Introduction

In common usage, the word “induction” means the generalization from particular cases or facts. The ability to formulate general hypotheses from a limited number of facts is a distinguishing characteristic of a creative mathematician. The creative process does not stop here, however. These hypotheses must then be proved or disproved. In mathematics, a special method of proof called mathematical induction ranks among the most important basic tools in a mathematician’s toolbox. In this section mathematical induction will be used to prove a variety of mathematical statements, some new and some that up to now we have just assumed to be true.

We illustrate the process of formulating hypotheses by an example. Suppose we are interested in the sum of the first \( n \) consecutive odd integers, where \( n \) is a positive integer. We begin by writing the sums for the first few values of \( n \) to see if we can observe a pattern:

\[
1 = 1 \quad n = 1 \\
1 + 3 = 4 \quad n = 2 \\
1 + 3 + 5 = 9 \quad n = 3 \\
1 + 3 + 5 + 7 = 16 \quad n = 4 \\
1 + 3 + 5 + 7 + 9 = 25 \quad n = 5 
\]

Is there any pattern to the sums 1, 4, 9, 16, and 25? You no doubt observed that each is a perfect square and, in fact, each is the square of the number of terms in the sum. Thus, the following conjecture seems reasonable:

Conjecture \( P_n \): For each positive integer \( n \),

\[
1 + 3 + 5 + \cdots + (2n - 1) = n^2
\]

That is, the sum of the first \( n \) odd integers is \( n^2 \) for each positive integer \( n \).

So far ordinary induction has been used to generalize the pattern observed in the first few cases listed above. But at this point conjecture \( P_n \) is simply that—a conjecture. How do we prove that \( P_n \) is a true statement? Continuing to list specific cases will never provide a general proof—not in your lifetime or all your descendants’ lifetimes! Mathematical induction is the tool we will use to establish the validity of conjecture \( P_n \).
Before discussing this method of proof, let’s consider another conjecture:

**Conjecture Qₙ:** For each positive integer \( n \), the number \( n^2 - n + 41 \) is a prime number.

It is important to recognize that a conjecture can be proved false if it fails for only one case. A single case or example for which a conjecture fails is called a **counterexample**. We check the conjecture for a few particular cases in Table 1. From the table, it certainly appears that conjecture \( Qₙ \) has a good chance of being true. You may want to check a few more cases. If you persist, you will find that conjecture \( Qₙ \) is true for \( n \) up to 41. What happens at \( n = 41 \)?

\[
41^2 - 41 + 41 = 41^2
\]

which is not prime. Thus, since \( n = 41 \) provides a counterexample, conjecture \( Qₙ \) is false. Here we see the danger of generalizing without proof from a few special cases. This example was discovered by Euler (1707–1783).

### EXPLORE-DISCUSS 1

Prove that the following statement is false by finding a counterexample: If \( n \geq 2 \), then at least one-third of the positive integers less than or equal to \( n \) are prime.

### Mathematical Induction

We begin by stating the **principle of mathematical induction**, which forms the basis for all our work in this section.

#### Theorem 1

**Principle of Mathematical Induction**

Let \( Pᵢ \) be a statement associated with each positive integer \( n \), and suppose the following conditions are satisfied:

1. \( P₁ \) is true.
2. For any positive integer \( k \), if \( Pᵢ \) is true, then \( Pᵢ₊₁ \) is also true.

Then the statement \( Pᵢ \) is true for all positive integers \( n \).

Theorem 1 must be read very carefully. At first glance, it seems to say that if we assume a statement is true, then it is true. But that is not the case at all. If the two conditions in Theorem 1 are satisfied, then we can reason as follows:

- \( P₁ \) is true. \hspace{1cm} \text{Condition 1}
- \( P₂ \) is true, because \( P₁ \) is true. \hspace{1cm} \text{Condition 2}
- \( P₃ \) is true, because \( P₂ \) is true. \hspace{1cm} \text{Condition 2}
- \( P₄ \) is true, because \( P₃ \) is true. \hspace{1cm} \text{Condition 2}
  \[ \vdots \]
  \[ \vdots \]
Since this chain of implications never ends, we will eventually reach \( P_n \) for any positive integer \( n \).

To help visualize this process, picture a row of dominoes that goes on forever (see Fig. 1) and interpret the conditions in Theorem 1 as follows: Condition 1 says that the first domino can be pushed over. Condition 2 says that if the \( k \)th domino falls, then so does the \( (k + 1) \)st domino. Together, these two conditions imply that all the dominoes must fall.

Now, to illustrate the process of proof by mathematical induction, we return to the conjecture \( P_n \) discussed earlier, which we restate below:

\[
P_n: 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad n \text{ any positive integer}
\]

We already know that \( P_1 \) is a true statement. In fact, we demonstrated that \( P_1 \) through \( P_5 \) are all true by direct calculation. Thus, condition 1 in Theorem 1 is satisfied. To show that condition 2 is satisfied, we assume that \( P_k \) is a true statement:

\[
P_k: 1 + 3 + 5 + \cdots + (2k - 1) = k^2
\]

Now we must show that this assumption implies that \( P_{k+1} \) is also a true statement:

\[
P_{k+1}: 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2
\]

Since we have assumed that \( P_k \) is true, we can perform operations on this equation. Note that the left side of \( P_{k+1} \) is the left side of \( P_k \) plus \( (2k + 1) \). So we start by adding \( (2k + 1) \) to both sides of \( P_k \):

\[
1 + 3 + 5 + \cdots + (2k - 1) = k^2 \quad \text{ } \quad P_k
\]

\[
1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) \quad \text{Add } (2k + 1) \text{ to both sides.}
\]

Factoring the right side of this equation, we have

\[
1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2 \quad P_{k+1}
\]

But this last equation is \( P_{k+1} \). Thus, we have started with \( P_k \), the statement we assumed true, and performed valid operations to produce \( P_{k+1} \), the statement we want to be true. In other words, we have shown that if \( P_k \) is true, then \( P_{k+1} \) is also true. Since both conditions in Theorem 1 are satisfied, \( P_n \) is true for all positive integers \( n \).

Now we will consider some additional examples of proof by induction. The first is another summation formula. Mathematical induction is the primary tool for proving that formulas of this type are true.
**Proof**

State the conjecture:

\[ P_n : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} \]

**Part 1**

Show that \( P_1 \) is true.

\[ P_1 : \frac{1}{2} = \frac{2^1 - 1}{2^1} = \frac{1}{2} \]

Thus, \( P_1 \) is true.

**Part 2**

Show that if \( P_k \) is true, then \( P_{k+1} \) is true. It is a good practice to always write out both \( P_k \) and \( P_{k+1} \) at the beginning of any induction proof to see what is assumed and what must be proved:

\[ P_k : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k} \quad \text{We assume} \quad P_k \text{ is true.} \]

\[ P_{k+1} : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}} \quad \text{We must show that} \quad P_{k+1} \text{ follows from } P_k. \]

We start with the true statement \( P_k \), add \( 1/2^{k+1} \) to both sides, and simplify the right side:

\[
\begin{align*}
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} \\
&= \frac{2^k - 1}{2^k} \cdot \frac{2}{2} + \frac{1}{2^{k+1}} \\
&= \frac{2^{k+1} - 2 + 1}{2^{k+1}} \\
&= \frac{2^{k+1} - 1}{2^{k+1}}
\end{align*}
\]

Thus,

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}} \quad P_{k+1}
\]

and we have shown that if \( P_k \) is true, then \( P_{k+1} \) is true.

**Conclusion**

Both conditions in Theorem 1 are satisfied. Thus, \( P_n \) is true for all positive integers \( n \).
Matched Problem 1

Prove that for all positive integers $n$

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

The next example provides a proof of a law of exponents that previously we had to assume was true. First we redefine $a^n$ for $n$ a positive integer, using a recursion formula:

**DEFINITION 1**

*Recursive Definition of $a^n$*

For $n$ a positive integer

$$a^1 = a$$

$$a^{n+1} = a^n a \quad n \geq 1$$

**EXAMPLE 2**

*Proving a Law of Exponents*

Prove that $(xy)^n = x^n y^n$ for all positive integers $n$.

**Proof**

State the conjecture:

$$P_n: \quad (xy)^n = x^n y^n$$

**Part 1**

Show that $P_1$ is true.

$$(xy)^1 = xy \quad \text{Definition 1}$$

$$= x^1 y^1 \quad \text{Definition 1}$$

Thus, $P_1$ is true.

**Part 2**

Show that if $P_k$ is true, then $P_{k+1}$ is true.

$$P_k: \quad (xy)^k = x^k y^k \quad \text{Assume $P_k$ is true.}$$

$$P_{k+1}: \quad (xy)^{k+1} = x^{k+1} y^{k+1} \quad \text{Show that $P_{k+1}$ follows from $P_k$.}$$

Here we start with the left side of $P_{k+1}$ and use $P_k$ to find the right side of $P_{k+1}$:

$$(xy)^{k+1} = (xy)^k (xy)^1 \quad \text{Definition 1}$$

$$= x^k y^k xy \quad \text{Use $P_k$: $(xy)^k = x^k y^k$}$$

$$= (x^k x) (y^k y) \quad \text{Property of real numbers}$$

$$= x^{k+1} y^{k+1} \quad \text{Definition 1}$$
Thus, \((xy)^{k+1} = x^{k+1}y^{k+1}\), and we have shown that if \(P_k\) is true, then \(P_{k+1}\) is true.

**Conclusion**

Both conditions in Theorem 1 are satisfied. Thus, \(P_n\) is true for all positive integers \(n\).

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**Matched Problem 2**

Prove that \((x/y)^n = x^n/y^n\) for all positive integers \(n\).

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Our last example deals with factors of integers. Before we start, recall that an integer \(p\) is *divisible* by an integer \(q\) if \(p = qr\) for some integer \(r\).

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**EXAMPLE 3** **Proving a Divisibility Property**

Prove that \(4^{2n} - 1\) is divisible by 5 for all positive integers \(n\).

**Proof**

Use the definition of divisibility to state the conjecture as follows:

\[ P_n: \quad 4^{2n} - 1 = 5r \quad \text{for some integer } r \]

**Part 1** Show that \(P_1\) is true.

\[ P_1: \quad 4^2 - 1 = 15 = 5 \cdot 3 \]

Thus, \(P_1\) is true.

**Part 2** Show that if \(P_k\) is true, then \(P_{k+1}\) is true.

\[ P_k: \quad 4^{2k} - 1 = 5r \quad \text{for some integer } r \quad \text{Assume } P_k \text{ is true.} \]

\[ P_{k+1}: \quad 4^{2(k+1)} - 1 = 5s \quad \text{for some integer } s \quad \text{Show that } P_{k+1} \text{ must follow.} \]

As before, we start with the true statement \(P_k\):

\[
\begin{align*}
4^{2k} - 1 &= 5r \\
4^2(4^{2k} - 1) &= 4^2(5r) \quad \text{Multiply both sides by } 4^2. \\
4^{2k+2} - 16 &= 80r \quad \text{Simplify.} \\
4^{2(k+1)} - 1 &= 80r + 15 \quad \text{Add } 15 \text{ to both sides.} \\
&= 5(16r + 3) \quad \text{Factor out } 5.
\end{align*}
\]

Thus,

\[ 4^{2(k+1)} - 1 = 5s \quad P_{k+1} \]

where \(s = 16r + 3\) is an integer, and we have shown that if \(P_k\) is true, then \(P_{k+1}\) is true.
Both conditions in Theorem 1 are satisfied. Thus, $P_n$ is true for all positive integers $n$.

**Matched Problem 3**

Prove that $8^n - 1$ is divisible by 7 for all positive integers $n$.

In some cases, a conjecture may be true only for $n \geq m$, where $m$ is a positive integer, rather than for all $n \geq 0$. For example, see Problems 49 and 50 in Exercise 8-2. The principle of mathematical induction can be extended to cover cases like this as follows:

**Theorem 2**

**Extended Principle of Mathematical Induction**

Let $m$ be a positive integer, let $P_n$ be a statement associated with each integer $n \geq m$, and suppose the following conditions are satisfied:

1. $P_m$ is true.
2. For any integer $k \geq m$, if $P_k$ is true, then $P_{k+1}$ is also true.

Then the statement $P_n$ is true for all integers $n \geq m$.

**Three Famous Problems**

The problem of determining whether a certain statement about the positive integers is true may be extremely difficult. Proofs may require remarkable insight and ingenuity and the development of techniques far more advanced than mathematical induction. Consider, for example, the famous problems of proving the following statements:

1. **Lagrange’s Four Square Theorem, 1772**: Each positive integer can be expressed as the sum of four or fewer squares of positive integers.
2. **Fermat’s Last Theorem, 1637**: For $n > 2$, $x^n + y^n = z^n$ does not have solutions in the natural numbers.
3. **Goldbach’s Conjecture, 1742**: Every positive even integer greater than 2 is the sum of two prime numbers.

The first statement was considered by the early Greeks and finally proved in 1772 by Lagrange. Fermat’s last theorem, defying the best mathematical minds for over 350 years, finally succumbed to a 200-page proof by Prof. Andrew Wiles of Princeton University in 1993. To this date no one has been able to prove or disprove Goldbach’s conjecture.

**EXPLORE-DISCUSS 2**

(A) Explain the difference between a theorem and a conjecture.

(B) Why is “Fermat’s last theorem” a misnomer? Suggest more accurate names for the result.
Answers to Matched Problems

1. Sketch of proof. State the conjecture: \( P_n: \quad 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \)

Part 1. \( 1 = \frac{1(1 + 1)}{2} \); \( P_1 \) is true.

Part 2. Show that if \( P_k \) is true, then \( P_{k+1} \) is true.

\[
1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2} \quad P_k
\]

\[
1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)
\]

\[
= \frac{(k + 1)(k + 2)}{2} \quad P_{k+1}
\]

Conclusion: \( P_n \) is true.

2. Sketch of proof. State the conjecture: \( P_n: \quad \left( \frac{x}{y} \right)^n = \frac{x^n}{y^n} \)

Part 1. \( \left( \frac{x}{y} \right)^1 = \frac{x}{y} = \frac{x^1}{y^1} \); \( P_1 \) is true.

Part 2. Show that if \( P_k \) is true, then \( P_{k+1} \) is true.

\[
\left( \frac{x}{y} \right)^{k+1} = \left( \frac{x}{y} \right)^k \cdot \left( \frac{x}{y} \right) = \frac{x^k}{y^k} \cdot \frac{x}{y} = \frac{x^{k+1}}{y^{k+1}}
\]

Conclusion: \( P_n \) is true.

3. Sketch of proof. State the conjecture: \( P_n: \quad 8^n - 1 = 7r \quad \text{for some integer } r \)

Part 1. \( 8^1 - 1 = 7 = 7 \cdot 1 \); \( P_1 \) is true.

Part 2. Show that if \( P_k \) is true, then \( P_{k+1} \) is true.

\[
8^k - 1 = 7r \quad P_k
\]

\[
8(8^k - 1) = 8(7r)
\]

\[
8^{k+1} - 1 = 56r + 7 = 7(8r + 1) = 7s \quad P_{k+1}
\]

Conclusion: \( P_n \) is true.

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**EXERCISE 8-2**

**A**

In Problems 1–4, find the first positive integer \( n \) that causes the statement to fail.

1. \( 3^n + 4^n \geq 5^n \)
2. \( n^2 - 3n < 100 \)
3. \( 17^n - 1 \) is divisible by \( 2^n \)
4. \( n^2 = 5n - 6 \)

Verify each statement \( P_n \) in Problems 5–10 for \( n = 1, 2, \) and \( 3. \)

5. \( P_n: \quad 2 + 6 + 10 + \cdots + (4n - 2) = 2n^2 \)
6. \( P_n: \quad 4 + 8 + 12 + \cdots + 4n = 2n(n + 1) \)
7. \( P_n: \quad a^n a^m = a^{n+m} \)
8. \( P_n: \quad (a^n)^m = a^{nm} \)
9. \( P_n: \quad 9^n - 1 \) is divisible by \( 4 \)
10. \( P_n: \quad 4^n - 1 \) is divisible by \( 3 \)

Write \( P_n \) and \( P_{k+1} \) for \( P_n \) as indicated in Problems 11–16.

11. \( P_n \) in Problem 5
12. \( P_n \) in Problem 6
13. \( P_n \) in Problem 7
14. \( P_n \) in Problem 8
15. \( P_n \) in Problem 9
16. \( P_n \) in Problem 10

In Problems 17–22, use mathematical induction to prove that each \( P_n \) holds for all positive integers \( n. \)

17. \( P_n \) in Problem 5
18. \( P_n \) in Problem 6
19. \( P_n \) in Problem 7
20. \( P_n \) in Problem 8
21. \( P_n \) in Problem 9
22. \( P_n \) in Problem 10
In Problems 23–26, prove the statement is false by finding a counterexample.

23. If \( n > 2 \), then any polynomial of degree \( n \) has at least one real zero.

24. Any positive integer \( n > 7 \) can be written as the sum of three or fewer squares of positive integers.

25. If \( n \) is a positive integer, then there is at least one prime number \( p \) such that \( n < p < n + 6 \).

26. If \( a, b, c, d \) are positive integers such that \( a^2 + b^2 = c^2 + d^2 \), then \( a = c \) or \( a = d \).

In Problems 27–42, use mathematical induction to prove each proposition for all positive integers \( n \), unless restricted otherwise.

27. \( 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2 \)

28. \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \left(\frac{1}{2}\right)^n \)

29. \( 1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}(4n^3 - n) \)

30. \( 1 + 8 + 16 + \cdots + 8(n - 1) = (2n - 1); n > 1 \)

31. \( 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \)

32. \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3} \)

33. \( a^n = a^n; n > 3 \)

34. \( a^n = \frac{1}{a^{-n}}; n > 5 \)

35. \( a^m a^n = a^{m+n}; m, n \in N \) [Hint: Choose \( m \) as an arbitrary element of \( N \), and then use induction on \( n \).]

36. \( (a^n) = a^m; m, n \in N \)

37. \( x^n - 1 \) is divisible by \( x - 1; x \neq 1 \) [Hint: Divisible means that \( x^n - 1 = (x - 1)Q(x) \) for some polynomial \( Q(x) \).]

38. \( x^n - y^n \) is divisible by \( x - y; x \neq y \)

39. \( x^{2n} - 1 \) is divisible by \( x - 1; x \neq 1 \)

40. \( x^{2n} - 1 \) is divisible by \( x + 1; x \neq -1 \)

41. \( 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 \) [Hint: See Matched Problem 1 following Example 1.]

42. \( \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n + 1)(n + 2)} = \frac{n(n + 3)}{4(n + 1)(n + 2)} \)

In Problems 43–46, suggest a formula for each expression, and prove your hypothesis using mathematical induction, \( n \in N \).

43. \( 2 + 4 + 6 + \cdots + 2n \)

44. \( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} \)

45. The number of lines determined by \( n \) points in a plane, no three of which are collinear

46. The number of diagonals in a polygon with \( n \) sides

In Problems 47–50, prove the statement is true for all integers \( n \) as specified.

47. \( a > 1 \Rightarrow a^n > 1; n \in N \)

48. \( 0 < a < 1 \Rightarrow 0 < a^n < 1; n \in N \)

49. \( n^2 > 2n; n \geq 3 \)

50. \( 2^n > n^2; n \geq 5 \)

51. Prove or disprove the generalization of the following two facts:

\[ 3^2 + 4^2 = 5^2 \]
\[ 3^3 + 4^3 + 5^3 = 6^3 \]

52. Prove or disprove: \( n^2 + 21n + 1 \) is a prime number for all natural numbers \( n \).

\[ (a^n) = a^m; m, n \in N \]

If \( \{a_n\} \) and \( \{b_n\} \) are two sequences, we write \( \{a_n\} = \{b_n\} \) if and only if \( a_n = b_n, n \in N \) in Problems 53–56, use mathematical induction to show that \( \{a_n\} = \{b_n\} \).

53. \( a_1 = 1, a_n = a_{n-1} + 2; b_n = 2n - 1 \)

54. \( a_1 = 2, a_n = a_{n-1} + 2; b_n = 2n \)

55. \( a_1 = 2, a_n = 2a_{n-1}; b_n = 2^{n-1} \)

56. \( a_1 = 2, a_n = 3a_{n-1}; b_n = 2 \cdot 3^{n-1} \)

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**SECTION 8-3** Arithmetic and Geometric Sequences

- Arithmetic and Geometric Sequences
- nth-Term Formulas
- Sum Formulas for Finite Arithmetic Series
- Sum Formulas for Finite Geometric Series
- Sum Formula for Infinite Geometric Series