(G) If you have a TI-85 or TI-86 graphing calculator, enter INTERP in your calculator exactly as shown in Table 4. To use the program, enter the x values in L1 and the corresponding y values in L2 (see the output in Table 4) and then execute the program. If you have some other graphing utility that can store and execute programs, consult your manual and modify the statements in INTERP so that the program works on your graphing utility. Use INTERP to check your answers to part F.

CHAPTER 3 REVIEW

3-1 POLYNOMIAL FUNCTIONS AND GRAPHS

In this chapter, unless indicated otherwise, the coefficients of the nth-degree polynomial function \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) are complex numbers and the domain is the set of complex numbers. The number \( r \) is said to be a zero of the function \( P \), or a zero of the polynomial \( P(x) \), or a solution or root of the equation \( P(x) = 0 \), if \( P(r) = 0 \). If the coefficients of \( P(x) \) are real numbers, then the x intercepts of the graph of \( y = P(x) \) are real zeros of \( P \) and \( P(x) \) and real solutions or roots for the equation \( P(x) = 0 \).

Synthetic division is an efficient method for dividing polynomials by linear terms of the form \( x - r \) that is well-suited to calculator use.

Let \( P(x) \) be a polynomial of degree greater than 0 and let \( r \) be a real number. Then we have the following important theorems:

**Division Algorithm.** \( P(x) = (x - r)Q(x) + R \), where \( x - r \) is the divisor; \( Q(x) \), a unique polynomial of degree 1 less than \( P(x) \), is the quotient; and \( R \), a unique real number, is the remainder.

**Remainder Theorem.** \( P(r) = R \).

The left and right behavior of an nth-degree polynomial \( P(x) \) with real coefficients is determined by its highest degree or leading term. As \( x \to \pm \infty \), \( a_n x^n \) and \( P(x) \) both approach \( \pm \infty \), depending on \( n \) and the sign of \( a_n \). The points on a continuous graph where local extrema occur are called turning points. Important graph properties are

1. \( P \) is continuous for all real numbers.
2. The graph of \( P \) is a smooth curve.
3. The graph of \( P \) has at most \( n \) x intercepts.
4. \( P \) has at most \( n - 1 \) turning points.

3-2 FINDING RATIONAL ZEROS OF POLYNOMIALS

If \( P(x) \) is a polynomial of degree \( n > 0 \), then we have the following important theorems:

**Factor Theorem.** The number \( r \) is a zero of \( P(x) \) if and only if \( (x - r) \) is a factor of \( P(x) \).

**Fundamental Theorem of Algebra.** \( P(x) \) has at least one zero.

**n Zeros Theorem.** \( P(x) \) can be expressed as a product of \( n \) linear factors and has \( n \) zeros, not necessarily distinct.

If \( P(x) \) is represented as the product of linear factors and \( x - r \) occurs \( m \) times, then \( r \) is called a zero of multiplicity \( m \).

**Imaginary Zeros Theorem.** If \( P(x) \) has real coefficients, then imaginary zeros of \( P(x) \), if they exist, must occur in conjugate pairs.

**Real Zeros and Odd-Degree Polynomials.** If \( P(x) \) has real coefficients and is of odd degree, then \( P(x) \) always has at least one real zero.
Rational Zero Theorem. If the rational number \( \frac{b}{c} \), in lowest terms, is a zero of the polynomial
\[
P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad a_n \neq 0
\]
with integer coefficients, then \( b \) must be an integer factor of \( a_0 \) and \( c \) must be an integer factor of \( a_n \).

If \( P(x) = (x - r)Q(x) \), then \( Q(x) \) is called a reduced polynomial for \( P(x) \).

3-3 Approximating Real Zeros of Polynomials

The following theorems are important tools for locating the real zeros of a polynomial with real coefficients. Once located, a graphing utility can be used to approximate the zeros.

Location Theorem. If \( f \) is continuous on an interval \( I \), \( a \) and \( b \) are two numbers in \( I \), and \( f(a) \) and \( f(b) \) are of opposite sign, then there is at least one \( x \) intercept between \( a \) and \( b \).

Upper and Lower Bounds of Real Zeros. If \( n > 0 \), \( a_n > 0 \), and \( P(x) \) is divided by \( x - r \) using synthetic division:

1. If \( r > 0 \) and all numbers in the quotient row of the synthetic division, including the remainder, are nonnegative, then \( r \) is greater than or equal to the largest zero of \( P(x) \) and is called an upper bound of the real zeros of \( P(x) \).
2. If \( r < 0 \) and all numbers in the quotient row of the synthetic division, including the remainder, alternate in sign, then \( r \) is less than or equal to the smallest zero of \( P(x) \) and is called a lower bound of the real zeros of \( P(x) \).

Zeros of Even and Odd Multiplicity. If \( P(x) \) is a polynomial with real coefficients, then

1. If \( r \) is a zero of odd multiplicity, then \( P(x) \) changes sign at \( r \) and does not have a local extremum at \( x = r \).
2. If \( r \) is a zero of even multiplicity, then \( P(x) \) does not change sign at \( r \) and has a local extremum at \( x = r \).

3-4 Rational Functions

A function of the form \( f(x) = \frac{n(x)}{d(x)} \), where \( n(x) \) and \( d(x) \) are polynomials is a rational function. The line \( x = a \) is a vertical asymptote for the graph of \( y = f(x) \) if \( f(x) \to \infty \) or \( f(x) \to -\infty \) as \( x \to a^+ \) or \( x \to a^- \). If \( d(a) = 0 \) and \( n(a) \neq 0 \), then the line \( x = a \) is a vertical asymptote. The line \( y = b \) is a horizontal asymptote for the graph of \( y = f(x) \) if \( f(x) \to b \) as \( x \to \infty \) or \( x \to -\infty \). The line \( y = mx + b \) is an oblique asymptote if \( \left\{ f(x) - (mx + b) \right\} \to 0 \) as \( x \to \infty \) or \( x \to -\infty \).

Let \( f(x) = \frac{a_nx^n + \cdots + a_1x + a_0}{b_mx^m + \cdots + b_1x + b_0} \), \( a_n, b_m \neq 0 \).

The behavior of the graph of \( f \) as \( x \to \infty \) or \( x \to -\infty \) is determined by the ratio of the leading terms of the numerator and denominator, \( \frac{a_n}{b_m}x^\circ \).

1. If \( m < n \), then the \( x \) axis is a horizontal asymptote.
2. If \( m = n \), then the line \( y = a_n/b_m \) is a horizontal asymptote.
3. If \( m > n \), then there are no horizontal asymptotes.

Analyzing and Sketching the Graph of a Rational Function: \( f(x) = n(x)/d(x) \)

Step 1. Intercepts. Find the real solutions of the equation \( n(x) = 0 \) and use these solutions to plot any \( x \) intercepts of the graph of \( f \). Evaluate \( f(0) \), if it exists, and plot the \( y \) intercept.

Step 2. Vertical Asymptotes. Find the real solutions of the equation \( d(x) = 0 \) and use these solutions to determine the domain of \( f \), the points of discontinuity, and the vertical asymptotes. Sketch any vertical asymptotes as dashed lines.

Step 3. Horizontal Asymptotes. Determine whether there is a horizontal asymptote and if so, sketch it as a dashed line.

Step 4. Complete the Sketch. Using a graphing utility graph as an aid and the information determined in steps 1–3, sketch the graph.

Chapter 3 Review Exercises

Work through all the problems in this chapter review, and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

A

1. Use synthetic division to divide \( P(x) = 2x^3 + 3x^2 - 1 \) by \( D(x) = x + 2 \), and write the answer in the form \( P(x) = D(x)Q(x) + R \).
2. If \( P(x) = x^3 - 4x^2 + 9x - 8 \), find \( P(3) \) using the remainder theorem and synthetic division.
3. What are the zeros of \( P(x) = 3(x - 2)(x + 4)(x + 1) \)?
4. If \( P(x) = x^2 - 2x + 2 \) and \( P(1 + i) = 0 \), find another zero of \( P(x) \).
5. Let \( P(x) \) be the polynomial whose graph is shown in the figure.

(A) Assuming that \( P(x) \) has integer zeros and leading coefficient 1, find the lowest-degree equation that could produce this graph.

(B) Describe the left and right behavior of \( P(x) \).
6. According to the upper and lower bound theorem, which of the following are upper or lower bounds of the zeros of

\[ P(x) = x^3 - 4x^2 + 2? \]

-2, -1, 3, 4

7. How do you know that \( P(x) = 2x^3 - 3x^2 + x - 5 \) has at least one real zero between 1 and 2?

8. Write the possible rational zeros for

\[ P(x) = x^3 - 4x^2 + x + 6. \]

9. Find all rational zeros for \( P(x) = x^3 - 4x^2 + x + 6. \)

10. Find the domain and \( x \) intercept(s) for:

   (A) \( f(x) = \frac{2x - 3}{x + 4} \)

   (B) \( g(x) = \frac{3x}{x^2 - x - 6} \)

11. Find the horizontal and vertical asymptotes for the functions in Problem 10.

12. Let \( P(x) = x^3 - 3x^2 - 3x + 4. \)

   (A) Graph \( P(x) \) and describe the graph verbally, including the number of \( x \) intercepts, the number of turning points, and the left and right behavior.

   (B) Approximate the largest \( x \) intercept to two decimal places.

13. If \( P(x) = 8x^4 - 14x^3 - 13x^2 - 4x + 7 \), find \( Q(x) \) and \( R \) such that \( P(x) = (x + \frac{1}{4})Q(x) + R \). What is \( P(\frac{1}{4})? \)

14. If \( P(x) = 4x^3 - 8x^2 - 3x - 3 \), find \( P(-\frac{1}{2}) \) using the remainder theorem and synthetic division.

15. Use the quadratic formula and the factor theorem to factor \( P(x) = x^3 - 2x - 1. \)

16. Is \( x + 1 \) a factor of \( P(x) = 9x^{26} - 11x^{17} + 8x^{11} - 5x^4 - 7? \)

   Explain, without dividing or using synthetic division.

17. Determine all rational zeros of \( P(x) = 2x^4 - 3x^2 - 18x - 8. \)

18. Factor the polynomial in Problem 17 into linear factors.

19. Find all rational zeros of \( P(x) = x^3 - 3x^2 + 5. \)

20. Find all zeros (rational, irrational, and imaginary) exactly for \( P(x) = 2x^4 - x^3 + 2x - 1. \)

21. Factor the polynomial in Problem 20 into linear factors.

22. Let \( P(x) = x^5 - 10x^4 + 30x^3 - 20x^2 - 15x - 2. \)

   (A) Approximate the zeros of \( P(x) \) to two decimal places and state the multiplicity of each zero.

   (B) Can any of these zeros be approximated with the bisection method? A maximum routine? A minimum routine? Explain.

23. Let \( P(x) = x^3 - 2x^2 - 30x^2 - 25. \)

   (A) Find the smallest positive and largest negative integers that, by Theorem 2 in Section 3-3, are upper and lower bounds, respectively, for the real zeros of \( P(x) \).

   (B) If \( k, k + 1 \), \( k \) an integer, is the interval containing the largest real zero of \( P(x) \), determine how many additional intervals are required in the bisection method to approximate this zero to one decimal place.

   (C) Approximate the real zeros of \( P(x) \) to two decimal places.

24. Let \( f(x) = \frac{x - 1}{2x + 2}. \)

   (A) Find the domain and the intercepts for \( f \).

   (B) Find the vertical and horizontal asymptotes for \( f \).

   (C) Sketch a graph of \( f \). Draw vertical and horizontal asymptotes with dashed lines.

25. Use synthetic division to divide \( P(x) = x^3 + 3x + 2 \) by \( [x - (1 + i)] \), and write the answer in the form \( P(x) = D(x)Q(x) + R \).

26. Find a polynomial of lowest degree with leading coefficient 1 that has zeros \(-\frac{1}{2}\) (multiplicity 2), \(-3\), and 1 (multiplicity 3). (Leave the answer in factored form.) What is the degree of the polynomial?

27. Repeat Problem 26 for a polynomial \( P(x) \) with zeros \(-5, 2 - 3i, \) and \(2 + 3i.\)

28. Find all zeros (rational, irrational, and imaginary) exactly for \( P(x) = 2x^3 - 5x^4 - 8x^3 + 21x^2 - 4. \)

29. Factor the polynomial in Problem 28 into linear factors.

30. Let \( P(x) = x^4 + 16x^3 + 47x^2 - 137x + 73. \) Approximate (to two decimal places) the \( x \) intercepts and the local extrema.

31. What is the minimal degree of a polynomial \( P(x) \), given that \( P(-1) = -4, P(0) = 2, P(1) = -5, \) and \( P(2) = 3? \) Justify your conclusion.

32. If \( P(x) \) is a cubic polynomial with integer coefficients and if \( 1 + 2i \) is a zero of \( P(x) \), can \( P(x) \) have an irrational zero? Explain.

33. The solutions to the equation \( x^3 - 27 = 0 \) are the cube roots of 27.

   (A) How many cube roots of 27 are there?

   (B) 3 is obviously a cube root of 27; find all others.

34. Let \( P(x) = x^4 + 2x^3 - 500x^2 - 4,000. \)

   (A) Find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that, by Theorem 2 in Section 3-3, are upper and lower bounds, respectively, for the real zeros of \( P(x) \).

   (B) Approximate the real zeros of \( P(x) \) to two decimal places.

35. Graph

\[ f(x) = \frac{x^2 + 2x + 3}{x + 1} \]

   Indicate any vertical, horizontal, or oblique asymptotes with dashed lines.

36. Use a graphing utility to find any horizontal asymptotes for

\[ f(x) = \frac{2x}{\sqrt{x^2 + 3x + 4}} \]

**Applications**

In Problems 37–40, express the solutions as the roots of a polynomial equation of the form \( P(x) = 0. \) Find rational solutions exactly and irrational solutions to three decimal places.
37. **Architecture.** An entryway is formed by placing a rectangular door inside an arch in the shape of the parabola with graph \( y = 16 - x^2 \), \( x \) and \( y \) in feet (see the figure). If the area of the door is 48 square feet, find the dimensions of the door.

38. **Construction.** A grain silo is formed by attaching a hemisphere to the top of a right circular cylinder (see the figure). If the cylinder is 18 feet high and the volume of the silo is \( 486 \pi \) cubic feet, find the common radius of the cylinder and the hemisphere.

39. **Manufacturing.** A box is to be made out of a piece of cardboard that measures 15 by 20 inches. Squares, \( x \) inches on a side, will be cut from each corner, and then the ends and sides will be folded up (see the figure). Find the value of \( x \) that would result in a box with a volume of 300 cubic inches.

40. **Geometry.** Find all points on the graph of \( y = x^2 \) that are 3 units from the point \((1, 4)\).

41. **Advertising.** A chain of appliance stores uses television ads to promote the sale of refrigerators. Analyzing past records produced the data in the table, where \( x \) is the number of ads placed monthly and \( y \) is the number of refrigerators sold that month.

   (A) Find a cubic regression equation for this data using the number of ads as the independent variable.
   (B) Estimate (to the nearest integer) the number of refrigerators that would be sold if 15 ads are placed monthly.
   (C) Estimate (to the nearest integer) the number of ads that should be placed to sell 750 refrigerators monthly.

<table>
<thead>
<tr>
<th>Number of Ads ( x )</th>
<th>Number of Refrigerators ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>270</td>
</tr>
<tr>
<td>20</td>
<td>430</td>
</tr>
<tr>
<td>25</td>
<td>525</td>
</tr>
<tr>
<td>30</td>
<td>630</td>
</tr>
<tr>
<td>45</td>
<td>890</td>
</tr>
<tr>
<td>48</td>
<td>915</td>
</tr>
</tbody>
</table>

42. **Women in the Workforce.** It is reasonable to conjecture from the data given in the table that many Japanese women tend to leave the work force to marry and have children, but then reenter the workforce when the children are grown.

   (A) Explain why you might expect cubic regression to provide a better fit to the data than linear or quadratic regression.
   (B) Find a cubic regression for this data using age as the independent variable.
   (C) Use the regression equation to estimate (to the nearest year) the ages at which 65% of the women are in the workforce.

<table>
<thead>
<tr>
<th>Age</th>
<th>Percentage of Women Employed</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>75</td>
</tr>
<tr>
<td>27</td>
<td>64</td>
</tr>
<tr>
<td>32</td>
<td>52</td>
</tr>
<tr>
<td>37</td>
<td>61</td>
</tr>
<tr>
<td>42</td>
<td>70</td>
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<td>47</td>
<td>72</td>
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<tr>
<td>52</td>
<td>66</td>
</tr>
<tr>
<td>57</td>
<td>57</td>
</tr>
<tr>
<td>62</td>
<td>40</td>
</tr>
</tbody>
</table>
If \( x \) is the radius of the cup and \( h \) the height (Fig. 1), then the volume is

\[
V = \pi x^2 h = 65 \quad \text{Volume}
\]

The amount of aluminum used is the total surface area of the can.

\[
S = 2\pi x h + \pi x^2 \quad \text{Surface area}
\]

Solving the volume equation for \( h \) and substituting in the surface area equation, we have

\[
S = 2\pi x \left( \frac{65}{\pi x^2} \right) + \pi x^2 = \frac{130}{x} + \pi x^2
\]

Graphing this equation and using a built-in minimum routine (Fig. 2), we see that the minimum amount of aluminum is approximately 71 square inches when the radius is approximately 2.75 inches and the height is approximately \([65/(\pi \cdot 2.75^2)] \approx 2.74\) inches.