The methods for finding zeros discussed in the preceding section are designed to find as many exact real and imaginary zeros as possible. But there are zeros that cannot be found by using these methods. For example, the polynomial

\[ P(x) = x^3 + x - 1 \]

must have at least one real zero (Theorem 5 in Section 3-2). Since the only possible rational zeros are \(\pm 1\), and neither of these turns out to be a zero, \(P(x)\) must have at least one irrational zero. We cannot find the exact value of this zero, but it can be approximated using various well-known methods.

In this section we develop two important tools for locating real zeros, the location theorem and the upper and lower bound theorem. We will see that the upper and lower bound theorem is a useful tool for approximating real zeros on a graphing utility. Next we discuss how the location theorem leads to the bisection method, a simple approximation technique that forms the basis for the zero approximation routines on most graphing utilities. Finally, we investigate some of the problems that can be encountered when a polynomial has multiple zeros.

**In this section, we restrict our attention to the real zeros of polynomials with real coefficients.**

### Locating Real Zeros

Let us return to the polynomial function

\[ P(x) = x^3 + x - 1 \]

As we have found, \(P(x)\) has no rational zeros and at least one irrational zero. The graph of \(P(x)\) is shown in Figure 1.

Note that \(P(0) = -1\) and \(P(1) = 1\). Since the graph of a polynomial function is continuous, the graph of \(P(x)\) must cross the \(x\) axis at least once between \(x = 0\) and \(x = 1\). This observation is the basis for Theorem 1 and leads to a simple method for approximating zeros.

### Location Theorem

If \(f\) is continuous on an interval \(I\), \(a\) and \(b\) are two numbers in \(I\), and \(f(a)\) and \(f(b)\) are of opposite sign, then there is at least one \(x\) intercept between \(a\) and \(b\).
We will find Theorem 1 very useful when we are searching for real zeros, hence the name *location theorem*. It is important to remember that “at least” in Theorem 1 means “one or more.” Notice in Figure 2(a) that \( f(-3) = -15 < 0 \), \( f(3) = 15 > 0 \), and \( f \) has one zero between \(-3\) and 3. In Figure 2(b), \( f(-3) = -15 \) and \( f(3) = 15 \), but this time there are three zeros between \(-3\) and 3.

The converse to the location theorem (Theorem 1) is false; that is, if \( c \) is a zero of \( f \), then \( f \) may or may not change sign at \( c \). Compare Figures 2(a) and 2(c). Both functions have a zero at \( x = 0 \), but the first changes sign at 0 and the second does not.

**Example 1**

*Locating Real Zeros*

Let \( P(x) = x^3 - 6x^2 + 9x - 3 \). Use a table to locate the zeros of \( P(x) \) between successive integers and check graphically.

**Solution**

We construct a table and look for sign changes [Fig. 3(a)]. The table in Figure 3(a) shows three sign changes. According to Theorem 1, \( P(x) \) must have a real zero in each of the intervals \((0, 1)\), \((1, 2)\), and \((3, 4)\). The graph in Figure 3(b) confirms this. Since \( P(x) \) is a cubic polynomial, we have located all of its zeros.

**Matched Problem 1**

Let \( P(x) = x^3 - 8x^2 + 15x - 2 \). Use a table to locate the zeros of \( P(x) \) between successive integers and check graphically.

In the solution to Example 1, we located three zeros in a relatively few number of steps and could stop searching because we knew that a cubic polynomial could not have more than three zeros. But what if we had not found three zeros?
Some cubic polynomials have only one real zero. How can we tell when we have searched far enough? The next theorem tells us how to find upper and lower bounds for the real zeros of a polynomial. Any number that is greater than or equal to the largest zero of a polynomial is called an upper bound of the zeros of the polynomial. Similarly, any number that is less than or equal to the smallest zero of the polynomial is called a lower bound of the zeros of the polynomial. Theorem 2, based on the synthetic division process, enables us to determine upper and lower bounds of the real zeros of a polynomial with real coefficients.

**Theorem 2**

**Upper and Lower Bounds of Real Zeros**

Given an $n$th-degree polynomial $P(x)$ with real coefficients, $n > 0$, $a_n > 0$, and $P(x)$ divided by $x - r$ using synthetic division:

1. **Upper Bound.** If $r > 0$ and all numbers in the quotient row of the synthetic division, including the remainder, are nonnegative, then $r$ is an upper bound of the real zeros of $P(x)$.

2. **Lower Bound.** If $r < 0$ and all numbers in the quotient row of the synthetic division, including the remainder, alternate in sign, then $r$ is a lower bound of the real zeros of $P(x)$.

[Note: In the lower-bound test, if 0 appears in one or more places in the quotient row, including the remainder, the sign in front of it can be considered either positive or negative, but not both. For example, the numbers 1, 0, 1 can be considered to alternate in sign, while 1, 0, $-1$ cannot.]

We sketch a proof of part 1 of Theorem 2. The proof of part 2 is similar, only a little more difficult.

**Proof**

If all the numbers in the quotient row of the synthetic division are nonnegative after dividing $P(x)$ by $x - r$, then

$$P(x) = (x - r)Q(x) + R$$

where the coefficients of $Q(x)$ are nonnegative and $R$ is nonnegative. If $x > r > 0$, then $x - r > 0$ and $Q(x) > 0$; hence,

$$P(x) = (x - r)Q(x) + R > 0$$

Thus, $P(x)$ cannot be 0 for any $x$ greater than $r$, and $r$ is an upper bound for the real zeros of $P(x)$.

Theorem 2 requires performing synthetic division repeatedly until the desired patterns occur in the quotient row. This is a simple, but tedious, operation to carry out by hand. Table 1 shows a simple program for a graphing calculator that will perform synthetic division.
(A) If you have a TI-82, TI-83, TI-85, or TI-86 graphing calculator, enter the appropriate version of SYNDIV in your calculator exactly as shown in Table 1. If you have some other graphing utility that can store and execute programs, consult your manual and modify the statements in SYNDIV so that the program works on your graphing utility.

(B) Store the coefficients of \( P(x) = x^5 - 6x^2 + 9x - 3 \) in L1 (see the first line of output in Table 1) and execute the program. Type 0 at the “R5?” prompt and press ENTER to display the results of synthetic division with \( r = 0 \). To continue press ENTER again. Repeat this process for \( r = 1, 2, 3, \) and 4. Press QUIT at the “R=?” prompt to terminate the program. Compare the last number in each list with the values of \( P(x) \) in Figure 3(a).

### Example 2

#### Bounding Real Zeros

Let \( P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90 \). Find the smallest positive integer and the largest negative integer that, by Theorem 2, are upper and lower bounds, respectively, for the real zeros of \( P(x) \). Also note the location of any zeros discovered in the process of searching for upper and lower bounds.

#### Solution

We can use SYNDIV or hand calculations to perform synthetic division for \( r = 1, 2, 3, \ldots \) until the quotient row turns nonnegative; then repeat this process...
for \( r = -1, -2, -3, \ldots \) until the quotient row alternates in sign. We organize these results in the *synthetic division table* shown below. In a *synthetic division table* we dispense with writing the product of \( r \) with each coefficient in the quotient and simply list the results in the table. It is also useful to include \( r = 0 \) in the table to detect any sign changes between \( r = 0 \) and \( r = \pm 1 \).

\[
\begin{array}{cccc}
1 & -2 & -10 & 40 & -90 \\
0 & 1 & -2 & -10 & 40 & -90 \\
1 & 1 & -1 & -11 & 29 & -61 \\
2 & 1 & 0 & -10 & 20 & -50 \\
3 & 1 & 1 & -7 & 19 & -33 \\
4 & 1 & 2 & -2 & 32 & 38 \\
\hline
\text{UB} & 5 & 1 & 3 & 5 & 65 & 235 \{ \text{This quotient row is nonnegative; hence, 5 is an upper bound (UB).} \} \\
-1 & 1 & -3 & -7 & 47 & -137 \\
-2 & 1 & -4 & -2 & 44 & -178 \\
-3 & 1 & -5 & 5 & 25 & -165 \\
-4 & 1 & -6 & 14 & -16 & -26 \\
\text{LB} & -5 & 1 & -7 & 25 & -85 & 335 \{ \text{This quotient row alternates in sign; hence, -5 is a lower bound (LB).} \}
\end{array}
\]

The graph of \( P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90 \) for \(-5 \leq x \leq 5\) is shown in Figure 4. Theorem 2 implies that all the real zeros of \( P(x) \) are between \(-5\) and \(5\). We can be certain that the graph does not change direction and cross the \( x \) axis somewhere outside the viewing window in Figure 4. We also note that there is at least one zero in \((3, 4)\) and at least one in \((-5, -4)\), as indicated by the sign changes in the values of \( P(x) \) shown in the last column of the synthetic division table.

### MATCHED PROBLEM 2

Let \( P(x) = x^4 - 5x^3 - x^2 + 40x - 70 \). Find the smallest positive integer and the largest negative integer that, by Theorem 2, are upper and lower bounds, respectively, for the real zeros of \( P(x) \). Also note the location of any zeros discovered in the process of searching for upper and lower bounds.

### EXAMPLE 3

**Approximating Real Zeros with a Graphing Utility**

Let \( P(x) = x^3 - 30x^2 + 275x - 720 \).

(A) Find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that, by Theorem 2, are upper and lower bounds, respectively, for the real zeros of \( P(x) \).

(B) Approximate the real zeros of \( P(x) \) to two decimal places.
Solutions

(A) We construct a synthetic division table to search for bounds for the zeros of \( P(x) \). The size of the coefficients in \( P(x) \) indicates that we can speed up this search by choosing larger increments between test values.

\[
\begin{array}{cccc}
1 & -30 & 275 & -720 \\
10 & 1 & -20 & 75 & 30 \\
20 & 1 & -10 & 75 & 780 \\
UB & 30 & 1 & 0 & 275 & 7,530 \\
LB & -10 & 1 & -40 & 675 & -7,470 \\
\end{array}
\]

Thus, all real zeros of \( P(x) = x^3 - 30x^2 + 275x - 720 \) must lie between \(-10\) and 30.

(B) Graphing \( P(x) \) for \(-10 \leq x \leq 30 \) (Fig. 5) shows that \( P(x) \) has three zeros. The approximate values of these zeros (details omitted) are 4.48, 11.28, and 14.23.

MATCHED PROBLEM

3

Let \( P(x) = x^3 - 25x^2 + 170x - 170 \).

(A) Find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that, by Theorem 2, are upper and lower bounds, respectively, for the real zeros of \( P(x) \).

(B) Approximate the real zeros of \( P(x) \) to two decimal places.

Remark

One of the most frequently asked questions concerning graphing utilities is how to determine the correct viewing window. The upper and lower bound theorem provides an answer to this question for polynomial functions. As Example 3 illustrates, the upper and lower bound theorem and the zero approximation routine on a graphing utility are two important mathematical tools that work very well together.

The Bisection Method

Now that we know more about locating real zeros of a polynomial, we want to discuss a technique that can be used to approximate real zeros. Explore/Discuss 2 provides an introduction to the repeated systematic application of the location theorem (Theorem 1) called the bisection method. This method forms the basis for the zero approximation routines in many graphing utilities.

Explore/Discuss

2

Let \( P(x) = x^3 + x - 1 \). Since \( P(0) = -1 \) and \( P(1) = 1 \), the location theorem implies that \( P(x) \) must have at least one zero in \((0, 1)\).

(A) Is \( P(0.5) \) positive or negative? Is there a zero in \((0, 0.5)\) or in \((0.5, 1)\)?
(B) Let \( m \) be the midpoint of the interval from part A that contains a zero. Is \( P(m) \) positive or negative? What does this tell you about the location of the zero?

(C) Explain how this process could be used repeatedly to approximate a zero to any desired accuracy.

The bisection method used to approximate real zeros is straightforward: Let \( P(x) \) be a polynomial with real coefficients. If \( P(x) \) has opposite signs at the endpoints of the interval \((a, b)\), then a real zero \( r \) lies in this interval. We bisect this interval [find the midpoint \( m = (a + b)/2 \)], check the sign of \( P(m) \), and choose the interval \((a, m)\) or \((m, b)\) on which \( P(x) \) has opposite signs at the endpoints. We repeat this bisecting process (producing a set of “nested” intervals, each half the size of the preceding one and each containing the real zero \( r \)) until we get the desired decimal accuracy for the zero approximation. At any point in the process if \( P(m) = 0 \), we stop, since \( m \) is a real zero. An example will help clarify the process.

**Example 4**

**Approximating Real Zeros by Bisection**

For the polynomial \( P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90 \) in Example 2, we found that all the real zeros lie between \(-5\) and \(5\) and that each of the intervals \((-5, -4)\) and \((3, 4)\) contained at least one zero. Use bisection to approximate a real zero on the interval \((3, 4)\) to one-decimal-place accuracy.

We organize the results of our calculations in a table. Since the sign of \( P(x) \) changes at the endpoints of the interval \((3.5625, 3.625)\), we conclude that a real zero lies in this interval and is given by \( r = 3.6 \) to one-decimal-place accuracy (each endpoint rounds to 3.6).

![Figure 6](https://example.com/figure6.png)

Figure 6 illustrates the nested intervals produced by the bisection method in Table 2. Match each step in Table 2 with an interval in Figure 6. Note how each interval that contains a zero gets smaller and smaller and is contained in the preceding interval that contained the zero.

<table>
<thead>
<tr>
<th>Sign Change Interval ([a, b])</th>
<th>Midpoint (m)</th>
<th>(P[a])</th>
<th>(P[m])</th>
<th>(P[b])</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3, 4))</td>
<td>3.5</td>
<td>–</td>
<td>–</td>
<td>+</td>
</tr>
<tr>
<td>((3.5, 4))</td>
<td>3.75</td>
<td>–</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>((3.5, 3.75))</td>
<td>3.625</td>
<td>–</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>((3.5, 3.625))</td>
<td>3.5625</td>
<td>–</td>
<td>–</td>
<td>+</td>
</tr>
<tr>
<td>((3.5625, 3.625))</td>
<td>We stop here</td>
<td>–</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>
If we had wanted two-decimal-place accuracy, we would have continued the process in Table 2 until the endpoints of a sign change interval rounded to the same two-decimal-place number.

**MATCHED PROBLEM 4**

Use the bisection method to approximate to one-decimal-place accuracy a zero on the interval \((-5, -4)\) for the polynomial in Example 4.

The bisection method is easy to implement on a graphing utility. Table 3 shows a program that computes the sequence of nested intervals shown in Table 2.

### Table 3  Bisection on a Graphing Utility

<table>
<thead>
<tr>
<th><strong>Program BISECT</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>TI 82/83</strong></td>
</tr>
<tr>
<td><code>0→L1</code></td>
</tr>
<tr>
<td><code>Input &quot;LEFT BOUND: &quot;,L</code></td>
</tr>
<tr>
<td><code>Input &quot;RIGHT BOUND: &quot;,R</code></td>
</tr>
<tr>
<td><code>If L≥R</code></td>
</tr>
<tr>
<td><code>Then</code></td>
</tr>
<tr>
<td><code>Disp &quot;BOUND ERROR&quot;</code></td>
</tr>
<tr>
<td><code>Stop</code></td>
</tr>
<tr>
<td><code>End</code></td>
</tr>
<tr>
<td><code>If Y1(L)Y1(R)&gt;0</code></td>
</tr>
<tr>
<td><code>Then</code></td>
</tr>
<tr>
<td><code>Disp &quot;NO SIGN CHANGE&quot;</code></td>
</tr>
<tr>
<td><code>Stop</code></td>
</tr>
<tr>
<td><code>Else</code></td>
</tr>
<tr>
<td><code>Lbl A</code></td>
</tr>
<tr>
<td><code>L+L1(1):R+L1(2)</code></td>
</tr>
<tr>
<td><code>Pause L1</code></td>
</tr>
<tr>
<td><code>L+R)/2→M</code></td>
</tr>
<tr>
<td><code>If Y1(M)=0</code></td>
</tr>
<tr>
<td><code>Then</code></td>
</tr>
<tr>
<td><code>Disp &quot;Y = 0 AT X =&quot;,M</code></td>
</tr>
<tr>
<td><code>Stop</code></td>
</tr>
<tr>
<td><code>End</code></td>
</tr>
<tr>
<td><code>If Y1(L)Y1(M)&lt;0</code></td>
</tr>
<tr>
<td><code>Then</code></td>
</tr>
<tr>
<td><code>M→R</code></td>
</tr>
<tr>
<td><code>Else</code></td>
</tr>
<tr>
<td><code>M+L</code></td>
</tr>
<tr>
<td><code>End</code></td>
</tr>
<tr>
<td><code>Goto A</code></td>
</tr>
</tbody>
</table>

(A) If you have a TI-82, TI-83, TI-85, or TI-86 graphing calculator, enter the appropriate version of BISECT in your calculator exactly as shown in Table 3. If you have some other graphing utility that can store and execute programs, consult your manual and modify the statements in BISECT so that the program works on your graphing utility.
Approximating Multiple Zeros

Consider the polynomial

\[ P(x) = x(x - 1)^2(x + 1)^4 \]

which has a simple zero at 0, a zero of multiplicity 2 at 1 and a zero of multiplicity 4 at -1 (see Fig. 8). Notice that \( P(x) \) has a local maximum at \( x = -1 \) and does not change sign at \( x = -1 \). Also, \( x \) has a local minimum at \( x = 1 \) and does not change sign there either. Both of these are zeros of even multiplicity. On the other hand, 0 is a zero of odd multiplicity, \( P(x) \) does change sign at \( x = 0 \), and does not have a local extremum at \( x = 0 \). Theorem 3, which we state without proof, generalizes these observations.

Zeros of Even and Odd Multiplicity

If \( P(x) \) is a polynomial with real coefficients, then

1. If \( r \) is a zero of odd multiplicity, then \( P(x) \) changes sign at \( r \) and does not have a local extremum at \( x = r \).
2. If \( r \) is a zero of even multiplicity, then \( P(x) \) does not change sign at \( r \) and has a local extremum at \( x = r \).

Let \( P(x) = x(x - 1)^2(x + 1)^4 \).

(A) Use BISECT to approximate the zeros of \( P(x) \). Where does it fail to work? Why?

(B) Use the zero approximation routine on your graphing utility to approximate the zeros of \( P(x) \). Does this routine ever fail to work?

The bisection method shown in Table 3 requires that the function change sign at a zero to approximate that zero. Thus, this method will always fail at a zero of even multiplicity. Most graphing utilities use a more complicated routine that may or may not work at zeros of even multiplicity. For example, the TI-83 graphing calculator was able to approximate the zero of \( P(x) = x(x - 1)^2(x + 1)^4 \) at 1, but not the zero at -1. What are we to do if the zero routine fails? The ideas introduced in Theorem 3 provide the answer:
Zeros of even multiplicity can be approximated by using a maximum or minimum approximation routine, whichever applies.

The next example illustrates this approach.

**Example 5**

**Approximating Multiple Zeros**

Let \( P(x) = x^5 + 6x^4 + 4x^3 - 24x^2 - 16x + 32 \). Approximate the zeros of \( P(x) \) to two decimal places. Use maximum and minimum routines to approximate any zeros of even multiplicity. Determine the multiplicity of each zero.

**Solution**

Examining the graph of \( P(x) \), we see that there is a zero of odd multiplicity at \( x = -2 \) [Fig. 9(a)]. It appears that there may be zeros of even multiplicity near \( x = -3 \) and \( x = 1 \). Using a maximum routine near \( x = -3 \) [Fig. 9(b)] and a minimum routine near \( x = 1 \) [Fig. 9(c)], we find that \(-3.24\) and \(1.24\) are zeros of even multiplicity. Since \( P(x) \) is a fifth-degree polynomial, these zeros must be double zeros and \(-2\) must be a simple zero.

![Figure 9](image)

**Matched Problem 5**

Let \( P(x) = x^5 - 6x^4 + 40x^2 - 12x - 72 \). Approximate the zeros of \( P(x) \) to two decimal places. Use maximum and minimum routines to approximate any zeros of even multiplicity. Determine the multiplicity of each zero.

**Application**

**Example 6**

**Construction**

An oil tank is in the shape of a right circular cylinder with a hemisphere at each end (see Fig. 10). The cylinder is 55 inches long, and the volume of the tank is \( 11,000\pi \) cubic inches (approximately 20 cubic feet). Let \( x \) denote the common radius of the hemispheres and the cylinder.

![Figure 10](image)

(A) Find a polynomial equation that \( x \) must satisfy.

(B) Approximate \( x \) to one decimal place.
Solutions

(A) If \( x \) is the common radius of the hemispheres and the cylinder in inches, then

\[
\begin{align*}
\text{(Volume of tank)} &= \left( \frac{4}{3} \pi x^3 \right) + \left( 55 \pi x^2 \right) \\
11,000 \pi &= \frac{4}{3} \pi x^3 + 55 \pi x^2 \\
33,000 &= 4x^3 + 165x^2 \\
0 &= 4x^3 + 165x^2 - 33,000
\end{align*}
\]

Thus, \( x \) must be a positive zero of

\[ P(x) = 4x^3 + 165x^2 - 33,000 \]

(B) Since the coefficients of \( P(x) \) are large, we use larger increments in the synthetic division table:

\[
\begin{array}{c|cccc}
& 4 & 165 & 0 & -33,000 \\
10 & & 420 & 2050 & -12,500 \\
UB & 20 & 4245 & 4900 & 65,000 \\
\end{array}
\]

Graphing \( y = P(x) \) for \( 0 \leq x \leq 20 \) (Fig. 11), we see that \( x = 12.4 \) inches (to one decimal place).

**FIGURE 11**

\[ P(x) = 4x^3 + 165x^2 - 33,000. \]

**MATCHED PROBLEM 6**

Repeat Example 6 if the volume of the tank is \( 44,000 \pi \) cubic inches.

---

**Answers to Matched Problems**

1. Intervals containing zeros: \((0, 1), (2, 3), (5, 6)\)
2. Lower bound: \(-3\); upper bound: \(6\); intervals containing zeros: \((-3, -2), (3, 4)\)
3. (A) Lower bound: \(-10\); upper bound: \(30\) \hspace{1cm} (B) Real zeros: \(1.20, 11.46, 12.34\)
4. \(x = -4.1\) \hspace{1cm} 5. \(-1.65\) (double zero), \(2\) (simple zero), \(3.65\) (double zero)
6. (A) \(P(x) = 4x^3 + 165x^2 - 132,000 = 0\) \hspace{1cm} (B) 22.7 inches


In Problems 1–4, use the table of values for the polynomial function \( P \) to discuss the possible locations of the \( x \) intercepts of the graph of \( y = P(x) \).

1. \[
\begin{array}{cccccc}
  x & -7 & -5 & -1 & 3 & 5 & 8 \\
  P(x) & 9 & 4 & -3 & 6 & 4 & -2
\end{array}
\]

2. \[
\begin{array}{ccccc}
  x & -8 & -2 & 0 & 2 & 4 & 9 \\
  P(x) & -3 & 4 & 5 & 2 & -5 & 6
\end{array}
\]

3. \[
\begin{array}{ccccc}
  x & -6 & -4 & 0 & 2 & 4 & 7 \\
  P(x) & -5 & 3 & -4 & -6 & 3 & -5
\end{array}
\]

4. \[
\begin{array}{ccccc}
  x & -5 & -3 & -1 & 0 & 2 & 5 \\
  P(x) & 7 & 4 & 2 & -1 & 3 & -6
\end{array}
\]

In Problems 5–8, use a synthetic division table and Theorem 1 to locate each real zero between successive integers.

5. \( P(x) = x^3 - 9x^2 + 23x - 14 \)
6. \( P(x) = x^3 - 12x^2 + 44x - 49 \)
7. \( P(x) = x^3 + 3x^2 - x - 5 \)
8. \( P(x) = x^3 + x^2 - 4x - 3 \)

Find the smallest positive integer and largest negative integer that, by Theorem 2, are upper and lower bounds, respectively, for the real zeros of each of the polynomials given in Problems 9–14.

9. \( P(x) = x^3 - 3x + 1 \)
10. \( P(x) = x^3 - 4x^2 + 4 \)
11. \( P(x) = x^3 - 3x^2 + 4x^2 + 2x - 9 \)
12. \( P(x) = x^3 - 4x^2 + 6x^2 - 4x - 7 \)
13. \( P(x) = x^3 - 3x^3 + 3x^2 + 2x - 2 \)
14. \( P(x) = x^3 - 3x^2 + 3x^2 + 2x - 1 \)

B

In Problems 15–22,

(A) Find the smallest positive integer and largest negative integer that, by Theorem 2, are upper and lower bounds, respectively, for the real zeros of \( P(x) \). Also note the location of any zeros between successive integers.

(B) If \( (k, k + 1) \) is the interval determined in part A that contains the largest real zero of \( P(x) \), determine the number of additional intervals required by the bisection method to obtain a one-decimal-place approximation to this zero and state the approximate value of the zero.

15. \( P(x) = x^3 - 2x^2 - 5x + 4 \)
16. \( P(x) = x^3 + x^2 - 4x - 1 \)
17. \( P(x) = x^3 - 2x^2 - x + 5 \)
18. \( P(x) = x^3 - 3x^2 - x - 2 \)
19. \( P(x) = x^4 - 2x^3 - 7x^2 + 9x + 7 \)
20. \( P(x) = x^4 - x^3 - 9x^2 + 9x + 4 \)
21. \( P(x) = x^4 - x^3 - 4x^2 + 4x + 3 \)
22. \( P(x) = x^4 - 3x^3 - x^2 + 3x + 3 \)

In Problems 23–30,

(A) Find the smallest positive integer and largest negative integer that, by Theorem 2, are upper and lower bounds, respectively, for the real zeros of \( P(x) \).

(B) Approximate the real zeros of each polynomial to two decimal places.

23. \( P(x) = x^3 - 2x^2 + 3x - 8 \)
24. \( P(x) = x^3 + 3x^2 + 4x + 5 \)
25. \( P(x) = x^4 + x^3 - 5x^2 + 7x - 22 \)
26. \( P(x) = x^4 - x^3 - 8x^2 - 12x - 25 \)
27. \( P(x) = x^3 - 3x^3 - 4x + 4 \)
28. \( P(x) = x^3 - x^4 - 2x^2 - 4x - 5 \)
29. \( P(x) = x^4 + x^3 + 3x^2 + x^2 + 2x - 5 \)
30. \( P(x) = x^3 - 2x^4 - 6x^2 - 9x + 10 \)

Problems 31–34 refer to the polynomial \( P(x) = (x - 1)^2(x - 2)(x - 3)^4 \)

31. Can the zero at \( x = 1 \) be approximated by the bisection method? Explain.
32. Can the zero at \( x = 2 \) be approximated by the bisection method? Explain.
33. Can the zero at \( x = 3 \) be approximated by the bisection method? Explain.
34. Which of the zeros can be approximated by a maximum approximation routine? By a minimum approximation routine? By the zero approximation routine on your graphing utility?
For each polynomial $P(x)$ in Problems 35–40, use a maximum or minimum routine to approximate zeros of even multiplicity and a zero approximation routine to approximate zeros of odd multiplicity, all to two decimal places. State the multiplicity of each zero.

35. $P(x) = x^4 - 4x^3 - 10x^2 + 28x + 49$
36. $P(x) = x^4 + 4x^3 - 4x^2 - 16x + 16$
37. $P(x) = x^4 - 6x^3 + 4x^2 + 24x^2 - 16x - 32$
38. $P(x) = x^4 - 6x^3 + 2x^2 + 28x^2 - 15x + 2$
39. $P(x) = x^4 - 6x^3 + 11x^2 - 4x^2 - 3.75x - 0.5$
40. $P(x) = x^4 + 12x^3 + 47x^3 + 56x^2 - 15.75x + 1$

In Problems 41–50,

(A) Find the smallest positive integer multiple of 10 and largest negative integer multiple of 10 that, by Theorem 2, are upper and lower bounds, respectively, for the real zeros of each polynomial.

(B) Approximate the real zeros of each polynomial to two decimal places.

41. $P(x) = x^4 - 24x^2 - 25x + 10$
42. $P(x) = x^4 - 37x^2 + 70x - 20$
43. $P(x) = x^4 + 12x^3 - 900x^2 + 5,000$
44. $P(x) = x^4 - 12x^3 - 425x^2 + 7,000$
45. $P(x) = x^4 - 100x^2 - 1,000x - 5,000$
46. $P(x) = x^4 - 5x^3 - 50x^2 - 500x + 7,000$
47. $P(x) = 4x^4 - 40x^3 + 1,475x^2 + 7,875x - 10,000$
48. $P(x) = 9x^4 + 120x^3 - 3,083x^2 - 25,674x - 48,400$
49. $P(x) = 0.01x^5 - 0.1x^4 - 12x^3 + 9,000$
50. $P(x) = 0.1x^5 + 0.7x^4 - 18.775x^3 - 340x^2 - 1,645x - 2,450$

**Applications**

Express the solutions to Problems 51–56 as the roots of a polynomial equation of the form $P(x) = 0$ and approximate these solutions to three decimal places. Use a graphing utility, if available; otherwise, use the bisection method.

51. **Geometry**. Find all points on the graph of $y = x^2$ that are 1 unit away from the point $(1, 2)$. [Hint: Use the distance-between-two-points formula from Section 1-1.]

52. **Geometry**. Find all points on the graph of $y = x^2$ that are 1 unit away from the point $(2, 1)$.

53. **Manufacturing**. A box is to be made out of a piece of cardboard that measures 18 by 24 inches. Squares, $x$ inches on a side, will be cut from each corner, and then the ends and sides will be folded up (see the figure). Find the value of $x$ that would result in a box with a volume of 600 cubic inches.

54. **Manufacturing**. A box with a hinged lid is to be made out of a piece of cardboard that measures 20 by 40 inches. Six squares, $x$ inches on a side, will be cut from each corner and the middle, and then the ends and sides will be folded up to form the box and its lid (see the figure). Find the value of $x$ that would result in a box with a volume of 500 cubic inches.

55. **Construction**. A propane gas tank is in the shape of a right circular cylinder with a hemisphere at each end (see the figure). If the overall length of the tank is 10 feet and the volume is $20\pi$ cubic feet, find the common radius of the hemispheres and the cylinder.

56. **Shipping**. A shipping box is reinforced with steel bands in all three directions (see the figure). A total of 20.5 feet of steel tape is to be used, with 6 inches of waste because of a 2-inch overlap in each direction. If the box has a square base and a volume of 2 cubic feet, find its dimensions.