In Chapter 3 you were introduced to the basic functions

\[ f(x) = b \]  \hspace{1cm} \text{Constant function}

\[ f(x) = ax + b, \quad a \neq 0 \]  \hspace{1cm} \text{Linear function}

\[ f(x) = ax^2 + bx + c, \quad a \neq 0 \]  \hspace{1cm} \text{Quadratic function}

as well as some special cases of more complicated functions such as

\[ f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0 \]  \hspace{1cm} \text{Cubic function}

Notice the evolving pattern going from the constant function to the cubic function—the terms in each equation are of the form \( ax^n \), where \( n \) is a nonnegative integer and \( a \) is a real number. All these functions are special cases of the general class of functions called \textit{polynomial functions}. The function

\[ P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad a_n \neq 0 \]

is called an \textit{nth-degree polynomial function}. We will also refer to \( P(x) \) as a \textit{polynomial of degree \( n \)} or, more simply, as a \textit{polynomial}. The numbers \( a_n, a_{n-1}, \ldots, a_1, a_0 \) are called the \textit{coefficients of the function}. A nonzero constant function is a zero-degree polynomial, a linear function is a first-degree polynomial, and a quadratic function is a second-degree polynomial. The zero function \( Q(x) = 0 \) is also considered to be a polynomial but is not assigned a degree. The coefficients of a polynomial function may be complex numbers, or may be restricted to real numbers, rational numbers, or integers, depending on our interests. The domain of a polynomial function can be the set of complex numbers, the set of real numbers, or appropriate subsets of either, depending on our interests. In general, the context will dictate the choice of coefficients and domain.

The number \( r \) is said to be a \textit{zero of the function \( P \)}, or a \textit{zero of the polynomial \( P(x) \)}, or a \textit{solution or root of the equation \( P(x) = 0 \)}, if

\[ P(r) = 0 \]

A zero of a polynomial may or may not be the number 0. A zero of a polynomial is any number that makes the value of the polynomial 0. If the coefficients of a polynomial \( P(x) \) are real numbers, then a real zero is simply an \( x \) intercept for the graph of \( y = P(x) \). Consider the polynomial

\[ P(x) = x^2 - 4x + 3 \]
The graph of \( P \) is shown in Figure 1.

**FIGURE 1** Zeros, roots, and \( x \) intercepts.

In general:

**Zeros and Roots**

If the coefficients of a polynomial \( P(x) \) are real, then the \( x \) intercepts of the graph of \( y = P(x) \) are real zeros of \( P \) and \( P(x) \) and real solutions, or roots, for the equation \( P(x) = 0 \).

- **Polynomial Division**

  We can find quotients of polynomials by a long-division process similar to that used in arithmetic. An example will illustrate the process.

**EXAMPLE 1** Algebraic Long Division

Divide \( 5 + 4x^3 - 3x \) by \( 2x - 3 \).

**Solution**

\[
\begin{aligned}
2x - 3 & | \quad 4x^3 + 0x^2 - 3x + 5 \\
& \quad 2x^2 + 3x + 3 \\
& \quad 4x^3 - 6x^2 \\
& \quad 6x^2 - 3x \\
& \quad 6x^2 - 9x \\
& \quad 6x - 9 \\
& \quad 14 = R \\
& \text{Remainder}
\end{aligned}
\]

Arrange the dividend and the divisor in descending powers of the variable. Insert, with 0 coefficients, any missing terms of degree less than 3. Divide the first term of the divisor into the first term of the dividend. Multiply the divisor by \( 2x^2 \), line up like terms, subtract as in arithmetic, and bring down \(-3x\). Repeat the process until the degree of the remainder is less than that of the divisor.

Thus,

\[
\frac{4x^3 - 3x + 5}{2x - 3} = 2x^2 + 3x + 3 + \frac{14}{2x - 3}
\]

**Check**

\[
(2x - 3) \left[ (2x^2 + 3x + 3) + \frac{14}{2x - 3} \right] = (2x - 3)(2x^2 + 3x + 3) + 14 = 4x^3 - 3x + 5
\]
Matched Problem 1

Divide $6x^2 - 30 + 9x^3$ by $3x - 4$.

Being able to divide a polynomial $P(x)$ by a linear polynomial of the form $x - r$ quickly and accurately will be of great help in the search for zeros of higher-degree polynomial functions. This kind of division can be carried out more efficiently by a method called synthetic division. The method is most easily understood through an example. Let’s start by dividing $P(x) = 2x^4 + 3x^3 - x - 5$ by $x + 2$, using ordinary long division. The critical parts of the process are indicated in color.

```
Divisor  x + 2  Quotient  2x^3 - 1x^2 + 2x - 5
2x^4 + 3x^3 + 0x^2 - 1x - 5  Dividend

-1x^3 + 0x^2
-1x^3 - 2x^2
2x^2 - 1x
2x^2 + 4x
-5x - 5
-5x - 10
5  Remainder
```

The numerals printed in color, which represent the essential part of the division process, are arranged more conveniently as follows:

```
Dividend coefficients
2  3  0 -1 -5

Quotient coefficients
2  4 -2  4 -10

Remainder
2  -1  2  -5  5
```

Mechanically, we see that the second and third rows of numerals are generated as follows. The first coefficient, 2, of the dividend is brought down and multiplied by 2 from the divisor; and the product, 4, is placed under the second dividend coefficient, 3, and subtracted. The difference, −1, is again multiplied by the 2 from the divisor; and the product is placed under the third coefficient from the dividend and subtracted. This process is repeated until the remainder is reached. The process can be made a little faster, and less prone to sign errors, by changing +2 from the divisor to −2 and adding instead of subtracting. Thus

```
Dividend coefficients
2  3  0 -1 -5

Quotient coefficients
-2  -4  2 -4  10

Remainder
-2  -1  2  -5  5
```
Key Steps in the Synthetic Division Process

To divide the polynomial $P(x)$ by $x - r$:

Step 1. Arrange the coefficients of $P(x)$ in order of descending powers of $x$. Write 0 as the coefficient for each missing power.

Step 2. After writing the divisor in the form $x - r$, use $r$ to generate the second and third rows of numbers as follows. Bring down the first coefficient of the dividend and multiply it by $r$; then add the product to the second coefficient of the dividend. Multiply this sum by $r$, and add the product to the third coefficient of the dividend. Repeat the process until a product is added to the constant term of $P(x)$.

Step 3. The last number to the right in the third row of numbers is the remainder. The other numbers in the third row are the coefficients of the quotient, which is of degree 1 less than $P(x)$.

EXAMPLE 2 Synthetic Division

Use synthetic division to find the quotient and remainder resulting from dividing $P(x) = 4x^4 - 30x^3 - 50x - 2$ by $x + 3$. Write the answer in the form $Q(x) + R/(x - r)$, where $R$ is a constant.

Solution Since $x + 3 = x - (-3)$, we have $r = -3$, and

\[
\begin{array}{c|cccc}
4 & 0 & -30 & 0 & -50 \\
-12 & 36 & -18 & 54 & -12 \\
-3 & 4 & -12 & 6 & -18 \\
\hline
& 4 & -12 & 6 & -18 & 4 & -14
\end{array}
\]

The quotient is $4x^4 - 12x^3 + 6x^2 - 18x + 4$ with a remainder of $-14$. Thus,

\[
\frac{P(x)}{x + 3} = 4x^4 - 12x^3 + 6x^2 - 18x + 4 + \frac{-14}{x + 3}
\]

Matched Problem 2 Repeat Example 2 with $P(x) = 3x^4 - 11x^3 - 18x + 8$ and divisor $x - 4$.

A calculator is a convenient tool for performing synthetic division. Any type of calculator can be used, although one with a memory will save some keystrokes. The flowchart in Figure 2 shows the repetitive steps in the synthetic division process, and Figure 3 illustrates the results of applying this process to Example 2 on a graphing calculator.
• **Division Algorithm** If we divide \( P(x) = 2x^4 - 5x^3 - 4x^2 + 13 \) by \( x - 3 \), we obtain

\[
\frac{2x^4 - 5x^3 - 4x^2 + 13}{x - 3} = 2x^3 + x^2 - x - 3 + \frac{4}{x - 3} \quad x \neq 3
\]

If we multiply both sides of this equation by \( x - 3 \), then we get

\[
2x^4 - 5x^3 - 4x^2 + 13 = (x - 3)(2x^3 + x^2 - x - 3) + 4
\]

This last equation is an identity in that the left side is equal to the right side for all replacements of \( x \) by real or imaginary numbers, including \( x = 3 \). This example suggests the important division algorithm, which we state as Theorem 1 without proof.

---

**Theorem 1**

<table>
<thead>
<tr>
<th>Division Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>For each polynomial ( P(x) ) of degree greater than 0 and each number ( r ), there exists a unique polynomial ( Q(x) ) of degree 1 less than ( P(x) ) and a unique number ( R ) such that ( P(x) = (x - r)Q(x) + R ).</td>
</tr>
</tbody>
</table>

The polynomial \( Q(x) \) is called the **quotient**, \( x - r \) is the **divisor**, and \( R \) is the **remainder**. Note that \( R \) may be 0.
EXPLORE-DISCUSS 1  Let \( P(x) = x^3 - 3x^2 - 2x + 8 \).

(A) Evaluate \( P(x) \) for
(i) \( x = -2 \)  (ii) \( x = 1 \)  (iii) \( x = 3 \)

(B) Use synthetic division to find the remainder when \( P(x) \) is divided by
(i) \( x + 2 \)  (ii) \( x - 1 \)  (iii) \( x - 3 \)

What conclusion does a comparison of the results in parts A and B suggest?

**Remainder Theorem**

We now use the division algorithm in Theorem 1 to prove the *remainder theorem*. The equation in Theorem 1,

\[
P(x) = (x - r)Q(x) + R
\]

is an identity; that is, it is true for all real or imaginary replacements for \( x \). In particular, if we let \( x = r \), then we observe a very interesting and useful relationship:

\[
P(r) = (r - r)Q(r) + R
\]

\[
= 0 \cdot Q(r) + R
\]

\[
= 0 + R
\]

\[
= R
\]

In words, the value of a polynomial \( P(x) \) at \( x = r \) is the same as the remainder \( R \) obtained when we divide \( P(x) \) by \( x - r \). We have proved the well-known remainder theorem:

**Theorem 2  Remainder Theorem**

If \( R \) is the remainder after dividing the polynomial \( P(x) \) by \( x - r \), then

\[
P(r) = R
\]

**Example 3  Two Methods for Evaluating Polynomials**

If \( P(x) = 4x^4 + 10x^3 + 19x + 5 \), find \( P(-3) \) by:

(A) Using the remainder theorem and synthetic division
(B) Evaluating \( P(-3) \) directly

**Solutions**  (A) Use synthetic division to divide \( P(x) \) by \( x - (-3) \).
4-1 Polynomial Functions and Graphs

Matched Problem 3

Repeat Example 3 for \( P(x) = 3x^4 - 16x^2 - 3x + 7 \) and \( x = -2 \).

Graphing Polynomial Functions

The shape of the graph of a polynomial function is connected to the degree of the polynomial. The shapes of odd-degree polynomial functions have something in common, and the shapes of even-degree polynomial functions have something in common. Figure 4 shows graphs of representative polynomial functions from degrees 1 to 6 and suggests some general properties of graphs of polynomial functions.

Figure 4: Graphs of polynomial functions.

(a) \( f(x) = x - 2 \)
(b) \( g(x) = x^3 - 5x \)
(c) \( h(x) = x^5 - 6x^3 + 8x + 1 \)
(d) \( f(x) = x^2 - x + 1 \)
(e) \( G(x) = 2x^6 - 7x^2 + x + 3 \)
(f) \( H(x) = x^6 - 7x^4 + 12x^2 - x - 2 \)

Notice that the odd-degree polynomial graphs start negative, end positive, and cross the \( x \) axis at least once. The even-degree polynomial graphs start positive, end positive, and may not cross the \( x \) axis at all. In all cases in Figure 4, the coefficient of the highest-degree term was chosen positive. If any leading coefficient had been chosen negative, then we would have a similar graph but reflected in the \( x \) axis.

The shape of the graph of a polynomial is also related to the shape of the graph...
of the highest-degree or **leading term** of the polynomial. Figure 5 compares the graph of one of the polynomials from Figure 4 with the graph of its leading term. Although quite dissimilar for points close to the origin, as we “zoom out” to points distant from the origin, the graphs become quite similar. The leading term in the polynomial dominates all other terms combined.

**FIGURE 5**  
$p(x) = x^3, h(x) = x^3 - 6x^3 + 8x + 1.$

In general, the behavior of the graph of a polynomial function as $x$ decreases without bound to the left or as $x$ increases without bound to the right is determined by its leading term. We often use the symbols $-\infty$ and $\infty$ to help describe this left and right behavior.* The various possibilities are summarized in Theorem 3.

### Theorem 3  
**Left and Right Behavior of a Polynomial**

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

1. $a_n > 0$ and $n$ even  
   Graph of $P(x)$ increases without bound as $x$ decreases to the left and as $x$ increases to the right.

2. $a_n > 0$ and $n$ odd  
   Graph of $P(x)$ decreases without bound as $x$ decreases to the left and increases without bound as $x$ increases to the right.

$$P(x) \rightarrow \begin{cases} 
\infty & \text{as } x \rightarrow -\infty \\
\infty & \text{as } x \rightarrow \infty
\end{cases}$$

*Remember, the symbol $\infty$ does not represent a real number. Earlier, we used $\infty$ to denote unbounded intervals, such as $[0, \infty)$. Now we are using it to describe quantities that are growing with no upper limit on their size.
3. \(a_n < 0\) and \(n\) even  
Graph of \(P(x)\) decreases without bound as \(x\) decreases to the left and as \(x\) increases to the right.

4. \(a_n < 0\) and \(n\) odd  
Graph of \(P(x)\) increases without bound as \(x\) decreases to the left and decreases without bound as \(x\) increases to the right.

Figure 4 gives examples of polynomial functions with graphs containing the maximum number of turning points possible for a polynomial of that degree. A turning point on a continuous graph is a point that separates an increasing portion from a decreasing portion. Listed in Theorem 4 below are useful properties of polynomial functions we accept without proof. Property 3 is discussed in detail later in this chapter. The other properties are established in calculus.

**Theorem 4**  
**Graph Properties of Polynomial Functions**

Let \(P\) be an \(n\)th-degree polynomial function with real coefficients.

1. \(P\) is continuous for all real numbers.
2. The graph of \(P\) is a smooth curve.
3. The graph of \(P\) has at most \(n\) \(x\) intercepts.
4. \(P\) has at most \(n - 1\) turning points.

**EXPLORE-DISCUSS 2**  
(A) What is the least number of turning points an odd-degree polynomial function can have? An even-degree polynomial function?  
(B) What is the maximum number of \(x\) intercepts the graph of a polynomial function of degree \(n\) can have?  
(C) What is the maximum number of real solutions an \(n\)th-degree polynomial equation can have?  
(D) What is the least number of \(x\) intercepts the graph of a polynomial function of odd degree can have? Of even degree?
(E) What is the least number of real solutions a polynomial equation of odd degree can have? Of even degree?

EXAMPLE 4   Graphing a Polynomial

Graph \( P(x) = x^3 - 12x + 2, \ -4 \leq x \leq 4 \). Find points by using synthetic division and the remainder theorem. How many \( x \) intercepts does the graph have? How many turning points? Describe the left and right behavior of \( P(x) \).

Solution

We evaluate \( P(x) \) from \( x = -4 \) to \( x = 4 \) for integer values of \( x \). The process is speeded up by forming a synthetic division table. To simplify the form of the table, we dispense with writing the product of \( r \) with each coefficient in the quotient and perform the calculations mentally or with a calculator. A calculator becomes increasingly useful as the coefficients become more numerous or complicated. The table also provides other important information, as will be seen in subsequent sections.

\[
\begin{array}{c|ccccc}
1 & 0 & -12 & 2 \\
-4 & 1 & -4 & 4 & -14 = P(-4) \\
-3 & 1 & -3 & -3 & 11 = P(-3) \\
-2 & 1 & -2 & -8 & 18 = P(-2) \\
-1 & 1 & -1 & -11 & 13 = P(-1) \\
0 & 1 & 0 & -12 & 2 = P(0) \\
1 & 1 & 1 & -11 & -9 = P(1) \\
2 & 1 & 2 & -8 & -14 = P(2) \\
3 & 1 & 3 & -3 & -7 = P(3) \\
4 & 1 & 4 & 4 & 18 = P(4) \\
\end{array}
\]

Next we plot the points found in the table and connect them with a smooth curve (Fig. 6). As we draw this curve, we notice that the graph crosses the \( x \) axis three times and changes direction twice. The next two sections will address the question of determining precisely where a graph crosses the \( x \) axis. Precise determination of the location of turning points requires calculus techniques. Lacking this precise information, we simply change direction at \( x = -2 \) and \( x = 2 \).

The leading term of \( P(x) \) is \( x^3 \). From case 2 in Theorem 3 we see that \( P(x) \to -\infty \) as \( x \to -\infty \) and \( P(x) \to \infty \) as \( x \to \infty \).

Matched Problem 4

Graph \( P(x) = x^3 - 4x^2 - 4x + 16, \ -3 \leq x \leq 5 \). Find points by using synthetic division and the remainder theorem. How many \( x \) intercepts does the graph have? How many turning points? Describe the left and right behavior of \( P(x) \).
Remark. A graphing utility can quickly produce a table of values, without using synthetic division, and can graph a polynomial just as quickly. In Section 4-3 we will find that a synthetic division table is a valuable tool when used in conjunction with a graphing utility. Thus, students with graphing utilities should also learn to construct synthetic division tables. (See Table 2 in Section 4-3 for a more efficient way to construct a synthetic division table on a graphing utility.)

Answers to Matched Problems

1. \(3x^2 + 6x + 8 + \frac{2}{3x - 4}\)

2. \(\frac{P(x)}{x - 4} = 3x^3 + x^2 + 4x - 2 + \frac{0}{x - 4} = 3x^3 + x^2 + 4x - 2\)

3. \(P(-2) = -3\) for both parts, as it should.

4. Three x intercepts and two turning points; \(P(x) \to -\infty\) as \(x \to -\infty\); \(P(x) \to +\infty\) as \(x \to +\infty\).

<table>
<thead>
<tr>
<th>x</th>
<th>P(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-35</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>15</td>
</tr>
<tr>
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<td>16</td>
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<tr>
<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-5</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>21</td>
</tr>
</tbody>
</table>

**EXERCISE 4-1**

A

*In Problems 1–4, a is a positive real number. Match each function with one of graphs (a)–(d).*

1. \(f(x) = ax^3\)

2. \(g(x) = -ax^4\)

3. \(h(x) = ax^5\)

4. \(k(x) = -ax^3\)
Problems 5–8 refer to the graphs of functions f, g, h, and k shown below.

5. Which of these functions could be a second-degree polynomial?
6. Which of these functions could be a third-degree polynomial?
7. Which of these functions could be a fourth-degree polynomial?
8. Which of these functions is not a polynomial?

In Problems 9–16, divide, using algebraic long division. Write the quotient, and indicate the remainder.

9. \((a^2 - 4) \div (a + 2)\)
10. \((a^2 + 4) \div (a + 2)\)
11. \((b^2 - 6b - 9) \div (b - 3)\)
12. \((b^3 + b + 9) \div (b - 3)\)
13. \((3x + 2 + x^3 - x^2) \div (x - 1)\)
14. \((2x^2 + 4x - x^3 - 8) \div (x - 2)\)
15. \((1 + 8y^3 - 4y^2 - 2y) \div (2y + 1)\)
16. \((3 + 8y^3 - 6y^2 - 7y) \div (2y - 3)\)

In Problems 17–22, use synthetic division to write the quotient \(P(x) \div (x - r)\) in the form \(P(x)/(x - r) = Q(x) + R/(x - r)\), where \(R\) is a constant.

17. \((x^2 + 4x - 15) \div (x - 3)\)
18. \((x^2 - 2x - 1) \div (x - 4)\)
19. \((3x^2 - x - 7) \div (x + 2)\)
20. \((4x^2 + 18x + 4) \div (x + 5)\)
21. \((2x^3 + 3x^2 - 8x + 1) \div (x + 3)\)
22. \((3x^3 - 4x^2 - 7x + 9) \div (x - 2)\)

B

Use synthetic division and the remainder theorem in Problems 23–28.

23. Find \(P(-5)\), given \(P(x) = 2x^2 + 8x - 6\).
24. Find \(P(2)\), given \(P(x) = 3x^2 - 4x + 2\).
25. Find \(P(-4)\), given \(P(x) = 4x^3 + 12x^2 - 8x + 25\).
26. Find \(P(-3)\), given \(P(x) = 5x^3 + 14x^2 + 3x + 10\).
27. Find \(P(3)\), given \(P(x) = 2x^4 - 5x^3 + 2x^2 - 11x - 14\).
28. Find \(P(-6)\), given \(P(x) = 3x^4 + 18x^3 + x^2 + 4x - 7\).

In Problems 29–44, divide, using synthetic division. Write the quotient, and indicate the remainder. As coefficients get more involved, a calculator should prove helpful. Do not round off—all quantities are exact.

29. \((2x^3 - 5x^2 + 3) \div (x - 1)\)
30. \((3x^4 - 2x - 5) \div (x + 1)\)
31. \((x^4 - 16) \div (x + 4)\)
32. \((x^4 - 32) \div (x - 2)\)
33. \((4x^4 - 9x^3 - 8x^2 - 2x - 7) \div (x - 3)\)
34. \((2x^4 + 6x^3 - 4x^2 - 5x + 7) \div (x + 3)\)
35. \((x^6 + 7x^5 + 10x^4 - x^3 - 5x) \div (x + 5)\)
36. \((x^6 + 6x^5 + 2x^4 + 12x^3 - 3x - 18) \div (x + 6)\)
37. \((2x^4 + 9x^3 + 5x^2 - 4x + 3) \div (x + \frac{3}{2})\)
38. \((2x^4 + 5x^3 + 5x + 8) \div (x + \frac{1}{2})\)
39. \((3x^4 + 5x^3 - 5x^2 + 10x - 1) \div (x - \frac{1}{2})\)
40. \((4x^4 - 11x^3 + 18x^2 - 5x + 4) \div (x - \frac{3}{2})\)
41. \((5x^4 + 4x^3 + 2x - 5) \div (x - 0.2)\)
42. \((3x^4 - 4x^2 + 5x + 8) \div (x + 0.8)\)
43. \((5x^4 + 2x^4 + 4x^3 + 6x^2 - 6) \div (x + 0.6)\)
44. \((10x^5 - 4x^4 + 2x^3 + 4x - 1) \div (x - 0.4)\)

In Problems 45–52, graph each polynomial function using synthetic division and the remainder theorem. Then describe each graph verbally, including the number of \(x\) intercepts, the number of turning points, and the left and right behavior.

*Check your work in Problems 45–52 by graphing on a graphing utility.

45. \(P(x) = x^3 - 5x^2 + 2x + 8, -2 \leq x \leq 5\)
46. \(P(x) = x^3 + 2x^2 - 5x - 6, -4 \leq x \leq 3\)
47. \(P(x) = x^3 + 4x^2 - x - 4, -5 \leq x \leq 2\)
48. \(P(x) = x^3 - 2x^2 - 5x + 6, -3 \leq x \leq 4\)
49. \(P(x) = -x^3 + 2x^2 - 3, -2 \leq x \leq 3\)

*Please note that use of a graphing utility is not required to complete these exercises. Checking them with a g.u. is optional.
50. \( P(x) = -x^3 - x + 4, -2 \leq x \leq 2 \)
51. \( P(x) = -x^3 + 3x^2 - 3x + 2, -1 \leq x \leq 3 \)
52. \( P(x) = -x^3 + x^2 + 4x + 6, -3 \leq x \leq 4 \)

In Problems 53–56, either give an example of a polynomial with real coefficients that satisfies the given conditions or explain why such a polynomial cannot exist.

53. \( P(x) \) is a third-degree polynomial with one \( x \) intercept.
54. \( P(x) \) is a fourth-degree polynomial with no \( x \) intercepts.
55. \( P(x) \) is a third-degree polynomial with no \( x \) intercepts.
56. \( P(x) \) is a fourth-degree polynomial with no turning points.

C

In Problems 57–60, divide, using algebraic long division. Write the quotient, and indicate the remainder:

57. \( (x^4 + x^3 + x^2 - x - 2) \div (x^2 - 1) \)
58. \( (x^4 + 2x^3 - 3x^2 - 8x - 4) \div (x^2 - 4) \)
59. \( (x^4 + x^3 - 7x^2 + 8x + 1) \div (x^2 + 3x - 2) \)
60. \( (x^4 - 7x^3 + 15x^2 - 9x + 1) \div (x^2 - 4x + 1) \)

In Problems 61 and 62, divide, using synthetic division. Do not use a calculator.

61. \( (x^4 + 2x^3 - 2x^2 + 2x - 3) \div (x - i) \)
62. \( (x^4 + 2x^3 - 2x^2 + 2x - 3) \div (x + i) \)

63. Let \( P(x) = x^2 + 2ix - 10 \). Find:
   (A) \( P(2 - i) \)
   (B) \( P(5 - 5i) \)
   (C) \( P(3 - i) \)
   (D) \( P(-3 + i) \)

64. Let \( P(x) = x^2 - 4ix - 13 \). Find:
   (A) \( P(5 + 6i) \)
   (B) \( P(1 + 2i) \)
   (C) \( P(3 + 2i) \)
   (D) \( P(-3 + 2i) \)

In Problems 65–72, graph each polynomial function using synthetic division and the remainder theorem. Then describe each graph verbally, including the number of \( x \) intercepts, the number of turning points, and the left and right behavior.

Check your work in Problems 65–72 by graphing on a graphing utility.

65. \( P(x) = x^4 - 2x^3 - 2x^2 + 8x - 8 \)
66. \( P(x) = x^4 + x^3 - 3x^2 + 7x - 6 \)
67. \( P(x) = x^4 + 4x^3 - x^2 - 10x - 8 \)
68. \( P(x) = x^4 - 8x^2 - 4x + 10 \)
69. \( P(x) = -x^4 + 2x^3 + 10x^2 - 10x - 9 \)
70. \( P(x) = -x^4 - 5x^3 + x^2 + 20x + 5 \)
71. \( P(x) = x^5 - 6x^4 + 4x^3 + 17x^2 - 5x - 7 \)
72. \( P(x) = x^5 - 9x^4 + 4x^3 + 15x - 10 \)

73. (A) Divide \( P(x) = ax^2 + ax + a_0 \) by \( x - r \), using both synthetic division and the long-division process, and compare the coefficients of the quotient and the remainder produced by each method.
   (B) Expand the expression representing the remainder. What do you observe?

74. Repeat Problem 73 for
   \( P(x) = ax^3 + a_2x^2 + a_1x + a_0 \)

75. Polynomials also can be evaluated conveniently using a “nested factoring” scheme. For example, the polynomial \( P(x) = 2x^4 - 3x^3 + 2x^2 - 5x + 7 \) can be written in a nested factored form as follows:

\[
P(x) = 2x^4 - 3x^3 + 2x^2 - 5x + 7 = (2x - 3)x^3 + 2x^2 - 5x + 7 = [(2x - 3)x + 2]x^2 - 5x + 7 = \{(2x - 3)x + 2\}x - 5\}x + 7
\]

Use the nested factored form to find \( P(-2) \) and \( P(1.7) \). [Hint: To evaluate \( P(-2) \), store \(-2\) in your calculator’s memory and proceed from left to right recalling \(-2\) as needed.]

76. Find \( P(-2) \) and \( P(1.3) \) for \( P(x) = 3x^4 + x^3 - 10x^2 + 5x - 2 \) using the nested factoring scheme presented in Problem 75.

SECTION 4-2 Finding Rational Zeros of Polynomials

- Factor Theorem
- Fundamental Theorem of Algebra
- Imaginary Zeros
- Rational Zeros