Solving Quadratic Equations by Completing the Square

**OBJECTIVES**

1. Solve a quadratic equation by the square root method
2. Solve a quadratic equation by completing the square
3. Solve a geometric application involving a quadratic equation

In Section 6.9, we solved quadratic equations by factoring and using the zero product rule. However, not all equations are factorable over the integers. In this section, we will look at another method that can be used to solve a quadratic equation, called the square root method. First, we will solve a special type of equation using the factoring method of Chapter 6.

**Example 1**

**Solving Equations by Factoring**

Solve the quadratic equation \( x^2 = 16 \) by factoring.

We write the equation in standard form:

\[ x^2 - 16 = 0 \]

Factoring, we have

\( (x + 4)(x - 4) = 0 \)

Finally, the solutions are

\[ x = -4 \quad \text{or} \quad x = 4 \quad \text{or} \quad \{±4\} \]

**CHECK YOURSELF 1**

Solve each of the following quadratic equations.

(a) \( 5x^2 = 180 \) \hspace{1cm} (b) \( x^2 = 25 \)

The equation in Example 1 could have been solved in an alternative fashion. We could have used what is called the square root method. Again, given the equation

\[ x^2 = 16 \]

we can write the equivalent statement

\[ x = \sqrt{16} \quad \text{or} \quad x = -\sqrt{16} \]

This yields the solutions

\[ x = 4 \quad \text{or} \quad x = -4 \quad \text{or} \quad \{±4\} \]

This discussion leads us to the following general result.

**NOTE** Be sure to include both the positive and the negative square roots when you use the square root method.
Example 2 further illustrates the use of this property.

**Example 2**

**Using the Square Root Method**

Solve each equation by using the square root method.

**(a)** \(x^2 = 9\)

By the square root property,

\[ x = \sqrt{9} \quad \text{or} \quad x = -\sqrt{9} \]

\[ = 3 \quad \text{or} \quad = -3 \quad \text{or} \quad \{\pm 3\} \]

**(b)** \(x^2 - 17 = 0\)

Add 17 to both sides of the equation.

\[ x^2 = 17 \]

so \( x = \pm \sqrt{17} \quad \text{or} \quad \{\pm \sqrt{17}\} \quad \text{or} \quad \{-\sqrt{17}, \sqrt{17}\} \)

**(c)** \(2x^2 - 3 = 0\)

\[ 2x^2 = 3 \]

\[ x^2 = \frac{3}{2} \]

\[ x = \pm \sqrt{\frac{3}{2}} \]

\[ x = \pm \frac{\sqrt{6}}{2} \quad \text{or} \quad \{\pm \frac{\sqrt{6}}{2}\} \]

**(d)** \(x^2 + 1 = 0\)

\[ x^2 = -1 \]

\[ x = \pm \sqrt{-1} = \pm i \quad \text{or} \quad \{\pm i\} \]

**CHECK YOURSELF 2**

Solve each equation.

**(a)** \(x^2 = 5\)  **(b)** \(x^2 - 2 = 0\)  **(c)** \(3x^2 - 8 = 0\)  **(d)** \(x^2 + 9 = 0\)

We can also use the approach in Example 2 to solve an equation of the form

\[(x + 3)^2 = 16\]
As before, by the square root property we have
\[ x + 3 = \pm 4 \]
Subtract 3 from both sides of the equation.

Solving for \( x \) yields
\[ x = -3 \pm 4 \]
which means that there are two solutions:
\[ x = -3 + 4 \quad \text{or} \quad x = -3 - 4 \]
\[ = 1 \quad \text{or} \quad -7 \quad \text{or} \quad \{1, -7\} \]

Example 3

Using the Square Root Method

Use the square root method to solve each equation.

(a) \((x - 5)^2 - 5 = 0\)

\[(x - 5)^2 = 5\]
\[x - 5 = \pm \sqrt{5}\]
\[x = 5 \pm \sqrt{5} \quad \text{or} \quad \{5 \pm \sqrt{5}\}\]

(b) \(3(y + 1)^2 - 2 = 0\)

\[3(y + 1)^2 = 2\]
\[(y + 1)^2 = \frac{2}{3}\]
\[y + 1 = \pm \sqrt{\frac{2}{3}}\]
\[y = -1 \pm \frac{\sqrt{6}}{3}\]
\[= -3 \pm \frac{\sqrt{6}}{3} \quad \text{or} \quad \left\{-3 \pm \frac{\sqrt{6}}{3}\right\}\]

The approximate solutions are \{-1.82, -0.18\}.

CHECK YOURSELF 3

Using the square root method, solve each equation.

(a) \((x - 2)^2 - 3 = 0\)  \hspace{1cm} (b) \(2(x - 1)^2 = 1\)

Not all quadratic equations can be solved directly by factoring or using the square root method. We must extend our techniques.

The square root method is useful in this process because any quadratic equation can be written in the form
\[(x + h)^2 = k\]
which yields the solution
\[x = -h \pm \sqrt{k}\]
The process of changing an equation in standard form
\[ ax^2 + bx + c = 0 \]
to the form
\[ (x + h)^2 = k \]
is called the method of **completing the square**, and it is based on the relationship between the middle term and the last term of any perfect-square trinomial.

Let's look at three perfect-square trinomials to see whether we can detect a pattern:

1. \[ x^2 + 4x + 4 = (x + 2)^2 \]
2. \[ x^2 - 6x + 9 = (x - 3)^2 \]
3. \[ x^2 + 8x + 16 = (x + 4)^2 \]

Note that in each case the last (or constant) term is the square of one-half of the coefficient of \( x \) in the middle (or linear) term. For example, in equation (2),
\[ \frac{1}{2} \text{ of this coefficient is } -3, \text{ and } (-3)^2 = 9, \text{ the constant.} \]

Verify this relationship for yourself in equation (3). To summarize, in perfect-square trinomials, the constant is always the square of one-half the coefficient of \( x \).

We are now ready to use the above observation in the solution of quadratic equations by completing the square. Consider Example 4.

### Example 4

**Completing the Square to Solve an Equation**

Solve \( x^2 + 8x - 7 = 0 \) by completing the square.

First, we rewrite the equation with the constant on the right-hand side:
\[ x^2 + 8x = 7 \]

Our objective is to have a perfect-square trinomial on the left-hand side. We know that we must add the square of one-half of the \( x \) coefficient to complete the square. In this case, that value is 16, so now we add 16 to each side of the equation.
\[ x^2 + 8x + 16 = 7 + 16 \]

Factor the perfect-square trinomial on the left, and combine like terms on the right to yield \( (x + 4)^2 = 23 \)

Now the square root property yields
\[ x + 4 = \pm \sqrt{23} \]

Subtracting 4 from both sides of the equation gives
\[ x = -4 \pm \sqrt{23} \quad \text{or} \quad \{-4 \pm \sqrt{23}\} \]

As decimals, these solutions are approximated by \{-8.8, 0.8\}.

### Check Yourself 4

Solve \( x^2 - 6x - 2 = 0 \) by completing the square.
Example 5

Completing the Square to Solve an Equation

Solve \( x^2 + 5x - 3 = 0 \) by completing the square.

\[
egin{align*}
x^2 + 5x - 3 &= 0 & \text{Add 3 to both sides.} \\
x^2 + 5x &= 3 & \text{Make the left-hand side a perfect square.}
\end{align*}
\]

\[
\begin{align*}
x^2 + 5x + \left(\frac{5}{2}\right)^2 &= 3 + \left(\frac{5}{2}\right)^2 \\
\left(x + \frac{5}{2}\right)^2 &= \frac{37}{4} & \text{Take the square root of both sides.}
\end{align*}
\]

\[
x + \frac{5}{2} = \pm \frac{\sqrt{37}}{2} & \text{ Solve for } x.
\]

\[
x = -5 \pm \frac{\sqrt{37}}{2}
\]

or \( \left\{ -\frac{5}{2} \pm \frac{\sqrt{37}}{2} \right\} \)

The approximate solutions are \( \{-5.54, 0.54\} \).

Check Yourself 5

Solve \( x^2 + 3x - 7 = 0 \) by completing the square.

Some equations have nonreal complex solutions, as Example 6 illustrates.

Example 6

Completing the Square to Solve an Equation

Solve \( x^2 + 4x + 13 = 0 \) by completing the square.

\[
\begin{align*}
x^2 + 4x + 13 &= 0 & \text{Subtract 13 from both sides.} \\
x^2 + 4x &= -13 & \text{Add} \left(\frac{1}{2} \times 4\right)^2 \text{ to both sides.}
\end{align*}
\]

\[
\begin{align*}
x^2 + 4x + 4 &= -13 + 4 \\
(x + 2)^2 &= -9 & \text{Factor the left-hand side.}
\end{align*}
\]

\[
\begin{align*}
x + 2 &= \pm \sqrt{-9} & \text{Take the square root of both sides.} \\
x + 2 &= \pm \sqrt{9}i \\
x + 2 &= \pm 3i & \text{Simplify the radical.}
\end{align*}
\]

\[
x = -2 \pm 3i \quad \text{or} \quad \{-2 \pm 3i\}
\]

Check Yourself 6

Solve \( x^2 + 10x + 41 = 0 \).

Example 7 illustrates a situation in which the leading coefficient of the quadratic member is not equal to 1. As you will see, an extra step is required.
Completing the Square to Solve an Equation

Example 7

Solve $3x^2 + 6x - 7 = 0$ by completing the square.

$$3x^2 + 6x - 7 = 0$$  Add 7 to both sides.

$$3x^2 + 6x = 7$$  Divide both sides by 3.

$$x^2 + 2x = \frac{7}{3}$$  Now, complete the square on the left.

$$x^2 + 2x + 1 = \frac{7}{3} + 1$$  The left side is now a perfect square.

$$(x + 1)^2 = \frac{10}{3}$$

$$x + 1 = \pm \sqrt{\frac{10}{3}}$$

$$x = -1 \pm \sqrt{\frac{10}{3}}$$

The following algorithm summarizes our work in this section with solving quadratic equations by completing the square.

Step by Step: Completing the Square

1. Isolate the constant on the right side of the equation.
2. Divide both sides of the equation by the coefficient of the $x^2$ term if that coefficient is not equal to 1.
3. Add the square of one-half of the coefficient of the linear term to both sides of the equation. This will give a perfect-square trinomial on the left side of the equation.
4. Write the left side of the equation as the square of a binomial, and simplify on the right side.
5. Use the square root property, and then solve the resulting linear equations.

Let's proceed now to applications involving geometry.

Example 8

Applying the Completing of a Square

The length of a rectangle is 4 cm greater than its width. If the area of the rectangle is 108 cm², what are the approximate dimensions of the rectangle?

Step 1  You are asked to find the dimensions (the length and the width) of the rectangle.
Solving Quadratic Equations by Completing the Square  

**Step 2** Whenever geometric figures are involved in an application, start by drawing, and then labeling, a sketch of the problem. Letting \( x \) represent the width and \( x + 4 \) the length, we have

![Diagram showing a rectangle with width \( x \) and length \( x + 4 \)]

**Step 3** The area of a rectangle is the product of its length and width, so

\[
x(x + 4) = 108
\]

**NOTE** Multiply and complete the square.

**Step 4**

\[
x(x + 4) = 108
\]

\[
x^2 + 4x = 108
\]

\[
x^2 + 4x + 4 = 108 + 4
\]

\[
(x + 2)^2 = 112
\]

\[
x + 2 = \pm \sqrt{112}
\]

\[
x = -2 \pm \sqrt{112}
\]

**Step 5** We reject \(-2 - \sqrt{112}\) (cm) as a solution. A length cannot be negative, and so we must consider only \(-2 + \sqrt{112}\) (cm) in finding the required dimensions.

The width \( x \) is approximately 8.6 cm, and the length \( x + 4 \) is 12.6 cm. Because (8.6 cm) (12.6 cm) gives a rectangle of area 108.36 cm\(^2\), the solution is verified.

**Example 8**

In a triangle, the base is 4 in. less than its height. If its area is 35 in.\(^2\), find the length of the base and the height of the triangle.

**Example 9**

Applying the Completing of a Square

An open box is formed from a rectangular piece of cardboard, whose length is 2 in. more than its width, by cutting 2-in. squares from each corner and folding up the sides. If the volume of the box is to be 100 in.\(^3\), what must be the size of the original piece of cardboard?

**Step 1** We are asked for the dimensions of the sheet of cardboard.

**Step 2** Again sketch the problem.
Step 3  To form an equation for volume, we sketch the completed box.

NOTE The original width of the cardboard was \(x\). Removing two 2-in. squares leaves \(x - 4\) for the width of the box. Similarly, the length of the box is \(x - 2\). Do you see why?

Because volume is the product of height, length, and width,

\[
2(x - 2)(x - 4) = 100
\]

Step 4

\[
2(x - 2)(x - 4) = 100
\]

\[
(x - 2)(x - 4) = 50
\]

\[
x^2 - 6x + 8 = 50
\]

\[
x^2 - 6x = 42
\]

\[
x^2 - 6x + 9 = 42 + 9
\]

\[
(x - 3)^2 = 51
\]

\[
x - 3 = \pm \sqrt{51}
\]

\[
x = 3 \pm \sqrt{51}
\]

Step 5  Again, we need consider only the positive solution. The width \(x\) of the original piece of cardboard is approximately 10.14 in., and its length \(x + 2\) is 12.14 in. The dimensions of the completed box will be 6.14 by 8.14 by 2 in., which gives volume of an approximate 100 in.³.

CHECK YOURSELF 9

A similar box is to be made by cutting 3-cm squares from a piece of cardboard that is 4 cm longer than it is wide. If the required volume is 300 cm³, find the dimensions of the original sheet of cardboard.

CHECK YOURSELF ANSWERS

1. (a) \{-6, 6\}; (b) \{-5, 5\}
2. (a) \{\sqrt{5}, -\sqrt{5}\}; (b) \{\sqrt{2}, -\sqrt{2}\}; (c) \left\{\frac{2\sqrt{6}}{3}, -\frac{2\sqrt{6}}{3}\right\}; (d) \{3i, -3i\}
3. (a) \{2 \pm \sqrt{3}\}; (b) \left\{\frac{2 \pm \sqrt{2}}{2}\right\}
4. \{3 \pm \sqrt{11}\}
5. \left\{-\frac{3 \pm \sqrt{37}}{2}\right\}
6. \{-5 \pm 4i\}
7. \left\{\frac{4 \pm \sqrt{10}}{2}\right\}
8. Base \approx 6.6\text{ in.}; height \approx 10.6\text{ in.}
9. 14.2\text{ cm} \times 18.2\text{ cm}
In exercises 1 to 8, solve by factoring or completing the square.

1. \( x^2 + 6x + 5 = 0 \)  
2. \( x^2 + 5x + 6 = 0 \)

3. \( z^2 - 2z - 35 = 0 \)  
4. \( q^2 - 5q - 24 = 0 \)

5. \( 2x^2 - 5x - 3 = 0 \)  
6. \( 3x^2 + 10x - 8 = 0 \)

7. \( 6y^2 - y - 2 = 0 \)  
8. \( 10z^2 + 3z - 1 = 0 \)

In exercises 9 to 20, use the square root method to find solutions for the equations.

9. \( x^2 = 36 \)  
10. \( x^2 = 144 \)

11. \( y^2 = 7 \)  
12. \( p^2 = 18 \)

13. \( 2x^2 - 12 = 0 \)  
14. \( 3x^2 - 66 = 0 \)

15. \( 2t^2 + 12 = 4 \)  
16. \( 3u^2 - 5 = -32 \)

17. \( (x + 1)^2 = 12 \)  
18. \( (2x - 3)^2 = 5 \)

19. \( (2z + 1)^2 - 3 = 0 \)  
20. \( (3p - 4)^2 + 9 = 0 \)
In exercises 21 to 32, find the constant that must be added to each binomial expression to form a perfect-square trinomial.

21. \(x^2 + 12x\)  
22. \(r^2 - 14r\)

23. \(y^2 - 8y\)  
24. \(w^2 + 16w\)

25. \(x^2 - 3x\)  
26. \(z^2 + 5z\)

27. \(n^2 + n\)  
28. \(x^2 - x\)

29. \(x^2 + \frac{1}{2}x\)  
30. \(x^2 - \frac{1}{3}x\)

31. \(x^2 + 2ax\)  
32. \(y^2 - 4ay\)

In exercises 33 to 54, solve each equation by completing the square.

33. \(x^2 + 12x - 2 = 0\)  
34. \(x^2 - 14x - 7 = 0\)

35. \(y^2 - 2y = 8\)  
36. \(z^2 + 4z - 72 = 0\)

37. \(x^2 - 2x - 5 = 0\)  
38. \(x^2 - 2x = 3\)

39. \(x^2 + 10x + 13 = 0\)  
40. \(x^2 + 3x - 17 = 0\)

41. \(z^2 - 5z - 7 = 0\)  
42. \(q^2 - 8q + 20 = 0\)

43. \(m^2 - m - 3 = 0\)  
44. \(y^2 + y - 5 = 0\)

45. \(x^2 + \frac{1}{2}x = 1\)  
46. \(x^2 - \frac{1}{3}x = 2\)

47. \(2x^2 + 2x - 1 = 0\)  
48. \(3x^2 - 3x = 1\)
49. \( 3x^2 - 6x = 2 \)

50. \( 4x^2 + 8x - 1 = 0 \)

51. \( 3x^2 - 2x + 12 = 0 \)

52. \( 7y^2 - 2y + 3 = 0 \)

53. \( x^2 + 8x + 20 = 0 \)

54. \( x^2 - 2x + 10 = 0 \)

In exercises 55 to 60, find the constant that must be added to each binomial to form a perfect-square trinomial. Let \( x \) be the variable; other letters represent constants.

55. \( x^2 + 2ax \)

56. \( x^2 + 2abx \)

57. \( x^2 + 3ax \)

58. \( x^2 + abx \)

59. \( a^2x^2 + 2ax \)

60. \( a^2x^2 + 4abx \)

In exercises 61 and 62, solve each equation by completing the square.

61. \( x^2 + 2ax = 4 \)

62. \( x^2 + 2ax - 8 = 0 \)

Solve the following applications.

63. The width of a rectangle is 3 ft less than its length. If the area of the rectangle is 70 ft\(^2\), what are the dimensions of the rectangle?

64. The length of a rectangle is 5 cm more than its width. If the area of the rectangle is 84 cm\(^2\), find the dimensions of the rectangle.

65. The length of a rectangle is 2 cm more than 3 times its width. If the area of the rectangle is 85 cm\(^2\), find the dimensions of the rectangle.

66. If the length of a rectangle is 3 ft less than twice its width and the area of the rectangle is 54 ft\(^2\), what are the dimensions of the rectangle?

67. The length of a rectangle is 1 cm more than its width. If the length of the rectangle is doubled, the area of the rectangle is increased by 30 cm\(^2\). What were the dimensions of the original rectangle?
68. A box is to be made from a rectangular piece of tin that is twice as long as it is wide. To accomplish this, a 10-cm square is cut from each corner, and the sides are folded up. The volume of the finished box is to be 5000 cm$^3$. Find the dimensions of the original piece of tin.

Hint 1: To solve this equation, you will want to use the following sketch of the piece of tin. Note that the original dimensions are represented by $x$ and $2x$. Do you see why? Also recall that the volume of the resulting box will be the product of the length, width, and height.

\[ (2x - 20)(x - 20)10 = 5000 \]

69. An open box is formed from a square piece of material by cutting 2-in. squares from each corner of the material and folding up the sides. If the volume of the box that is formed is 72 in.$^3$, what was the size of the original piece of material?

70. An open carton is formed from a rectangular piece of cardboard that is 4 ft longer than it is wide, by removing 1-ft squares from each corner and folding up the sides. If the volume of the carton is then 12 ft$^3$, what were the dimensions of the original piece of cardboard?

71. A box that has a volume of 2000 in.$^3$ was made from a square piece of tin. The square piece cut from each corner had sides of length 4 in. What were the original dimensions of the square?

72. A square piece of cardboard is to be formed into a box. After 5-cm squares are cut from each corner and the sides are folded up, the resulting box will have a volume of 400 cm$^3$. Find the length of a side of the original piece of cardboard.

73. Why must the leading coefficient of the quadratic member be set equal to 1 before using the technique of completing the square?

74. What relationship exists between the solutions of a quadratic equation and the graph of a quadratic function?
In exercises 75 to 78, use your graphing utility to find the graph. For each graph, approximate the x-intercepts to the nearest tenth. (You may have to adjust the viewing window to see both intercepts.)

75. \( y = x^2 + 12x - 2 \)

76. \( y = x^2 - 14x - 7 \)

77. \( y = x^2 - 2x - 8 \)

78. \( y = x^2 + 4x - 72 \)

79. On your graphing calculator, view the graph of \( f(x) = x^2 + 1 \).
   
   (a) What can you say about the x-intercepts of the graph?
   
   (b) Determine the zeros of the function, using the square root method.
   
   (c) How does your answer to part (a) relate to your answer to part (b)?

80. Consider the following representation of “completing the square”: Suppose we wish to complete the square for \( x^2 + 10x \). A square with dimensions \( x \) by \( x \) has area equal to \( x^2 \).

   \[
   \begin{array}{c}
   \hline
   x & x^2 \\
   \hline
   x & x \\
   \hline
   \end{array}
   \]

   We divide the quantity 10x by 2 and get 5x. If we extend the base \( x \) by 5 units, and draw the rectangle attached to the square, the rectangle’s dimensions are 5 by \( x \) with an area of 5x.
Now we extend the height by 5 units, and draw another rectangle whose area is $5x$.

(a) What is the total area represented in the figure so far?
(b) How much area must be added to the figure to “complete the square”?
(c) Write the area of the completed square as a binomial squared.

81. Repeat the process described in exercise 80 with $x^2 + 16x$.

Answers

1. $\{-5, -1\}$  
3. $\{-5, 7\}$  
5. $\left\{-\frac{1}{2}, 3\right\}$  
7. $\left\{-\frac{1}{2},\frac{2}{3}\right\}$  
9. $\{±6\}$  
11. $\{-\sqrt{7}, \sqrt{7}\}$  
13. $\{-\sqrt{6}, \sqrt{6}\}$  
15. $\{±2i\}$  
17. $\{-1 ± 2\sqrt{3}\}$  
19. $\left\{-\frac{1 ± \sqrt{3}}{2}\right\}$  
21. 36  
23. 16  
25. $\frac{9}{4}$  
27. $\frac{1}{4}$  
29. $\frac{1}{16}$  
31. $a^2$  
33. $\{-6 ± \sqrt{38}\}$  
35. $\{-2, 4\}$  
37. $\{1 ± \sqrt{5}\}$  
39. $\{-5 ± 2\sqrt{3}\}$  
41. $\left\{\frac{5 ± \sqrt{53}}{2}\right\}$  
43. $\left\{\frac{1 ± \sqrt{13}}{2}\right\}$  
45. $\left\{-\frac{1 ± \sqrt{17}}{4}\right\}$  
47. $\left\{-\frac{1 ± \sqrt{3}}{2}\right\}$  
49. $\left\{\frac{3 ± \sqrt{15}}{3}\right\}$  
51. $\left\{\frac{1 ± i\sqrt{25}}{3}\right\}$  
53. $\{-4 ± 2i\}$  
55. $a^2$  
57. $\frac{9}{4}a^2$  
59. 1

61. $\{-a ± \sqrt{4 + a^2}\}$  
63. Width 7 ft, length 10 ft
65. Width 5 cm, length 17 cm  
67. Width 5 cm, length 6 cm
69. 10 in. by 10 in.  
71. $8 + 10\sqrt{5}$ in. or 30.4 in.  
73.

75.  
77.

79. (a) There are none; (b) $±i$; (c) If the graph of $f(x)$ has no $x$ intercepts, the zeros of the function are not real.

81. (a) $x^2 + 16x$; (b) 64; (c) $x^2 + 16x + 64 = (x + 8)^2$