(B) Find the cycles per second for C, three notes higher than A.

91. Puzzle. If you place 1¢ on the first square of a chessboard, 2¢ on the second square, 4¢ on the third, and so on, continuing to double the amount until all 64 squares are covered, how much money will be on the sixty-fourth square? How much money will there be on the whole board?

92. Puzzle. If a sheet of very thin paper 0.001 inch thick is torn in half, and each half is again torn in half, and this process is repeated for a total of 32 times, how high will the stack of paper be if the pieces are placed one on top of the other? Give the answer to the nearest mile.

93. Atmospheric Pressure. If atmospheric pressure decreases roughly by a factor of 10 for each 10-mile increase in altitude up to 60 miles, and if the pressure is 15 pounds per square inch at sea level, what will the pressure be 40 miles up?

94. Zeno’s Paradox. Visualize a hypothetical 440-yard oval racetrack that has tapes stretched across the track at the halfway point and at each point that marks the halfway point of each remaining distance thereafter. A runner running around the track has to break the first tape before the second, the second before the third, and so on. From this point of view it appears that he will never finish the race. This famous paradox is attributed to the Greek philosopher Zeno (495–435 B.C.). If we assume the runner runs at 440 yards per minute, the times between tape breakings form an infinite geometric progression. What is the sum of this progression?

95. Geometry. If the midpoints of the sides of an equilateral triangle are joined by straight lines, the new figure will be an equilateral triangle with a perimeter equal to half the original. If we start with an equilateral triangle with perimeter 1 and form a sequence of “nested” equilateral triangles proceeding as described, what will be the total perimeter of all the triangles that can be formed in this way?

96. Photography. The shutter speeds and f-stops on a camera are given as follows:

<table>
<thead>
<tr>
<th>Shutter speeds:</th>
<th>1, 1/2, 1/4, 1/8, 1/15, 1/30, 1/60, 1/125, 1/250, 1/500</th>
</tr>
</thead>
<tbody>
<tr>
<td>f-stops:</td>
<td>1.4, 2, 2.8, 4, 5.6, 8, 11, 16, 22</td>
</tr>
</tbody>
</table>

These are very close to being geometric progressions. Estimate their common ratios.

97. Geometry. We know that the sum of the interior angles of a triangle is 180°. Show that the sums of the interior angles of polygons with 3, 4, 5, 6, . . . sides form an arithmetic sequence. Find the sum of the interior angles for a 21-sided polygon.

Section 6-4 | Multiplication Principle, Permutations, and Combinations

- Multiplication Principle
- Factorial
- Permutations
- Combinations

This section introduces some new mathematical tools that are usually referred to as counting techniques. In general, a counting technique is a mathematical method of determining the number of objects in a set without actually enumerating the objects in the set as 1, 2, 3, . . . . For example, we can count the number
of squares in a checker board (see Fig. 1) by counting 1, 2, 3,..., 64. This is enumeration. Or we can note that there are 8 rows with 8 squares in each row. Thus, the total number of squares must be $8 \times 8 = 64$. This is a very simple counting technique.

Now consider the problem of assigning telephone numbers. How many different seven-digit telephone numbers can be formed? As we will soon see, the answer is $10^7 = 10,000,000$, a number that is much too large to obtain by enumeration. Thus, counting techniques are essential tools if the number of elements in a set is very large. The techniques developed in this section will be applied to a brief introduction to probability theory in Section 6-5, and to a famous algebraic formula in Section 6-6.

**Multiplication Principle**

We start with an example.

**Example 1**

**Combined Outcomes**

Suppose we flip a coin and then throw a single die (see Fig. 2). What are the possible combined outcomes?

**Solution**

To solve this problem, we use a tree diagram:

Thus, there are 12 possible combined outcomes—two ways in which the coin can come up followed by six ways in which the die can come up.

**Matched Problem 1**

Use a tree diagram to determine the number of possible outcomes of throwing a single die followed by flipping a coin.

Now suppose you are asked, “From the 26 letters in the alphabet, how many ways can 3 letters appear in a row on a license plate if no letter is repeated?” To try to count the possibilities using a tree diagram would be extremely tedious, to say the least. The following multiplication principle, also called the fundamental counting principle, enables us to solve this problem easily. In addition, it forms the basis for several other counting techniques developed later in this section.
MULTIPLICATION PRINCIPLE

1. If two operations $O_1$ and $O_2$ are performed in order with $N_1$ possible outcomes for the first operation and $N_2$ possible outcomes for the second operation, then there are

$$N_1 \cdot N_2$$

possible combined outcomes of the first operation followed by the second.

2. In general, if $n$ operations $O_1, O_2, \ldots, O_n$ are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$, respectively, then there are

$$N_1 \cdot N_2 \cdot \ldots \cdot N_n$$

possible combined outcomes of the operations performed in the given order.

In Example 1, we see that there are two possible outcomes from the first operation of flipping a coin and six possible outcomes from the second operation of throwing a die. Hence, by the multiplication principle, there are $2 \cdot 6 = 12$ possible combined outcomes of flipping a coin followed by throwing a die. Use the multiplication principle to solve Matched Problem 1.

To answer the license plate question, we reason as follows: There are 26 ways the first letter can be chosen. After a first letter is chosen, 25 letters remain; hence there are 25 ways a second letter can be chosen. And after 2 letters are chosen, there are 24 ways a third letter can be chosen. Hence, using the multiplication principle, there are $26 \cdot 25 \cdot 24 = 15,600$ possible ways 3 letters can be chosen from the alphabet without allowing any letter to repeat. By not allowing any letter to repeat, earlier selections affect the choice of subsequent selections. If we allow letters to repeat, then earlier selections do not affect the choice in subsequent selections, and there are 26 possible choices for each of the 3 letters. Thus, if we allow letters to repeat, there are $26 \cdot 26 \cdot 26 = 26^3 = 17,576$ possible ways the 3 letters can be chosen from the alphabet.

**Example 2**

**Computer-Generated Tests**

Many universities and colleges are now using computer-assisted testing procedures. Suppose a screening test is to consist of 5 questions, and a computer stores 5 equivalent questions for the first test question, 8 equivalent questions for the second, 6 for the third, 5 for the fourth, and 10 for the fifth. How many different 5-question tests can the computer select? Two tests are considered different if they differ in one or more questions.

**Solution**

$O_1$: Select the first question $N_1$: 5 ways

$O_2$: Select the second question $N_2$: 8 ways

$O_3$: Select the third question $N_3$: 6 ways
6-4 Multiplication Principle, Permutations, and Combinations

\(O_4\): Select the fourth question \(N_4\): 5 ways
\(O_5\): Select the fifth question \(N_5\): 10 ways

Thus, the computer can generate

\[5 \cdot 8 \cdot 6 \cdot 5 \cdot 10 = 12,000\] different tests

**Matched Problem 2**

Each question on a multiple-choice test has 5 choices. If there are 5 such questions on a test, how many different response sheets are possible if only 1 choice is marked for each question?

**Example 3**

**Counting Code Words**

How many 3-letter code words are possible using the first 8 letters of the alphabet if:

(A) No letter can be repeated?  (B) Letters can be repeated?
(C) Adjacent letters cannot be alike?

**Solutions**

(A) No letter can be repeated.

\(O_1\): Select first letter \(N_1\): 8 ways
\(O_2\): Select second letter \(N_2\): 7 ways \(\text{Because 1 letter has been used}\)
\(O_3\): Select third letter \(N_3\): 6 ways \(\text{Because 2 letters have been used}\)

Thus, there are

\[8 \cdot 7 \cdot 6 = 336\] possible code words

(B) Letters can be repeated.

\(O_1\): Select first letter \(N_1\): 8 ways
\(O_2\): Select second letter \(N_2\): 8 ways \(\text{Repeats are allowed.}\)
\(O_3\): Select third letter \(N_3\): 8 ways \(\text{Repeats are allowed.}\)

Thus, there are

\[8 \cdot 8 \cdot 8 = 8^3 = 512\] possible code words

(C) Adjacent letters cannot be alike.

\(O_1\): Select first letter \(N_1\): 8 ways
\(O_2\): Select second letter \(N_2\): 7 ways \(\text{Cannot be the same as the first}\)
Explore/Discuss

The multiplication principle can be used to develop two additional counting techniques that are extremely useful in more complicated counting problems. Both of these methods use the factorial function, which we introduce next.

**Factorial**

For \( n \) a natural number, \( n \) factorial—denoted by \( n! \)—is the product of the first \( n \) natural numbers. Zero factorial is defined to be 1.

\[
\text{n FACTORIAL}
\]

For \( n \) a natural number

\[
n! = n(n - 1) \cdots 2 \cdot 1
\]

\[
1! = 1
\]

\[
0! = 1
\]

It is also useful to note that

**Recursion Formula for \( n \) Factorial**

\[
n! = n \cdot (n - 1)!
\]
**Example 4**

**Evaluating Factorials**

(A) \(4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2! = 4 \cdot 3 \cdot 2 \cdot 1! = 4 \cdot 3 \cdot 2 \cdot 1 = 24\)

(B) \(5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120\)

(C) \(\frac{7!}{6!} = \frac{7 \cdot 6!}{6!} = 7\)

(D) \(\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5!} = 336\)

(E) \(\frac{9!}{6!3!} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 2 \cdot 1} = 84\)

---

**Matched Problem 4**

Find (A) 6! (B) \(\frac{6!}{5!}\) (C) \(\frac{9!}{6!}\) (D) \(\frac{10!}{7!3!}\)

---

**Caution**

When reducing fractions involving factorials, don’t confuse the single integer \(n\) with the symbol \(n!\), which represents the product of \(n\) consecutive integers.

\[
\frac{6!}{3!} \neq 2! \quad \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = 120
\]

---

**Explore/Discuss 2**

A student used a calculator to solve Matched Problem 4, as shown in Figure 3. Check these answers. If any are incorrect, explain why and find a correct calculator solution.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Calculator Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>6!</td>
<td>720</td>
</tr>
<tr>
<td>(6! / 5!)</td>
<td>6</td>
</tr>
<tr>
<td>9! / 6!</td>
<td>504</td>
</tr>
<tr>
<td>(10! / 7!3!)</td>
<td>4320</td>
</tr>
</tbody>
</table>

**Figure 3**

It is interesting and useful to note that \(n!\) grows very rapidly. Compare the following:

\(5! = 120\) \quad \(10! = 3,628,800\) \quad \(15! = 1,307,674,368,000\)
If \( n! \) is too large for a calculator to store and display, an error message is displayed. Find the value of \( n \) such that your calculator will evaluate \( n! \), but not \((n + 1)!\).

**Permutations**

Suppose 4 pictures are to be arranged from left to right on one wall of an art gallery. How many arrangements are possible? Using the multiplication principle, there are 4 ways of selecting the first picture. After the first picture is selected, there are 3 ways of selecting the second picture. After the first 2 pictures are selected, there are 2 ways of selecting the third picture. And after the first 3 pictures are selected, there is only 1 way to select the fourth. Thus, the number of arrangements possible for the 4 pictures is

\[
4 \cdot 3 \cdot 2 \cdot 1 = 4! \quad \text{or} \quad 24
\]

In general, we refer to a particular arrangement, or ordering, of \( n \) objects without repetition as a **permutation** of the \( n \) objects. How many permutations of \( n \) objects are there? From the reasoning above, there are \( n \) ways in which the first object can be chosen, there are \( n - 1 \) ways in which the second object can be chosen, and so on. Applying the multiplication principle, we have Theorem 2.

**Theorem 2**

The number of permutations of \( n \) objects, denoted by \( P_{n,n} \), is given by

\[
P_{n,n} = n \cdot (n-1) \cdot \ldots \cdot 1 = n!
\]

Now suppose the director of the art gallery decides to use only 2 of the 4 available pictures on the wall, arranged from left to right. How many arrangements of 2 pictures can be formed from the 4? There are 4 ways the first picture can be selected. After selecting the first picture, there are 3 ways the second picture can be selected. Thus, the number of arrangements of 2 pictures from 4 pictures, denoted by \( P_{4,2} \), is given by

\[
P_{4,2} = 4 \cdot 3 = 12
\]

Or, in terms of factorials, multiplying \( 4 \cdot 3 \) by \( 1 \) in the form \( 2!/2! \), we have

\[
P_{4,2} = 4 \cdot 3 = \frac{4 \cdot 3 \cdot 2!}{2!} = \frac{4!}{2!}
\]

This last form gives \( P_{4,2} \) in terms of factorials, which is useful in some cases.

A **permutation of a set of \( n \) objects taken \( r \) at a time** is an arrangement of the \( r \) objects in a specific order. Thus, reasoning in the same way as in the example above, we find that the number of permutations of \( n \) objects taken \( r \) at a time, \( 0 \leq r \leq n \), denoted by \( P_{n,r} \), is given by

\[
P_{n,r} = n(n - 1)(n - 2) \cdot \ldots \cdot (n - r + 1)
\]
Multiplying the right side of this equation by 1 in the form \((n - r)!/(n - r)!\), we obtain a factorial form for \(P_{n,r}\):

\[
P_{n,r} = n(n - 1)(n - 2) \cdots (n - r + 1) \frac{(n - r)!}{(n - r)!}
\]

But

\[
n(n - 1)(n - 2) \cdots (n - r + 1)(n - r)! = n!
\]

Hence, we have Theorem 3.

### Theorem 3

**PERMUTATION OF \(n\) OBJECTS TAKEN \(r\) AT A TIME**

The number of permutations of \(n\) objects taken \(r\) at a time is given by

\[
P_{n,r} = n(n - 1)(n - 2) \cdots (n - r + 1)
\]

or

\[
P_{n,r} = \frac{n!}{(n - r)!} \quad 0 \leq r \leq n
\]

Note that if \(r = n\), then the number of permutations of \(n\) objects taken \(n\) at a time is

\[
P_{n,n} = \frac{n!}{(n - n)!} = \frac{n!}{0!} = n! \quad \text{Recall, } 0! = 1.
\]

which agrees with Theorem 1, as it should.

The permutation symbol \(P_{n,r}\) also can be denoted by \(P_{n}^{r}\), \(P_{n,r}\), or \(P(n, r)\). Many calculators use \(P_{n}^{r}\) to denote the function that evaluates the permutation symbol.

### Example 5

**Selecting Officers**

From a committee of 8 people, in how many ways can we choose a chair and a vice-chair, assuming one person cannot hold more than one position?

**Solution**

We are actually asking for the number of permutations of 8 objects taken 2 at a time—that is, \(P_{8,2}\):

\[
P_{8,2} = \frac{8!}{(8 - 2)!} = \frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 56
\]

### Matched Problem 5

From a committee of 10 people, in how many ways can we choose a chair, vice-chair, and secretary, assuming one person cannot hold more than one position?
Evaluating $P_{n,r}$

Find the number of permutations of 25 objects taken
(A) 2 at a time  (B) 4 at a time  (C) 8 at a time

Solution

Figure 4 shows the solution on a graphing utility.

\[
\begin{array}{c|c}
25 \text{ nPr 2} & 600 \\
25 \text{ nPr 4} & 303600 \\
25 \text{ nPr 8} & 4.3609104 \times 10^{10}
\end{array}
\]

Matched Problem

Find the number of permutations of 30 objects taken
(A) 2 at a time  (B) 4 at a time  (C) 6 at a time

Combinations

Now suppose that an art museum owns 8 paintings by a given artist and another art museum wishes to borrow 3 of these paintings for a special show. How many ways can 3 paintings be selected for shipment out of the 8 available? Here, the order of the items selected doesn’t matter. What we are actually interested in is how many subsets of 3 objects can be formed from a set of 8 objects. We call such a subset a combination of 8 objects taken 3 at a time. The total number of combinations is denoted by the symbol

\[C_{8,3} \quad \text{or} \quad \binom{8}{3}\]

To find the number of combinations of 8 objects taken 3 at a time, $C_{8,3}$, we make use of the formula for $P_{n,r}$ and the multiplication principle. We know that the number of permutations of 8 objects taken 3 at a time is given by $P_{8,3}$, and we have a formula for computing this quantity. Now suppose we think of $P_{8,3}$ in terms of two operations:

1. $O_1$: Select a subset of 3 objects (paintings)
2. $N_1$: $C_{8,3}$ ways
3. $O_2$: Arrange the subset in a given order
4. $N_2$: 3! ways

The combined operation, $O_1$, followed by $O_2$, produces a permutation of 8 objects taken 3 at a time. Thus,

\[P_{8,3} = C_{8,3} \cdot 3!\]
To find \( C_{8,3} \), we replace \( P_{8,3} \) in the preceding equation with \( 8!(8 - 3)! \) and solve for \( C_{8,3} \):

\[
\frac{8!}{(8 - 3)!} = C_{8,3} \cdot 3!
\]

\[
C_{8,3} = \frac{8!}{3!(8 - 3)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 1 \cdot 5!} = 56
\]

Thus, the museum can make 56 different selections of 3 paintings from the 8 available.

A **combination of a set of \( n \) objects taken \( r \) at a time** is an \( r \)-element subset of the \( n \) objects. Reasoning in the same way as in the example, the number of combinations of \( n \) objects taken \( r \) at a time, \( 0 \leq r \leq n \), denoted by \( C_{n,r} \), can be obtained by solving for \( C_{n,r} \) in the relationship

\[
P_{n,r} = C_{n,r} \cdot r!
\]

\[
C_{n,r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n - r)!}
\]

\[
P_{n,r} = \frac{n!}{(n - r)!}
\]

**Theorem 4**

**Combination of \( n \) objects taken \( r \) at a time**

The number of combinations of \( n \) objects taken \( r \) at a time is given by

\[
C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n - r)!} \quad 0 \leq r \leq n
\]

The combination symbols \( C_{n,r} \) and \( \binom{n}{r} \) also can be denoted by \( C_r^n \), \( \, C_r^r \), or \( C(n, r) \).

**Example 7**

**Selecting Subcommittees**

From a committee of 8 people, in how many ways can we choose a subcommittee of 2 people?

**Solution**

Notice how this example differs from Example 5, where we wanted to know how many ways a chair and a vice-chair can be chosen from a committee of 8 people. In Example 5, ordering matters. In choosing a subcommittee of 2 people, the ordering does not matter. Thus, we are actually asking for the number of combinations of 8 objects taken 2 at a time. The number is given by

\[
C_{8,2} = \binom{8}{2} = \frac{8!}{2!(8 - 2)!} = \frac{8 \cdot 7 \cdot 6!}{2 \cdot 1 \cdot 6!} = 28
\]
How many subcommittees of 3 people can be chosen from a committee of 8 people?

**Example 8**

**Evaluating Cₙᵣ**

Find the number of combinations of 25 objects taken

(A) 2 at a time  
(B) 4 at a time  
(C) 8 at a time

**Solution**

Figure 5 shows the solution on a graphing utility. Compare these results with Example 6.

**Figure 5**

<table>
<thead>
<tr>
<th>nCr</th>
<th>2</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>4</td>
<td>12650</td>
</tr>
<tr>
<td>25</td>
<td>8</td>
<td>1081575</td>
</tr>
</tbody>
</table>

**Matched Problem 8**

Find the number of combinations of 30 objects taken

(A) 2 at a time  
(B) 4 at a time  
(C) 6 at a time

**Remember: In a permutation, order counts. In a combination, order does not count.**

To determine whether a permutation or combination is needed, decide whether rearranging the collection or listing makes a difference. If so, use permutations. If not, use combinations.

**Explore/Discuss 3**

Each of the following is a selection without repetition. Would you consider the selection to be a combination? A permutation? Discuss your reasoning.

(A) A student checks out three books from the library.
(B) A baseball manager names his starting lineup.
(C) The newly elected president names his cabinet members.
(D) The president selects a delegation of three cabinet members to attend the funeral of a head of state.
(E) An orchestra conductor chooses three pieces of music for a symphony program.

A **standard deck** of 52 cards (see Fig. 6) has four 13-card suits: diamonds, hearts, clubs, and spades. Each 13-card suit contains cards numbered from 2 to 10, a jack, a queen, a king, and an ace. The jack, queen, and king are called **face cards**. Depending on the game, the ace may be counted as the lowest and/or the
highest card in the suit. Example 9, as well as other examples and exercises in this chapter, refer to this standard deck.

**Example 9**  
**Counting Card Hands**

Out of a standard 52-card deck, how many 5-card hands will have 3 aces and 2 kings?

**Solution**

- **$O_1$:** Choose 3 aces out of 4 possible  
  Order is not important.
- **$N_1$:** $C_{4,3}$
- **$O_2$:** Choose 2 kings out of 4 possible  
  Order is not important.
- **$N_2$:** $C_{4,2}$

Using the multiplication principle, we have

$$\text{Number of hands} = C_{4,3} \cdot C_{4,2} = 4 \cdot 6 = 24$$

**Matched Problem**

From a standard 52-card deck, how many 5-card hands will have 3 hearts and 2 spades?

**Example 10**  
**Counting Serial Numbers**

Serial numbers for a product are to be made using 2 letters followed by 3 numbers. If the letters are to be taken from the first 8 letters of the alphabet with no repeats and the numbers from the 10 digits 0 through 9 with no repeats, how many serial numbers are possible?

**Solution**

- **$O_1$:** Choose 2 letters out of 8 available  
  Order is important.
- **$N_1$:** $P_{8,2}$
- **$O_2$:** Choose 3 numbers out of 10 available  
  Order is important.
- **$N_2$:** $P_{10,3}$

FIGURE 6
A standard deck of cards.
Using the multiplication principle, we have

\[ \text{Number of serial numbers} = P_{8,2} \cdot P_{10,3} = 40,320 \]

**MATCHED PROBLEM 10**

Repeat Example 10 under the same conditions, except the serial numbers are now to have 3 letters followed by 2 digits with no repeats.

**Answers to Matched Problems**

1. 2. \(5^5\), or 3,125

3. (A) \(10 \cdot 9 \cdot 8 \cdot 7 = 5,040\) (B) \(10 \cdot 10 \cdot 10 \cdot 10 = 10,000\) (C) \(10 \cdot 9 \cdot 9 \cdot 9 = 7,290\)

4. (A) 720 (B) 6 (C) 504 (D) 120

5. \(P_{10,3} = \frac{10!}{(10-3)!} = 720\)

6. (A) 870 (B) 657,720 (C) 427,518,000

7. \(C_{8,3} = \frac{8!}{3!(8-3)!} = 56\)

8. (A) 435 (B) 27,405 (C) 593,775

9. \(C_{13,3} \cdot C_{13,2} = 22,308\)

10. \(P_{8,3} \cdot P_{10,2} = 30,240\)

**EXERCISE 6-4**

**A**


1. \(9!\)

2. \(10!\)

3. \(11!\)

4. \(12!\)

5. \(\frac{11!}{8!}\)

6. \(\frac{14!}{12!}\)

7. \(\frac{5!}{2!3!}\)

8. \(\frac{6!}{4!2!}\)

9. \(\frac{7!}{4!(7-4)!}\)

10. \(\frac{8!}{3!(8-3)!}\)

11. \(\frac{7!}{7!(7-7)!}\)

12. \(\frac{8!}{0!(8-0)!}\)

13. \(P_{3,3}\)

14. \(P_{4,2}\)

15. \(P_{5,4}\)

16. \(P_{32,2}\)

17. \(C_{5,3}\)

18. \(C_{4,2}\)

19. \(C_{32,4}\)

20. \(C_{32,2}\)

**In Problems 21 and 22, would you consider the selection to be a combination or a permutation? Explain your reasoning.**

21. (A) The recently elected chief executive officer (CEO) of a company named 3 new vice-presidents, of marketing, research, and manufacturing.

(B) The CEO selected 3 of her vice-presidents to attend the dedication ceremony of a new plant.

22. (A) An individual rented 4 videos from a rental store to watch over a weekend.

(B) The same individual did some holiday shopping by buying 4 videos, 1 for his father, 1 for his mother, 1 for his younger sister, and 1 for his older brother.

23. A particular new car model is available with 5 choices of color, 3 choices of transmission, 4 types of interior, and 2 types of engine. How many different variations of this model car are possible?

24. A deli serves sandwiches with the following options: 3 kinds of bread, 5 kinds of meat, and lettuce or sprouts. How many different sandwiches are possible, assuming one item is used out of each category?

25. In a horse race, how many different finishes among the first 3 places are possible for a 10-horse race? Exclude ties.

26. In a long-distance foot race, how many different finishes among the first 5 places are possible for a 50-person race? Exclude ties.

27. How many ways can a subcommittee of 3 people be selected from a committee of 7 people? How many ways can a president, vice president, and secretary be chosen from a committee of 7 people?

28. Suppose 9 cards are numbered with the 9 digits from 1 to 9. A 3-card hand is dealt, 1 card at a time. How many hands are possible where:

(A) Order is taken into consideration?

(B) Order is not taken into consideration?
29. There are 10 teams in a league. If each team is to play every other team exactly once, how many games must be scheduled?

30. Given 7 points, no 3 of which are on a straight line, how many lines can be drawn joining 2 points at a time?

31. How many 4-letter code words are possible from the first 6 letters of the alphabet, with no letter repeated? Allowing letters to repeat?

32. How many 5-letter code words are possible from the first 7 letters of the alphabet, with no letter repeated? Allowing letters to repeat?

33. A combination lock has 5 wheels, each labeled with the 10 digits from 0 to 9. How many opening combinations of 5 numbers are possible, assuming no digit is repeated? Assuming digits can be repeated?

34. A small combination lock on a suitcase has 3 wheels, each labeled with digits from 0 to 9. How many opening combinations of 3 numbers are possible, assuming no digit is repeated? Assuming digits can be repeated?

35. From a standard 52-card deck, how many 5-card hands will have all hearts?

36. From a standard 52-card deck, how many 5-card hands will have all face cards? All face cards, but no kings? Consider only jacks, queens, and kings to be face cards.

37. How many different license plates are possible if each contains 3 letters followed by 3 digits? How many of these license plates contain no repeated letters and no repeated digits?

38. How may 5-digit zip codes are possible? How many of these codes contain no repeated digits?

39. From a standard 52-card deck, how many 7-card hands have exactly 5 spades and 2 hearts?

40. From a standard 52-card deck, how many 5-card hands will have 2 clubs and 3 hearts?

41. A catering service offers 8 appetizers, 10 main courses, and 7 desserts. A banquet chairperson is to select 3 appetizers, 4 main courses, and 2 desserts for a banquet. How many ways can this be done?

42. Three research departments have 12, 15, and 18 members, respectively. If each department is to select a delegate and an alternate to represent the department at a conference, how many ways can this be done?

43. (A) Use a graphing utility to display the sequences $P_{10,0}$, $P_{10,1}$, $P_{10,2}$, $P_{10,3}, \ldots, P_{10,10}$ and 0!, 1!, 2!, 3!, 4!, 5!, 6!, 7!, 8!, 9!, 10! in table form, and show that $P_{10,0} = 0!$ for $r = 0, 1, \ldots, 10$.

(B) Find all values of $r$ such that $P_{10,r} = r!$

(C) Explain why $P_{n,r} \geq r!$ whenever $0 \leq r \leq n$.

44. (A) How are the sequences $\frac{P_{10,0}}{0!}, \frac{P_{10,1}}{1!}, \ldots, \frac{P_{10,10}}{10!}$ and $C_{10,1}, \ldots, C_{10,10}$ related?

(B) Use a graphing utility to graph each sequence and confirm the relationship of part A.

45. A sporting goods store has 12 pairs of ski gloves of 12 different brands thrown loosely in a bin. The gloves are all the same size. In how many ways can a left-hand glove and a right-hand glove be selected that do not match relative to brand?

46. A sporting goods store has 6 pairs of running shoes of 6 different styles thrown loosely in a basket. The shoes are all the same size. In how many ways can a left shoe and a right shoe be selected that do not match?

47. Eight distinct points are selected on the circumference of a circle.

(A) How many chords can be drawn by joining the points in all possible ways?

(B) How many triangles can be drawn using these 8 points as vertices?

(C) How many quadrilaterals can be drawn using these 8 points as vertices?

48. Five distinct points are selected on the circumference of a circle.

(A) How many chords can be drawn by joining the points in all possible ways?

(B) How many triangles can be drawn using these 5 points as vertices?

49. How many ways can 2 people be seated in a row of 5 chairs? 3 people? 4 people? 5 people?

50. Each of 2 countries sends 5 delegates to a negotiating conference. A rectangular table is used with 5 chairs on each long side. If each country is assigned a long side of the table, how many seating arrangements are possible? [Hint: Operation 1 is assigning a long side of the table to each country.]

51. A basketball team has 5 distinct positions. Out of 8 players, how many starting teams are possible if

(A) The distinct positions are taken into consideration?

(B) The distinct positions are not taken into consideration?

(C) The distinct positions are not taken into consideration, but either Mike or Ken, but not both, must start?
52. How many committees of 4 people are possible from a group of 9 people if
   (A) There are no restrictions?
   (B) Both Juan and Mary must be on the committee?
   (C) Either Juan or Mary, but not both, must be on the committee?

53. A 5-card hand is dealt from a standard 52-card deck.
   Which is more likely: the hand contains exactly 1 king or the hand contains no hearts?

54. A 10-card hand is dealt from a standard 52-card deck.
   Which is more likely: all cards in the hand are red or the hand contains all four aces?

Section 6-5 Sample Spaces and Probability

This section provides an introduction to probability, a topic that has whole books and courses devoted to it. Probability studies involve many subtle notions, and care must be taken at the beginning to understand the fundamental concepts on which the studies are based. First, we develop a mathematical model for probability studies. Our development, because of space, must be somewhat informal. More formal and precise treatments can be found in books on probability.

Experiments

Our first step in constructing a mathematical model for probability studies is to describe the type of experiments on which probability studies are based. Some types of experiments do not yield the same results, no matter how carefully they are repeated under the same conditions. These experiments are called random experiments. Familiar examples of random experiments are flipping coins, rolling dice, observing the frequency of defective items from an assembly line, or observing the frequency of deaths in a certain age group.

Probability theory is a branch of mathematics that has been developed to deal with outcomes of random experiments, both real and conceptual. In the work that follows, the word experiment will be used to mean a random experiment.

Sample Spaces and Events

Associated with outcomes of experiments are sample spaces and events. Our second step in constructing a mathematical model for probability studies is to define these two terms. Set concepts will be useful in this regard.

Consider the experiment, “A single six-sided die is rolled.” What outcomes might we observe? We might be interested in the number of dots facing up, or whether the number of dots facing up is an even number, or whether the number of dots facing up is divisible by 3, and so on. The list of possible outcomes appears endless. In general, there is no unique method of analyzing all possible outcomes of an experiment. Therefore, before conducting an experiment, it is important to decide just what outcomes are of interest.

In the die experiment, suppose we limit our interest to the number of dots facing up when the die comes to rest. Having decided what to observe, we make a