4.5 Solving Absolute Value Equations and Inequalities Graphically

4.5 OBJECTIVES

1. Draw the graph of an absolute value function
2. Solve an absolute value equation graphically
3. Solve an absolute value inequality graphically

Equations may contain absolute value notation in their statements. In this section, we will look at graphically solving statements that include absolute values.

To look at a graphical approach to solving, we must first look at the graph of an absolute value function. We will start by looking at the graph of the function \( f(x) = |x| \). All other graphs of absolute value functions are variations of this graph.

The graph can be found using a graphing calculator (most graphing calculators use \texttt{abs} to represent the absolute value). We will develop the graph from a table of values.

\[
\begin{array}{|c|c|}
\hline
x & f(x) = |x| \\
\hline
-3 & 3 \\
-2 & 2 \\
-1 & 1 \\
0 & 0 \\
1 & 1 \\
2 & 2 \\
\hline
\end{array}
\]

Plotting these ordered pairs, we see a pattern emerge. The graph is like a large V that has its vertex at the origin. The slope of the line to the right of 0 is 1, and the slope of the line to the left of 0 is \(-1\).

Let us now see what happens to the graph when we add or subtract some constant inside the absolute value bars.
Example 1

Graphing an Absolute Value Function

Graph each function.

(a) \( f(x) = |x - 3| \)

Again, we start with a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Then, we plot the points associated with the set of ordered pairs.

The graph of the function \( f(x) = |x - 3| \) is the same shape as the graph of the function \( f(x) = |x| \); it has just shifted to the right 3 units.

(b) \( f(x) = |x + 1| \)

We begin with a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>
SOLVING ABSOLUTE VALUE EQUATIONS AND INEQUALITIES GRAPHICALLY

SECTION 4.5

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CHECK YOURSELF 1

Graph each function.

(a) \( f(x) = |x - 2| \)  
(b) \( f(x) = |x + 3| \)

We can summarize what we have discovered about the horizontal shift of the graph of an absolute value function.

CHECK YOURSELF 2

Graph each function.

(a) \( f(x) = |x - 2| \)  
(b) \( f(x) = |x + 3| \)

We can summarize what we have discovered about the horizontal shift of the graph of an absolute value function.

Rules and Properties: Graphing Absolute Value Functions

The graph of the function \( f(x) = |x - a| \) will be the same shape as the graph of \( f(x) = |x| \) except that the graph will be shifted \( a \) units

to the right if \( a \) is positive

to the left if \( a \) is negative

We will now use these methods to solve equations that contain an absolute value expression.

Example 2

Solving an Absolute Value Equation Graphically

Graphically, find the solution set for the equation.

\( |x - 3| = 4 \)

\( x - 3 = 4 \) or \( x - 3 = -4 \)

\( x = 7 \) or \( x = -1 \)
We graph the function associated with each side of the equation.

\[ f(x) = |x - 3| \quad \text{and} \quad g(x) = 4 \]

Then, we draw a vertical line through each of the intersection points.

Looking at the \( x \) values of the two vertical lines, we find the solutions to the original equation. There are two \( x \) values that make the statement true: \(-1\) and \(7\). The solution set is \(-1, 7\).

**CHECK YOURSELF 2**

**Graphically find the solution set for the equation.**

\[ |x - 2| = 3 \]

Example 3 demonstrates a graphical approach to solving an absolute value inequality.

**Example 3**

**Solving an Absolute Value Inequality Graphically**

Graphically solve

\[ |x| < 6 \]

As we did in previous sections, we begin by letting each side of the inequality represent a function. Here

\[ f(x) = |x| \quad \text{and} \quad g(x) = 6 \]
Now we graph both functions on the same set of axes.

We next draw a dotted line (equality is not included) through the points of intersection of the two graphs.

The solution set is any value of \( x \) for which the graph of \( f(x) \) is below the graph of \( g(x) \).

In set notation, we write \( \{x | -6 < x < 6\} \).

**CHECK YOURSELF 3**

Graphically solve the inequality

\[ |x| > 3 \]
CHECK YOURSELF ANSWERS

1. (a) \( f(x) = |x - 2| \)

![Graph of \( f(x) = |x - 2| \)]

(b) \( f(x) = |x + 3| \)

![Graph of \( f(x) = |x + 3| \)]

2. \([-1, 5]\)

![Graph showing \([-1, 5]\)]

3. \(\{x| x < -3 \text{ or } x > 3\}\)

![Graph showing \(\{-3, 3\}\)]
4.5 Exercises

In exercises 1 to 6, graph each function.

1. \( f(x) = |x - 3| \)  
   ![Graph of \( f(x) = |x - 3| \)]

2. \( f(x) = |x + 2| \)  
   ![Graph of \( f(x) = |x + 2| \)]

3. \( f(x) = |x + 3| \)  
   ![Graph of \( f(x) = |x + 3| \)]

4. \( f(x) = |x - 4| \)  
   ![Graph of \( f(x) = |x - 4| \)]

5. \( f(x) = |x - (-3)| \)  
   ![Graph of \( f(x) = |x - (-3)| \)]

6. \( f(x) = |x - (-5)| \)  
   ![Graph of \( f(x) = |x - (-5)| \)]

In exercises 7 to 12, solve the equations graphically.

7. \( |x| = 3 \)  
   ![Graph of \( |x| = 3 \)]

8. \( |x| = 5 \)  
   ![Graph of \( |x| = 5 \)]
In exercises 13 to 16, determine the function represented by each graph.

13. 

14. 

15. 

16. 

9. \( |x - 2| = \frac{7}{2} \) 

10. \( |x - 5| = 3 \) 

11. \( |x + 2| = 4 \) 

12. \( |x + 4| = 2 \)
In exercises 17 to 28, solve each inequality graphically.

17. $|x| < 4$

18. $|x| < 6$

19. $|x| \geq 5$

20. $|x| \geq 2$

21. $|x - 3| < 4$

22. $|x - 1| < 5$

23. $|x - 2| \geq 5$

24. $|x + 2| > 4$
25. \(|x + 1| \leq 5\)

26. \(|x + 4| > 1\)

27. \(|x + 2| \geq -2\)

28. \(|x - 4| > -1\)

29. **Assessing Piston Design.** Combustion engines get their power from the force exerted by burning fuel on a piston inside a chamber. The piston is forced down out of the cylinder by the force of a small explosion caused by burning fuel mixed with air. The piston in turn moves a piston rod, which transfers the motion to the work of the engine. The rod is attached to a flywheel, which pushes the piston back into the cylinder to begin the process all over. Cars usually have four to eight of these cylinders and pistons. It is crucial that the piston and the cylinder fit well together, with just a thin film of oil separating the sides of the piston and the sides of the cylinder. When these are manufactured, the measurements for each part must be accurate. But, there is always some error. How much error is a matter for the engineers to set and for the quality control department to check.

Suppose the diameter of the cylinder is meant to be 7.6 cm, and the engineer specifies that this part must be manufactured to within 0.1 mm of that measurement. This figure is called the **tolerance.** As parts come off the assembly line, someone in quality control takes samples and measures the cylinders and the pistons. Given this information, complete the following.

1. Write an absolute value statement about the diameter, \(d_c\), of the cylinder.
2. If the diameter of the piston is to be 7.59 cm with a tolerance of 0.1 mm, write an absolute value statement about the diameter, \(d_p\), of the piston.
3. Investigate all the possible ways these two parts will fit together. If the two parts have to be within 0.1 mm of each other for the engine to run well, is there a problem with the way the parts may be paired together? Write your answer and use a graph to explain.
4. Accuracy in machining the parts is expensive, so the tolerance should be close enough to make sure the engine runs correctly, but not so close that the cost is prohibitive. If you think a tolerance of 0.1 mm is too large, find another that you think would work better. If it is too small, how much can it be enlarged and still have the engine run according to design? (That is, so \(|d_e - d_p| \leq 0.1\) mm.) Write the tolerance using absolute value signs. Explain your reasoning if you think a tolerance of 0.1 mm is not workable.

5. After you have decided on the appropriate tolerance for these parts, think about the quality control engineer’s job. Hazard a few educated opinions to answer these questions: How many parts should be pulled off the line and measured? How often? How many parts can reasonably be expected to be outside the expected tolerance before the whole line is shut down and the tools corrected?

**Answers**

1. 

3. 

5. 

7. 

9. 

11. 

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13. \( f(x) = |x - 2| \)
15. \( f(x) = |x + 2| \)
17. \( f(x) = |x| \)
   \( g(x) = 4 \)
   \( -4 < x < 4 \)
19. \( f(x) = |x| \)
   \( g(x) = 5 \)
   \( x \leq -5 \) or \( x \geq 5 \)
21. \( f(x) = |x - 3| \)
   \( g(x) = 4 \)
   \( -1 < x < 7 \)
23. \( f(x) = |x - 2| \)
   \( g(x) = 5 \)
   \( x \leq -3 \) or \( x \geq 7 \)
25. \( f(x) = |x + 1| \)
   \( g(x) = 5 \)
   \( -6 \leq x \leq 4 \)
27. \( f(x) = |x + 2| \)
   \( g(x) = -2 \)
   All real numbers