Equations that Are Quadratic in Form

OBJECTIVES

1. Solve a radical equation that is quadratic in form
2. Solve a fourth degree equation that is quadratic in form

Consider the following equations:

\[ \begin{align*}
2x - 5\sqrt{x} + 3 &= 0 \\
x^4 - 4x^2 + 3 &= 0 \\
(x^2 - x)^2 - 8(x^2 - x) + 12 &= 0
\end{align*} \]

None of these equations are quadratic, yet each can be readily solved by using quadratic methods.

Compare the following quadratic equations to the original three equations.

\[ \begin{align*}
2u^2 - 5u + 3 &= 0 \\
u^2 - 4u + 3 &= 0 \\
u^2 - 8u + 12 &= 0
\end{align*} \]

In each case, a simple substitution has been made that resulted in a quadratic equation. Equations that can be rewritten in this manner are said to be equations in quadratic form.

Example 1

Solving a Radical Equation

Solve.

\[ 2x - 5\sqrt{x} + 3 = 0 \]

By substituting \( u = \sqrt{x} \), we have

\[ 2u^2 - 5u + 3 = 0 \]

Factoring yields

\[ (2u - 3)(u - 1) = 0 \]

which gives the intermediate solutions

\[ u = \frac{3}{2} \quad \text{or} \quad u = 1 \]

We must now solve for \( x \) and then check our solutions. Because \( \sqrt{x} = u \), we can write

\[ \sqrt{x} = \frac{3}{2} \quad \text{or} \quad \sqrt{x} = 1 \]

\[ x = \frac{9}{4} \quad x = 1 \]

To check these solutions, we again simply substitute these values into the original equation. You should verify that each is a valid solution.
For certain equations in quadratic form, we can either solve by substitution (as we have done above) or solve directly by treating the equation as quadratic in some other power of the variable (in the case of the equation of the following example, $x^3$). In our next example, we show both methods of solution.

**Check Yourself 1**

Solve $3x - 8\sqrt{x} + 4 = 0$.

**Example 2**

Solving a Fourth Degree Equation

(a) Solve $x^4 - 4x^2 + 3 = 0$ by substitution:

Let $u = x^2$. Then

\[ u^2 - 4u + 3 = 0 \]

Factoring, we have

\[(u - 1)(u - 3) = 0\]

so

\[ u = 1 \quad \text{or} \quad u = 3 \]

Given these intermediate solutions, because $u = x^2$, we can write

\[ x^2 = 1 \quad \text{or} \quad x^2 = 3 \]

which, by using the square root method, yields the four solutions

\[ x = \pm 1 \quad \text{or} \quad x = \pm \sqrt{3} \]

We can check each of these solutions by substituting into the original equation. When we do so, we find that all four are valid solutions to the original equation.

(b) Solve $x^4 - 4x^2 + 3 = 0$ directly.

By treating the equation as quadratic in $x^2$, we can factor the left member, to write

\[(x^2 - 1)(x^2 - 3) = 0\]

This gives us the two equations

\[ x^2 - 1 = 0 \quad \text{or} \quad x^2 - 3 = 0 \]

Now

\[ x^2 = 1 \quad \text{or} \quad x^2 = 3 \]

Again, we have the four possible solutions

\[ x = \pm 1 \quad \text{or} \quad x = \pm \sqrt{3} \]

All check when they are substituted into the original equation.

**Check Yourself 2**

Solve $x^4 - 9x^2 + 20 = 0$ by substitution and by factoring directly.
In the following example, a binomial is replaced with $u$ to make it easier to proceed with the solution.

**Example 3**

**Solving a Fourth Degree Equation**

Solve.

$$(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$$

Because of the repeated factor $x^2 - x$, we substitute $u$ for $x^2 - x$. Factoring the resulting equation

$$u^2 - 8u + 12 = 0$$

gives

$$(u - 6)(u - 2) = 0$$

So

$$u = 6 \quad \text{or} \quad u = 2$$

We now have two intermediate solutions to work with. Because $u = x^2 - x$, we have two cases:

If $u = 6$, then

- $x^2 - x = 6$
- $x^2 - x - 6 = 0$
- $(x - 3)(x + 2) = 0$
- $x = 3 \quad \text{or} \quad x = -2$

If $u = 2$, then

- $x^2 - x = 2$
- $x^2 - x - 2 = 0$
- $(x - 2)(x + 1) = 0$
- $x = 2 \quad \text{or} \quad x = -1$

The quadratic equations now yield four solutions that we must check. Substituting into the original equation, you will find that all four are valid solutions.

**CHECK YOURSELF 3**

Solve for $x$:

$$(x^2 - 2x)^2 - 11(x^2 - 2x) + 24 = 0$$

To summarize our work with equations in quadratic form, two approaches are commonly used. The first involves substitution of a new intermediate variable to make the original equation quadratic. The second solves the original equation directly by treating the equation as quadratic in some other power of the original variable. The following algorithms outline the two approaches.

**Step by Step: Solving by Substitution**

1. **Step 1** Make an appropriate substitution so that the equation becomes quadratic.
2. **Step 2** Solve the resulting equation for the intermediate variable.
3. **Step 3** Use the intermediate values found in step 2 to find possible solutions for the original variable.
4. **Step 4** Check the solutions of step 3 by substitution into the original equation.
Step by Step: Solving by Factoring

Step 1  Treat the original equation as quadratic in some power of the variable, and factor.
Step 2  Solve the resulting equations.
Step 3  Check the solutions of step 2 by substitution into the original equation.

CHECK YOURSELF ANSWERS

1. \( \left\{ \frac{4}{9}, 4 \right\} \)  
2. \( \{\pm 2, \pm \sqrt{3}\} \)  
3. \( \{-1, -2, 3, 4\} \)
9.4 Exercises

Solve each of the following equations by factoring directly and then applying the zero product rule.

1. \( x^4 - 9x^2 + 20 = 0 \)
2. \( t^4 - 7t^2 + 12 = 0 \)

3. \( x^4 + x^2 - 12 = 0 \)
4. \( x^4 - 7x^2 - 18 = 0 \)

5. \( 2w^4 - 9w^2 + 4 = 0 \)
6. \( 3x^4 - 5x^2 + 2 = 0 \)

7. \( x^4 - 4x^2 + 4 = 0 \)
8. \( y^4 - 6y^2 + 9 = 0 \)

9. \( 3x^4 + 16x^2 - 12 = 0 \)
10. \( 2x^4 + 9x^2 - 5 = 0 \)

11. \( 2z^4 + 4z^2 - 70 = 0 \)
12. \( 3y^4 + 10y^2 - 8 = 0 \)

13. \( 4t^4 - 20t^2 = 0 \)
14. \( r^4 - 81 = 0 \)

Solve each of the following equations.

15. \( x^4 - x^2 - 12 = 0 \)
16. \( w^4 + w^2 - 12 = 0 \)

17. \( 2y^4 + y^2 - 15 = 0 \)
18. \( 4x^4 - 5x^2 - 9 = 0 \)

19. \( b - 20\sqrt{b} + 64 = 0 \)
20. \( z - 6\sqrt{z} + 8 = 0 \)
Solve each of the following equations by any method.

21. \( t - 8\sqrt{t} - 9 = 0 \)
22. \( y - 24\sqrt{y} - 25 = 0 \)
23. \( (x - 2)^2 - 3(x - 2) - 10 = 0 \)
24. \( (w + 1)^2 - 5(w + 1) + 6 = 0 \)
25. \( (x^2 + 2x)^2 + 3(x^2 + 2x) + 2 = 0 \)
26. \( (x^2 - 4x)^2 - (x^2 - 4x) - 12 = 0 \)

An equation involving rational exponents may sometimes be solved by substitution. For instance, to solve an equation of the form \( ax^{1/2} + bx^{1/4} + c = 0 \), make the substitution \( u = x^{1/4} \). Note that \( u^2 = (x^{1/4})^2 = x^{2/4} = x^{1/2} \).

Use the suggestion above to solve each of the following equations. Be sure to check your solutions.

27. \( 7m - 41\sqrt{m} - 6 = 0 \)
28. \( (x + 1) - 6\sqrt{x + 1} + 8 = 0 \)
29. \( (w - 3)^2 - 2(w - 3) = 15 \)
30. \( (x^2 - 4x)^2 + 7(x^2 - 4x) + 12 = 0 \)
31. \( 2y^4 - 5y^2 = 12 \)
32. \( 4t^4 - 29t^2 + 25 = 0 \)
33. \( x^{1/2} - 4x^{1/4} + 3 = 0 \)
34. \( x^{1/2} - 5x^{1/4} + 6 = 0 \)
35. \( x^{1/2} - x^{1/4} = 2 \)
36. \( 2x^{1/2} + x^{1/4} - 1 = 0 \)
37. \( x^{2/3} + 2x^{1/3} - 3 = 0 \)
38. \( x^{2/5} - x^{1/5} = 6 \)
Certain equations involving rational expressions can also be solved by the method of substitution. For instance, to solve an equation of the form

\[ \frac{a}{x^2} + \frac{b}{x} + c = 0 \]

make the substitution \( u = \frac{1}{x} \). Note that \( u^2 = \left( \frac{1}{x} \right)^2 = \frac{1}{x^2} \).

Use the suggestion above to solve the following equations.

39. \( \frac{1}{x^2} - \frac{6}{x} + 8 = 0 \)  
40. \( \frac{2}{x^2} - \frac{1}{x} = 3 \)

41. \( \frac{3}{x^2} - \frac{5}{x} = 2 \)  
42. \( \frac{1}{(x + 1)^2} - \frac{5}{x + 1} + 4 = 0 \)

43. \( \frac{1}{(x - 2)^2} + \frac{1}{x - 2} = 6 \)  
44. \( \frac{8}{(x - 3)^2} - \frac{2}{x - 3} = 1 \)

Solve each of the following applications.

45. Number problem. The sum of an integer and twice its square root is 24. What is the integer?

46. Number problem. The sum of an integer and 3 times its square root is 40. Find the integer.

47. Number problem. The sum of the reciprocal of an integer and the square of its reciprocal is \( \frac{3}{4} \). What is the integer?

48. Number problem. The difference between the reciprocal of an integer and the square of its reciprocal is \( \frac{2}{9} \). Find the integer.
Answers

1. \{\pm \sqrt{5}, \pm 2\}  
2. \{\pm \sqrt{3}, \pm 2i\}  
3. \{\pm \sqrt{5}, \pm 2i\}  
5. \left\{ \frac{\pm \sqrt{2}}{2}, \pm 2 \right\}  
7. \{\pm \sqrt{2}\}  
9. \left\{ \frac{\pm \sqrt{6}}{3}, \pm i\sqrt{6} \right\}  
11. \{\pm \sqrt{5}, \pm i\sqrt{7}\}  
13. \{0, \pm \sqrt{5}\}  
15. \{\pm 2, \pm i\sqrt{3}\}  
17. \left\{ \frac{\pm \sqrt{10}}{2}, \pm i\sqrt{3} \right\}  
19. \{16, 256\}  
21. \{81\}  
23. \{7, 0\}  
25. \{-1, -1 \pm i\}  
27. \{36\}  
29. \{0, 8\}  
31. \left\{ \pm 2, \frac{\pm i\sqrt{6}}{2} \right\}  
33. \{1, 81\}  
35. \{16\}  
37. \{-27, 1\}  
39. \left\{ \frac{1}{4}, \frac{1}{2} \right\}  
41. \left\{ -3, \frac{1}{2} \right\}  
43. \left\{ \frac{5}{3}, \frac{5}{2} \right\}  
45. \{16\}  
47. \{2\}