 3x| < 11$  
60. $|3 - 2x| \geq 11$

61. $|3 - 5x| \geq 7$  
62. $|7 - 3x| < 13$

63. $\left|\frac{3}{4}x - 5\right| < 7$  
64. $\left|\frac{2}{3}x + 5\right| \geq 3$

On some popular calculators there is a special absolute value function key. It is usually labeled “abs.” To register an absolute value, you press this key and then put the desired expression in parentheses. For the expression $|x + 3|$, enter abs$(x + 3)$. Rewrite each expression in calculator form.

65. $|x + 2|$  
66. $|x - 2|$

67. $|2x - 3|$  
68. $|5x + 7|$

69. $|3x + 2| - 4$  
70. $|4x - 7| + 2$

71. $2|3x - 1|$  
72. $-3|2x + 8|$
Answers

1. \{-5, 5\}  
2. \{-1, 5\}  
3. \{-6\}  
4. \{-10\}  
5. \{-3, 6\}  
6. \{16, 16\}  
7. \{-20, 12\}  
8. \{1, 7\}  
9. \{0, 4\}  
10. \{2, 5\}  
11. \{-5, 5\}  
12. \{0, 4\}  
13. \{-4, 5\}  
14. \{1, 7\}  
15. \{-16, 16\}  
16. No solution  
17. No solution  
18. \{\frac{2}{3}, 4\}  
19. \{-20, 12\}  
20. \{\frac{1}{2}\}  
21. \{2\}  
22. \{3, 6\}  
23. \{4, 10\}  
24. \{5, 5\}  
25. \{6\}  
26. \{7\}  
27. \{8\}  
28. \{9\}  
29. \{10\}  
30. \{11\}  
31. \{12\}  
32. \{13\}  
33. \{14\}  
34. \{15\}  
35. \{16\}  
36. \{17\}  
37. \{18\}  
38. \{19\}  
39. \{20\}  
40. \{21\}  
41. \{22\}  
42. \{23\}  
43. \{24\}  
44. \{25\}  
45. \{26\}  
46. \{27\}  
47. \{28\}  
48. \{29\}  
49. \{30\}  
50. \{31\}  
51. \{32\}  
52. \{33\}  
53. \{34\}  
54. \{35\}  
55. \{36\}  
56. \{37\}  
57. \{38\}  
58. \{39\}  
59. \{40\}  
60. \{41\}  
61. \{42\}  
62. \{43\}  
63. \{44\}  
64. \{45\}  
65. \{46\}  
66. \{47\}  
67. \{48\}  
68. \{49\}  
69. \{50\}  
70. \{51\}  
71. \{52\}  
72. \{53\}  
73. \{54\}  
74. \{55\}  
75. \{56\}  
76. \{57\}  
77. \{58\}  
78. \{59\}  
79. \{60\}  
80. \{61\}  
81. \{62\}  
82. \{63\}  
83. \{64\}  
84. \{65\}  
85. \{66\}  
86. \{67\}  
87. \{68\}  
88. \{69\}  
89. \{70\}  
90. \{71\}  
91. \{72\}  
92. \{73\}  
93. \{74\}  
94. \{75\}  
95. \{76\}  
96. \{77\}  
97. \{78\}  
98. \{79\}  
99. \{80\}  
100. \{81\}