63. Show that \( y = \frac{4x^2 - 3}{2x^2 + x - 3} \) is not defined for all values of \( x \).

64. Show that \( y = \frac{7x^2 - 4}{x^3 - 2x + 4} \) is not defined for all values of \( x \).

65. For \( f(x) = 2\sqrt{x - 1} \) and \( g(x) = x^3 - 1.2 \), find \( f(g(2.3)) \). Use two decimal places.

66. For \( f(x) = 2\sqrt{x - 1} \) and \( g(x) = x^3 - 1.2 \), find \( g(f(4.8)) \). Use two decimal places.

Graphs have visual impact. They also reveal information that may not be evident from verbal or algebraic descriptions. Two graphs depicting practical relationships are shown in Figure 1.3.

The graph in Figure 1.3a describes the variation in total industrial production in a certain country over a 4-year period of time. Notice that the highest point on the graph occurs near the end of the third year, indicating that production was greatest at that time.

The graph in Figure 1.3b represents population growth when environmental factors impose an upper bound on the possible size of the population. It indicates that the rate of population growth increases at first and then decreases as the size of the population gets closer and closer to the upper bound.

To represent a function \( y = f(x) \) geometrically as a graph, it is common practice to use a rectangular coordinate system on which units for the independent variable \( x \) are marked on the horizontal axis and those for the dependent variable \( y \) are marked on the vertical axis.
In Chapter 3, you will see efficient techniques involving calculus that can be used to draw accurate graphs of functions. For many functions, however, you can make a fairly good sketch by the elementary method of plotting points, which can be summarized as follows.

How to Sketch the Graph of a Function $f$ by Plotting Points

1. Choose a representative collection of numbers $x$ from the domain of $f$ and construct a table of function values $y = f(x)$ for those numbers.

2. Plot the corresponding points $(x, y)$.

3. Connect the plotted points with a smooth curve.

The Graph of a Function

The graph of a function $f$ consists of all points $(x, y)$ where $x$ is in the domain of $f$ and $y = f(x)$; that is, all points of the form $(x, f(x))$.

Explore!

Graph $f(x) = x^2$, $f(x) = x^2 + 2$, and $f(x) = x^2 - 3$ on your graphing calculator. Compare these graphs with those of $f(x - 2) = (x - 2)^2$ and $f(x - 3) = (x - 3)^2$.

Example 2.1

Graph the function $f(x) = x^2$.

Solution

Begin by constructing the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$\frac{1}{2}$</th>
<th>$0$</th>
<th>$\frac{1}{2}$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$9$</td>
<td>$4$</td>
<td>$1$</td>
<td>$\frac{1}{4}$</td>
<td>$0$</td>
<td>$\frac{1}{4}$</td>
<td>$1$</td>
<td>$4$</td>
<td>$9$</td>
</tr>
</tbody>
</table>

Then plot the points $(x, y)$ and connect them with the smooth curve shown in Figure 1.4a.
Many different curves pass through the points in Example 2.1. Several of these curves are shown in Figure 1.4b. There is no way to guarantee that the curve we pass through the plotted points is the actual graph of \( f \). However, in general, the more points that are plotted, the more likely the graph is to be reasonably accurate.

The next example illustrates how to sketch the graph of a function defined by more than one formula.

**EXAMPLE 2.2**

Graph the function

\[
f(x) = \begin{cases} 
2x & \text{if } 0 \leq x < 1 \\
2/x & \text{if } 1 \leq x < 4 \\
3 & \text{if } x \geq 4
\end{cases}
\]

**Solution**

When making a table of values for this function, remember to use the formula that is appropriate for each particular value of \( x \). Using the formula \( f(x) = 2x \) when \( 0 \leq x < 1 \), the formula \( f(x) = \frac{2}{x} \) when \( 1 \leq x < 4 \), and the formula \( f(x) = 3 \) when \( x \geq 4 \), you can compile the following table:
Now plot the corresponding points \((x, f(x))\) and draw the graph as in Figure 1.5. Notice that the pieces for \(0 \leq x < 1\) and \(1 \leq x < 4\) are connected to one another at \((1, 2)\) but that the piece for \(x \geq 4\) is separated from the rest of the graph. [The “open dot” at \((4, \frac{1}{2})\) indicates that the graph approaches this point but that the point is not actually on the graph.]

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>(\frac{2}{3})</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

**FIGURE 1.5** The graph of \(f(x) = \begin{cases} 2x & 0 \leq x < 1 \\ \frac{2}{x} & 1 \leq x < 4 \\ 3 & x \geq 4 \end{cases}\).

The points (if any) where a graph crosses the \(x\) axis are called \(x\) intercepts, and similarly, the \(y\) intercepts are where the graph crosses the \(y\) axis. Intercepts are key features of a graph and can be determined using algebra or technology in conjunction with the following criteria.

**How to Find the \(x\) and \(y\) Intercepts**  ■ To find any \(y\) intercept of \(y = f(x)\), set \(x = 0\) and solve for \(y\). To find any \(x\) intercept of \(y = f(x)\), set \(y = 0\) and solve for \(x\). Finding \(y\) intercepts is usually easy, but \(x\) intercepts may be hard to obtain.
Graph the function \( f(x) = -x^2 + x + 2 \). Include all \( x \) and \( y \) intercepts.

**Solution**

The \( y \) intercept is \( f(0) = 2 \). To find the \( x \) intercepts, solve the equation \( f(x) = 0 \). Factoring, we find that

\[
-x^2 + x + 2 = 0 \\
-(x + 1)(x - 2) = 0 \\
x = -1, x = 2
\]

Thus, the \( x \) intercepts are \((-1, 0)\) and \((2, 0)\).

Next, make a table of values and plot the corresponding points \((x, f(x))\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -3 )</th>
<th>(-2 )</th>
<th>(-1 )</th>
<th>(0 )</th>
<th>(1 )</th>
<th>(2 )</th>
<th>(3 )</th>
<th>(4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>(-10)</td>
<td>(-4)</td>
<td>(0)</td>
<td>(2)</td>
<td>(2)</td>
<td>(0)</td>
<td>(-4)</td>
<td>(-10)</td>
</tr>
</tbody>
</table>

The graph of \( f \) is shown in Figure 1.6.

The factoring in Example 2.3 is fairly straightforward, but in other problems, you may need to review the factoring procedure provided in Appendix A2.

**EXAMPLE 2.4**

Graph the function \( f(x) = x^3 - x^2 - 6x \). Include all \( x \) and \( y \) intercepts.

**Solution**

The \( y \) intercept is \( f(0) = 0 \). To find the \( x \) intercepts, set \( f(x) \) equal to 0 and solve for \( x \) by factoring the equation as follows:

\[
0 = x^3 - x^2 - 6x = x(x^2 - x - 6) \\
= x(x - 3)(x + 2)
\]

so that \( x = 0, x = 3, \) or \( x = -2 \)

It follows that the \( x \) intercepts are \((0, 0)\), \((3, 0)\), and \((-2, 0)\).
Next, make a table of values (including the intercepts) and plot the corresponding points \((x, f(x)):\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>(-18)</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>(-6)</td>
<td>(-8)</td>
<td>0</td>
<td>24</td>
</tr>
</tbody>
</table>

The graph is shown in Figure 1.7.

**FIGURE 1.7** The graph of the function \(y = x^3 - x^2 - 6x\).

The graphs in Figures 1.4a and 1.6 are called parabolas. In general, the graph of \(y = Ax^2 + Bx + C\) is a parabola as long as \(A \neq 0\). All parabolas have a “U shape,” and the parabola \(y = Ax^2 + Bx + C\) opens up if \(A > 0\) and down if \(A < 0\). The “peak” or “valley” of the parabola is called its vertex, and it always occurs where \(x = \frac{-B}{2A}\) (Figure 1.8; see also Problem 41). These features of the parabola are derived by the methods of calculus developed in Chapter 3. Note that to get a reasonable sketch of the parabola \(y = Ax^2 + Bx + C\), you need only determine three key features:

1. The location of the vertex \(\left(x = \frac{-B}{2A}\right)\)
2. Whether the parabola opens up \((A > 0)\) or down \((A < 0)\)
3. Any intercepts
For instance, in Example 2.3, the parabola \( y = -x^2 + x + 2 \) opens downward (since \( A = -1 \) is negative) and has its vertex (high point) at \( x = \frac{-B}{2A} = \frac{-1}{2(-1)} = \frac{1}{2} \).

Parabolas play a useful role in applications involving business and economics. For instance, suppose it is known that \( 60 - x \) units of a certain commodity will be sold when the price is \( x \) dollars per unit. Then the revenue derived from selling the units is given by the function

\[
R(x) = x(60 - x) = -x^2 + 60x
\]

whose graph is a downward opening parabola with vertex (top point) at \( x = \frac{-60}{2(-1)} = 30 \) (see Figure 1.9). You can interpret this result as saying that the largest possible revenue may be obtained by charging $30 per unit. Optimization problems such as this will be examined further in Section 4 and again in Chapter 3.
Sometimes it is necessary to determine when two functions are equal. For instance, an economist may wish to compute the market price at which the consumer demand for a commodity will be equal to supply. Or a political analyst may wish to predict how long it will take for the popularity of a certain challenger to reach that of the incumbent. We shall examine some of these applications in Section 4.

In geometric terms, the values of \( x \) for which two functions \( f(x) \) and \( g(x) \) are equal are the \( x \) coordinates of the points where their graphs intersect. In Figure 1.10, the graph of \( y = f(x) \) intersects that of \( y = g(x) \) at two points, labeled \( P \) and \( Q \). To find the points of intersection algebraically, set \( f(x) \) equal to \( g(x) \) and solve for \( x \). This procedure is illustrated in Example 2.5.

The quadratic formula is used in Example 2.5. Recall that this result says that the equation \( Ax^2 + Bx + C = 0 \) has real solutions if and only if \( D = B^2 - 4AC \geq 0 \), in which case, the solutions are

\[
\begin{align*}
    r_1 &= \frac{-B + \sqrt{B^2 - 4AC}}{2A} \\
    r_2 &= \frac{-B - \sqrt{B^2 - 4AC}}{2A}
\end{align*}
\]

A review of the quadratic formula may be found in Appendix A2.

**EXAMPLE 2.5**

Find all points of intersection of the graphs of \( f(x) = 3x + 2 \) and \( g(x) = x^2 \).

**Solution**

You must solve the equation \( x^2 = 3x + 2 \). Rewrite the equation as \( x^2 - 3x - 2 = 0 \) and apply the quadratic formula to obtain
The solutions are

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)} = \frac{3 \pm \sqrt{17}}{2}$$

The solutions are

$$x = \frac{3 + \sqrt{17}}{2} \approx 3.56 \quad \text{and} \quad x = \frac{3 - \sqrt{17}}{2} \approx -0.56$$

(The computations were done on a calculator, with results rounded off to two decimal places.)

Computing the corresponding $y$ coordinates from the equation $y = x^2$, you find that the points of intersection are approximately (3.56, 12.67) and (-0.56, 0.31). (As a result of round-off errors, you will get slightly different values for the $y$ coordinates if you substitute into the equation $y = x^2$. ) The graphs and the intersection points are shown in Figure 1.11.

A power function is a function of the form $f(x) = x^n$, where $n$ is a real number. For example, $f(x) = x^2$, $f(x) = x^{-3}$, and $f(x) = x^{1/2}$ are all power functions. So are $f(x) = \frac{1}{x^2}$ and $f(x) = \sqrt[3]{x}$ since they can be rewritten as $f(x) = x^{-2}$ and $f(x) = x^{1/3}$, respectively.

A polynomial is a function of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where $n$ is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are constants. If $a_n \neq 0$, the integer $n$ is called the degree of the polynomial. For example, the function $f(x) = 3x^5 - 6x^2 + 7$ is a polynomial of degree 5. It can be shown that the graph of a polynomial of degree $n$ is an unbroken curve that crosses the $x$ axis no more than $n$ times. To illustrate some of the possibilities, the graphs of three polynomials of degree 3 are shown in Figure 1.12.
A quotient \( \frac{p(x)}{q(x)} \) of two polynomials \( p(x) \) and \( q(x) \) is called a **rational function**. Such functions appear throughout this text in examples and exercises. Graphs of three rational functions are shown in Figure 1.13. You will learn how to sketch such graphs in Section 3 of Chapter 3.

**FIGURE 1.13** Graphs of three rational functions.

**THE VERTICAL LINE TEST**

It is important to realize that not every curve is the graph of a function (Figure 1.14). For instance, suppose the circle \( x^2 + y^2 = 5 \) were the graph of some function \( y = f(x) \). Then, since the points \((1, 2)\) and \((1, -2)\) both lie on the circle, we would have \( f(1) = 2 \) and \( f(1) = -2 \), contrary to the requirement that a function assigns one and only one value to each number in its domain. This example suggests the following geometric rule for determining whether a curve is the graph of a function.

**The Vertical Line Test** ▶ A curve is the graph of a function if and only if no vertical line intersects the curve more than once.

**FIGURE 1.14** The vertical line test.
P R O B L E M S 1.2

In Problems 1 through 16, sketch the graph of the given function. Include all x and y intercepts.

1. \( f(x) = x \)
2. \( f(x) = x^2 \)
3. \( f(x) = x^3 \)
4. \( f(x) = x^4 \)
5. \( f(x) = -x^3 + 1 \)
6. \( f(x) = 2x - 1 \)
7. \( f(x) = 2 - 3x \)
8. \( f(x) = \sqrt{x} \)
9. \( f(x) = \begin{cases} x - 1 & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases} \)
10. \( f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases} \)
11. \( f(x) = (x - 1)(x + 2) \)
12. \( f(x) = (x + 2)(x + 1) \)
13. \( f(x) = -x^2 - 2x + 15 \)
14. \( f(x) = x^2 + 2x - 8 \)
15. \( f(x) = 6x^2 + 13x - 5 \)
16. \( f(x) = -26x^2 + 10x + 21 \)

In Problems 17 through 24, find the points of intersection (if any) of the given pair of curves and draw the graphs.

17. \( y = 3x + 5 \) and \( y = -x + 3 \)
18. \( y = x^3 - 6x^2 \) and \( y = -x^2 \)
19. \( y = 3x + 8 \) and \( y = 3x - 2 \)
20. \( y = x^2 \) and \( y = 2x + 2 \)
21. \( y = x^2 \) and \( y = 6 - x \)
22. \( 3y - 2x = 5 \) and \( y + 3x = 9 \)
23. \( y = x^2 - x \) and \( y = x - 1 \)
24. \( 2x - 3y = -8 \) and \( 3x - 5y = -13 \)

MANUFACTURING COST
25. A manufacturer can produce cassette tape recorders at a cost of $40 apiece. It is estimated that if the tape recorders are sold for \( x \) dollars apiece, consumers will buy \( 120 - x \) of them a month. Express the manufacturer’s monthly profit as a function of price, graph this function, and use the graph to estimate the optimal selling price.

RETAIL SALES
26. A bookstore can obtain an atlas from the publisher at a cost of $10 per copy and estimates that if it sells the atlas for \( x \) dollars per copy, approximately \( 20(22 - x) \) copies will be sold each month. Express the bookstore’s monthly profit from the sale of the atlas as a function of price, graph this function, and use the graph to estimate the optimal selling price.

CONSUMER EXPENDITURE
27. The consumer demand for a certain commodity is \( D(p) = -200p + 12,000 \) units per month when the market price is \( p \) dollars per unit.
   (a) Graph this demand function.
   (b) Express consumers’ total monthly expenditure for the commodity as a function of \( p \). (The total monthly expenditure is the total amount of money consumers spend each month on the commodity.)
   (c) Graph the total monthly expenditure function.
(d) Discuss the economic significance of the \( p \) intercepts of the expenditure function.
(e) Use the graph in part (c) to estimate the market price that generates the greatest consumer expenditure.

**MOTION OF A PROJECTILE**

28. If an object is thrown vertically upward from the ground with an initial speed of 160 feet per second, its height (in feet) \( t \) seconds later is given by the function \( H(t) = -16t^2 + 160t \).

(a) Graph the function \( H(t) \).
(b) Use the graph in part (a) to determine when the object will hit the ground.
(c) Use the graph in part (a) to estimate how high the object will rise.

**AVERAGE COST**

29. Suppose the total cost of manufacturing \( x \) units of a certain commodity is \( C(x) = \frac{1}{6}x^3 + 2x + 5 \) dollars. Express the average cost per unit as a function of the number of units produced and, on the same set of axes, sketch the total cost and average cost functions. [Hint: Average cost is total cost divided by the number of units produced.]

**POISEUILLE’S LAW**

30. Recall from Problem 53, Section 1.1, that the speed of blood located \( r \) centimeters from the central axis of an artery is given by the function \( S(r) = C(R^2 - r^2) \), where \( C \) is a constant and \( R \) is the radius of the artery.* What is the domain of this function? Sketch the graph of \( S(r) \).

**MICROBIOLOGY**

31. A spherical cell of radius \( r \) has volume \( V = \frac{4}{3}\pi r^3 \) and surface area \( S = 4\pi r^2 \). Express \( V \) as a function of \( S \). If \( S \) is doubled, what happens to \( V \)?

32. (a) Graph the functions \( y = x^2 \) and \( y = x^2 + 3 \). How are the graphs related?
(b) Without further computation, graph the function \( y = x^2 - 5 \).
(c) Suppose \( g(x) = f(x) + c \), where \( c \) is a constant. How are the graphs of \( f \) and \( g \) related? Explain.

33. (a) Graph the functions \( y = x^2 \) and \( y = -x^2 \). How are the graphs related?
(b) Suppose \( g(x) = -f(x) \). How are the graphs of \( f \) and \( g \) related? Explain.

34. (a) Graph the functions \( y = x^2 \) and \( y = (x - 2)^2 \). How are the graphs related?
(b) Without further computation, graph the function \( y = (x + 1)^2 \).
(c) Suppose \( g(x) = f(x - c) \), where \( c \) is a constant. How are the graphs of \( f \) and \( g \) related? Explain.

35. It costs $90 to rent a piece of equipment plus $21 for every day of use.

(a) Make a table showing the number of days the equipment is rented and the cost of renting for 2 days, 5 days, 7 days, and 10 days.

---

(b) Write an algebraic expression representing the cost as a function of the number of days $x$.
(c) Graph the expression in part (b).

36. Graph $g(x) = -3x^2 - 7x - 4$ and find the $x$ intercepts.

37. Use your graphing utility to graph $y = x^4$, $y = x^4 - x$, $y = x^4 - 2x$, and $y = x^4 - 3x$ on the same coordinate axes, using $[-2, 2]$ by $[-2, 5]$. What effect does the added term involving $x$ have on the shape of the graph? Repeat using $y = x^4$, $y = x^4 - x^3$, $y = x^4 - 2x^3$, and $y = x^4 - 3x^3$. Adjust the viewing rectangle appropriately.

38. In each of the cases shown in the figure, use the vertical line test to determine whether the given curve is the graph of a function.

39. Show that the distance $d$ between the two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by the formula

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

[Hint: Apply the Pythagorean theorem to the right triangle whose hypotenuse is the line segment joining the two given points.] Then use the distance formula to find the distance between these points:
(a) $(5, -1)$ and $(2, 3)$
(b) $(2, 6)$ and $(2, -1)$

40. Use the distance formula in Problem 39 to show that the circle with center $(a, b)$ and radius $R$ has the equation

\[ (x - a)^2 + (y - b)^2 = R^2 \]
41. Show that the vertex of the parabola \( y = Ax^2 + Bx + C \) \((A \neq 0)\) occurs at the point where \( x = \frac{-B}{2A} \). [Hint: First verify that]

\[
Ax^2 + Bx + C = A\left[\left(x + \frac{B}{2A}\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)\right]
\]

Then note that the largest or smallest value of

\[
f(x) = Ax^2 + Bx + C
\]

must occur where \( x = \frac{-B}{2A} \).

42. Graph \( f(x) = \frac{-9x^2 - 3x - 4}{4x^2 + x - 1} \). Determine the values of \( x \) for which the function is defined.

43. Graph \( f(x) = \frac{8x^2 + 9x + 3}{x^2 + x - 1} \). Determine the values of \( x \) for which the function is defined.

Linear Functions

In many practical situations, the rate at which one quantity changes with respect to another is constant. Here is a simple example from economics.

**Example 3.1**

A manufacturer’s total cost consists of a fixed overhead of $200 plus production costs of $50 per unit. Express the total cost as a function of the number of units produced and draw the graph.