In this section we take another look at the graphs of linear equations, this time using the function concepts introduced in the preceding section. We also develop procedures for graphing functions defined by quadratic equations and functions formed by “piecing” together two or more other functions. We begin by discussing some general concepts related to the graphs of functions.

Each function that has a real number domain and range has a graph—the graph of the ordered pairs of real numbers that constitute the function. When functions are graphed, domain values usually are associated with the horizontal axis and range values with the vertical axis. Thus, the graph of a function $f$ is the same as the graph of the equation

$$y = f(x)$$

where $x$ is the independent variable and the abscissa of a point on the graph of $f$. The variables $y$ and $f(x)$ are dependent variables, and either is the ordinate of a point on the graph of $f$ (see Fig. 1).

The abscissa of a point where the graph of a function intersects the $x$ axis is called an $x$ intercept or zero of the function. The $x$ intercept is also a real solution or root of the equation $f(x) = 0$. The ordinate of a point where the graph of a function crosses the $y$ axis is called the $y$ intercept of the function. The $y$ intercept is given by $f(0)$, provided 0 is in the domain of $f$. Note that a function can have more than one $x$ intercept but can never have more than one $y$ intercept—a consequence of the vertical line test discussed in the preceding section.

The domain of a function is the set of all the $x$ coordinates of points on the graph of the function, and the range is the set of all the $y$ coordinates. It is instructive to view the domain and range as subsets of the coordinate axes as in Figure 2 on the next page. Note the effective use of interval notation in describing the domain and range of functions.
the functions in this figure. In Figure 2(a) a solid dot is used to indicate that a point is on the graph of the function and in Figure 2(b) an open dot to indicate that a point is not on the graph of the function. An open or solid dot at the end of a graph indicates that the graph terminates there, while an arrowhead indicates that the graph continues beyond the portion shown with no significant changes in shape [see Fig. 2(b)].

**FIGURE 2** Domain and range.

![Graph](image)

**EXAMPLE 1** Finding the Domain and Range from a Graph

Find the domain and range of the function $f$ in Figure 3:

**Solution**

The dots at each end of the graph of $f$ indicate that the graph terminates at these points. Thus, the $x$ coordinates of the points on the graph are between $-3$ and 6. The open dot at $(-3, 4)$ indicates that $-3$ is not in the domain of $f$, while the closed dot at $(6, -3)$ indicates that 6 is in the domain of $f$. That is,

$$\text{Domain: } -3 < x \leq 6 \quad \text{or} \quad (-3, 6)$$

The $y$ coordinates are between $-5$ and 4, and, as before, the open dot at $(-3, 4)$ indicates that 4 is not in the range of $f$ and the closed dot at $(3, -5)$ indicates that $-5$ is in the range of $f$. Thus,

$$\text{Range: } -5 \leq y < 4 \quad \text{or} \quad [-5, 4)$$

**Matched Problem 1**

Find the domain and range of the function $f$ given by the graph in Figure 4.

![Graph](image)
In this section we begin this process by introducing some of the language commonly used to describe the behavior of a graph.

We now take a look at increasing and decreasing properties of functions. Intuitively, a function is increasing over an interval $I$ in its domain if its graph rises as the independent variable increases over $I$. A function is decreasing over $I$ if its graph falls as the independent variable increases over $I$ (Fig. 5).

**DEFINITION 1 Increasing, Decreasing, and Constant Functions**

Let $I$ be an interval in the domain of a function $f$. Then:

1. $f$ is increasing on $I$ if $f(b) > f(a)$ whenever $b > a$ in $I$.
2. $f$ is decreasing on $I$ if $f(b) < f(a)$ whenever $b > a$ in $I$.
3. $f$ is constant on $I$ if $f(a) = f(b)$ for all $a$ and $b$ in $I$.

**Linear Functions** We now apply the general concepts discussed above to a specific class of functions known as linear functions.
DEFINITION 2

**Linear Function**

A function \( f \) is a **linear function** if

\[
f(x) = mx + b \quad m \neq 0
\]

where \( m \) and \( b \) are real numbers.

Graphing a linear function is equivalent to graphing the equation

\[
y = mx + b
\]

which we recognize as the equation of a line with slope \( m \) and \( y \) intercept \( b \). Since the expression \( mx + b \) represents a real number for all real number replacements of \( x \), the domain of a linear function is the set of all real numbers. The restriction \( m \neq 0 \) in the definition of a linear function implies that the graph is not a horizontal line. Hence, the range of a linear function is also the set of all real numbers.

**Graph of \( f(x) = mx + b, m \neq 0 \)**

The graph of a linear function \( f \) is a nonvertical and nonhorizontal straight line with slope \( m \) and \( y \) intercept \( b \).

Notice that two types of lines are not the graphs of linear functions. A vertical line with equation \( x = a \) does not pass the vertical line test and cannot define a function. A horizontal line with equation \( y = b \) does pass the vertical line test and does define a function. However, a function of the form

\[
f(x) = b \quad \text{Constant function}
\]

is called a **constant function**, not a linear function.


**EXPLORE-DISCUSS 1**

(A) Is it possible for a linear function to have two \(x\) intercepts? No \(x\) intercepts? If either of your answers is yes, give an example.

(B) Is it possible for a linear function to have two \(y\) intercepts? No \(y\) intercept? If either of your answers is yes, give an example.

(C) Discuss the possible number of \(x\) and \(y\) intercepts for a constant function.

---

**EXAMPLE 2  Graphing a Linear Function**

Find the slope and intercepts, and then sketch the graph of the linear function defined by

\[
f(x) = -\frac{2}{3}x + 4
\]

Check with a graphing utility.

The \(y\) intercept is \(f(0) = 4\), and the slope is \(-\frac{2}{3}\). To find the \(x\) intercept, we solve the equation \(f(x) = 0\) for \(x\):

\[
\begin{align*}
f(x) &= 0 \\
-\frac{2}{3}x + 4 &= 0 \\
-\frac{2}{3}x &= -4 \\
x &= (-\frac{3}{2})(-4) = 6 \quad \text{\(x\) intercept}
\end{align*}
\]

The graph of \(f\) is shown in Figure 6.

To find the \(y\) intercept with a graphing utility, simply evaluate the function at \(x = 0\) [Fig. 7(a)]. Most graphing utilities have a built-in procedure for approximating \(x\) intercepts, usually called root or zero [Fig. 7(b)].

---

**Matched Problem 2**

Find the slope and intercepts, and then sketch the graph of the linear function defined by

\[
f(x) = \frac{3}{2}x - 6
\]
Just as we used the first-degree polynomial \( mx + b, \ m \neq 0 \), to define a linear function, we use the second-degree polynomial \( ax^2 + bx + c, \ a \neq 0 \), to define a **quadratic function**.

**DEFINITION 3**

**Quadratic Function**

A function \( f \) is a **quadratic function** if

\[
 f(x) = ax^2 + bx + c \quad a \neq 0
\]  

(1)

where \( a, b, \) and \( c \) are real numbers.

The graphs of three quadratic functions are shown in Figure 8. The graph of a quadratic function is called a **parabola**.

**FIGURE 8** Graphs of quadratic functions.

Since the expression \( ax^2 + bx + c \) represents a real number for all real number replacements of \( x \):

**The domain of a quadratic function is the set of all real numbers.**

The range of a quadratic function and many important features of its graph can be determined by first transforming equation (1) by completing the square into the form

\[
 f(x) = a(x - h)^2 + k
\]

(2)

A brief review of completing the square, which we discussed in Section 1-6, might prove helpful at this point. We illustrate this method through an example and then generalize the results.

Consider the quadratic function given by

\[
 f(x) = 2x^2 - 8x + 4
\]

(3)

We start by transforming equation (3) into form (2) by completing the square as follows:
\[ f(x) = 2x^2 - 8x + 4 \]
\[ = 2(x^2 - 4x) + 4 \]
\[ = 2(x^2 - 4x + ?) + 4 \]
\[ = 2(x^2 - 4x + 4) + 4 - 8 \]
\[ = 2(x - 2)^2 - 4 \]

Thus,

\[ f(x) = 2(x - 2)^2 - 4 \quad (4) \]

If \( x = 2 \), then \( 2(x - 2)^2 = 0 \) and \( f(2) = -4 \). For any other value of \( x \), the positive number \( 2(x - 2)^2 \) is added to \(-4\), thus making \( f(x) \) larger. Therefore,

\[ f(2) = -4 \]

is the minimum value of \( f(x) \) for all \( x \)—a very important result! Furthermore, if we choose any two \( x \) values that are equidistant from the vertical line \( x = 2 \), we will obtain the same value for the function. For example, \( x = 1 \) and \( x = 3 \) are each one unit from \( x = 2 \), and the corresponding functional values are

\[ f(1) = 2(-1)^2 - 4 = -2 \]
\[ f(3) = 2(1)^2 - 4 = -2 \]

Thus, the vertical line \( x = 2 \) is a line of symmetry. That is, if the graph is drawn on a piece of paper and the paper is folded along the line \( x = 2 \), then the two sides of the parabola will match exactly. All these results are illustrated by graphing equations (3) or (4) and the line \( x = 2 \) in the same coordinate system. (Fig. 9).

From the above discussion, we see that as \( x \) moves from left to right, \( f(x) \) is decreasing on \( (-\infty, 2] \) and increasing on \([2, \infty) \). Furthermore, \( f(x) \) can assume any values greater than or equal to \(-4\), but no values less than \(-4\). Thus,

\[ \text{Range of } f: y \geq -4 \quad \text{or} \quad [-4, \infty) \]

In general, the graph of a quadratic function is a parabola with line of symmetry parallel to the vertical axis. The lowest or highest point on the parabola, whichever exists, is called the vertex. The maximum or minimum value of a quadratic function always occurs at the vertex of the parabola. The line of symmetry through the vertex is called the axis of the parabola. In the above example, \( x = 2 \) is the axis of the parabola and \( (2, -4) \) is its vertex.

Note the important results we have obtained by transforming equation (3) into equation (4):

The vertex of the parabola

The axis of the parabola
The minimum value of \( f(x) \)
The range of the function \( f \)

Now, let us explore the effect of changing the constants \( a, h, \) and \( k \) on the graph of \( y = a(x - h)^2 + k. \)

**EXPLORE-DISCUSS 2**

Explore the effect of changing the constants \( a, h, \) and \( k \) on the graph of \( f(x) = a(x - h)^2 + k. \)

(A) Let \( a = 1 \) and \( h = 5. \) Graph function \( f \) for \( k = -4, 0, \) and \( 3 \) simultaneously in the same coordinate system. Explain the effect of changing \( k \) on the graph of \( f. \)

(B) Let \( a = 1 \) and \( k = 2. \) Graph function \( f \) for \( h = -4, 0, \) and \( 5 \) simultaneously in the same coordinate system. Explain the effect of changing \( h \) on the graph of \( f. \)

(C) Let \( h = 5 \) and \( k = -2. \) Graph function \( f \) for \( a = 0.25, 1, \) and \( 3 \) simultaneously in the same coordinate system. Graph function \( f \) for \( a = 1, -1, \) and \( -0.25 \) simultaneously in the same coordinate system.

(D) Discuss parts A–C using a graphing utility and a standard viewing window.

The above discussion is generalized for all quadratic functions in the following box:

**Properties of a Quadratic Function and Its Graph**

Given a quadratic function and the form obtained by completing the square

\[
f(x) = ax^2 + bx + c = a(x - h)^2 + k \quad a \neq 0
\]

we summarize general properties as follows:

1. The graph of \( f \) is a parabola:

   - **Axis (of symmetry):** \( x = h \) (Parallel to \( y \) axis)
   - **Vertex:** \((h, k)\) (Parabola rises on one side of the vertex and falls on the other.)
   - **Opens up/down:**
     - \( a > 0 \) (Opens upward)
     - \( a < 0 \) (Opens downward)

2. **Axis**: \( x = h \) (Parallel to \( y \) axis)
4. \( f(h) = k \) is the minimum if \( a > 0 \) and the maximum if \( a < 0 \).

5. Domain: All real numbers
   
   Range: \((-\infty, k]\) if \( a < 0 \) or \([k, \infty)\) if \( a > 0 \)

---

**EXAMPLE 3**  **Graph of a Quadratic Function**

Graph, finding the vertex, axis, maximum or minimum of \( f(x) \), intervals where \( f \) is increasing or decreasing, and range.

\[
\begin{align*}
f(x) &= -0.5x^2 - x + 2
\end{align*}
\]

**Solution**

Complete the square:

\[
\begin{align*}
f(x) &= -0.5x^2 - x + 2 \\
&= -0.5(x^2 + 2x + ?) + 2 \\
&= -0.5(x^2 + 2x + 1) + 2 + 0.5 \\
&= -0.5(x + 1)^2 + 2.5
\end{align*}
\]

From this last form we see that \( h = -1 \) and \( k = 2.5 \). Thus, the vertex is \((-1, 2.5)\), the axis of symmetry is \( x = -1 \), and the maximum value is \( f(-1) = 2.5 \). To graph \( f \), locate the axis and vertex; then plot several points on either side of the axis (Fig. 10).

**FIGURE 10**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>2.5</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
</tr>
</tbody>
</table>

From the graph we see that \( f \) is increasing on \((-\infty, -1]\) and decreasing on \([-1, \infty)\). Also, \( y = f(x) \) can be any number less than or equal to 2.5. Thus, the range of \( f \) is \( y \leq 2.5 \) or \((-\infty, 2.5]\).

---

**Matched Problem 3**

Graph, finding the vertex, axis, maximum or minimum of \( f(x) \), intervals where \( f \) is increasing or decreasing, and range.

\[
\begin{align*}
f(x) &= -x^2 + 4x - 4
\end{align*}
\]
The **absolute value function** can be defined using the definition of absolute value from Section 1-4:

\[
 f(x) = |x| = \begin{cases} 
 -x & \text{if } x < 0 \\
 x & \text{if } x \geq 0
\end{cases}
\]

Notice that this function is defined by different formulas for different parts of its domain. Functions whose definitions involve more than one formula are called **piecewise-defined functions**. As the next example illustrates, piecewise-defined functions occur naturally in many applications.

**EXAMPLE 4 Rental Charges**

A car rental agency charges $0.25 per mile if the total mileage does not exceed 100. If the total mileage exceeds 100, the agency charges $0.25 per mile for the first 100 miles plus $0.15 per mile for the additional mileage. If \( x \) represents the number of miles a rented vehicle is driven, express the mileage charge \( C(x) \) as a function of \( x \). Find \( C(50) \) and \( C(150) \), and graph \( C \).

**Solution**

If \( 0 \leq x \leq 100 \), then

\[
 C(x) = 0.25x
\]

If \( x > 100 \), then

\[
 C(x) = 0.25(100) + 0.15(x - 100) = 25 + 0.15x - 15 = 10 + 0.15x
\]

Thus, we see that \( C \) is a piecewise-defined function:

\[
 C(x) = \begin{cases} 
 0.25x & \text{if } 0 \leq x \leq 100 \\
 10 + 0.15x & \text{if } x > 100
\end{cases}
\]

Piecewise-defined functions are evaluated by first determining which rule applies and then using the appropriate rule to find the value of the function. For example, to evaluate \( C(50) \), we use the first rule and obtain

\[
 C(50) = 0.25(50) = 12.50 \quad x = 50 \text{ satisfies } 0 \leq x \leq 100
\]

To evaluate \( C(150) \), we use the second rule and obtain

\[
 C(150) = 10 + 0.15(150) = 32.50 \quad x = 150 \text{ satisfies } x > 100
\]

To graph \( C \), we graph each rule in the definition for the indicated values of \( x \) (Fig. 11).
Notice that the two formulas produce the same value at \( x = 100 \) and that the graph of \( C \) contains no breaks. Informally, a graph (or portion of a graph) is said to be \textit{continuous} if it contains no breaks or gaps. (A formal presentation of continuity may be found in calculus texts.)

**Matched Problem 4**

Refer to Example 4. Find \( C(x) \) if the agency charges \$0.30 per mile when the total mileage does not exceed 75, and \$0.30 per mile for the first 75 miles plus \$0.20 per mile for additional mileage when the total mileage exceeds 75. Find \( C(50) \) and \( C(100) \), and graph \( C \).

---

**EXAMPLE 5  Graphing a Function Involving Absolute Value**

Graph the function \( f \) given by

\[
f(x) = x + \frac{x}{|x|}
\]

and find its domain and range.

Check with a graphing utility.

**Solution**

We use the piecewise definition of \(|x|\) to find a piecewise definition of \( f \) that does not involve \(|x|\).

If \( x < 0 \), then \(|x| = -x\) and

\[
f(x) = x + \frac{x}{|x|} = x + \frac{x}{-x} = x - 1
\]

If \( x = 0 \), then \( f \) is not defined, since division by 0 is not permissible.

If \( x > 0 \), then \(|x| = x\) and

\[
f(x) = x + \frac{x}{|x|} = x + \frac{x}{x} = x + 1
\]
Thus, a piecewise definition for $f$ is

$$f(x) = \begin{cases} x - 1 & \text{if } x < 0 \\ x + 1 & \text{if } x > 0 \end{cases}$$

Domain: $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$

We use this definition to graph $f$ as shown in Figure 12. Examining this graph, we see that $y = f(x)$ can be any number less than $-1$ or any number greater than $1$. Thus,

$$\text{Range: } y < -1 \quad \text{or} \quad y > 1 \quad \text{or} \quad (-\infty, -1) \cup (1, \infty)$$

Notice that we used open dots in the figure at $(0, -1)$ and $(0, 1)$ to indicate that these points do not belong to the graph of $f$. Because of the break in the graph at $x = 0$, we say that $f$ is **discontinuous** at $x = 0$.

The check is shown in Figure 13. Most graphing utilities denote the absolute value function by $\text{abs}(x)$—check your manual.

---

**Matched Problem 5**

Graph the function $f$ given by

$$f(x) = -\frac{2x}{|x|} - x$$

and find its domain and range.

---

**The Greatest Integer Function**

We conclude this section with a discussion of an interesting and useful function called the **greatest integer function**.

The **greatest integer** of a real number $x$, denoted by $\lfloor x \rfloor$, is the integer $n$ such that $n \leq x < n + 1$; that is, $\lfloor x \rfloor$ is the largest integer less than or equal to $x$. For example,

$$\lfloor 3.45 \rfloor = 3 \quad \lfloor -2.13 \rfloor = -3 \quad \text{Not } -2$$

$$\lfloor 7 \rfloor = 7 \quad \lfloor -8 \rfloor = -8$$

$$\lfloor 0 \rfloor = 0$$

The **greatest integer function** $f$ is defined by the equation $f(x) = \lfloor x \rfloor$. A piecewise definition of $f$ for $-2 \leq x < 3$ is shown at the top of the next page and a sketch of the graph of $f$ for $-5 \leq x \leq 5$ is shown in Figure 14. Since the domain of $f$ is all real numbers, the piecewise definition continues indefinitely in both directions, as does the stairstep pattern in the figure. Thus, the range of $f$ is the set of all integers. The greatest integer function is an example of a more general class of functions called **step functions**.
Notice in Figure 14 that at each integer value of \( x \) there is a break in the graph, and between integer values of \( x \) there is no break. Thus, the greatest integer function is discontinuous at each integer \( n \) and continuous on each interval of the form \([n, n + 1)\).

**EXPLORE-DISCUSS 3**

Most graphing utilities denote the greatest integer function as \( \text{int} (x) \), although not all define it the same way we have here. Graph \( y = \text{int} (x) \) for \( -5 \leq x \leq 5 \) and \( -5 \leq y \leq 5 \) and discuss any differences between your graph and Figure 14. If your graphing utility supports both a connected mode and a dot mode for graphing functions (consult your manual), which mode is preferable for this graph?

**EXAMPLE 6  Computer Science**

Let

\[
f(x) = \left\lfloor \frac{10x + 0.5}{10} \right\rfloor
\]

Find:

(A) \( f(6) \)  (B) \( f(1.8) \)  (C) \( f(3.24) \)  (D) \( f(4.582) \)  (E) \( f(-2.68) \)

What operation does this function perform?

**Solutions**

(A) \( f(6) = \left\lfloor \frac{60.5}{10} \right\rfloor = \frac{60}{10} = 6 \)

(B) \( f(1.8) = \left\lfloor \frac{18.5}{10} \right\rfloor = \frac{18}{10} = 1.8 \)

(C) \( f(3.24) = \left\lfloor \frac{32.9}{10} \right\rfloor = \frac{32}{10} = 3.2 \)

(D) \( f(4.582) = \left\lfloor \frac{46.32}{10} \right\rfloor = \frac{46}{10} = 4.6 \)

(E) \( f(-2.68) = \left\lfloor \frac{-26.3}{10} \right\rfloor = \frac{-27}{10} = -2.7 \)
Comparing the values of $x$ and $f(x)$ in Table 1 in the margin, we conclude that this function rounds decimal fractions to the nearest tenth.

**Matched Problem 6**

Let $f(x) = \lfloor x + 0.5 \rfloor$. Find:

(A) $f(6)$  
(B) $f(1.8)$  
(C) $f(3.24)$  
(D) $f(-4.3)$  
(E) $f(-2.69)$

What operation does this function perform?

**Answers to Matched Problems**

1. Domain: $-4 < x < 5$  
   or $(-4, 5)$  
   Range: $-4 < y \leq 3$  
   or $(-4, 3]$  

2. $y$ intercept: $f(0) = -6$  
   $x$ intercept: 4  
   Slope: $\frac{3}{2}$

3. Axis: $x = 2$  
   Vertex: $(2, f(2)) = (2, 0)$  
   Max $f(x)$: $f(2) = 0$  
   Increasing: $(-\infty, 2]$  
   Decreasing: $[2, \infty)$  
   Range: $(-\infty, f(2)] = (-\infty, 0]$  

4. $C(x) = \begin{cases} 
0.3x & \text{if } 0 \leq x \leq 75 \\
7.5 + 0.2x & \text{if } x > 75 
\end{cases}$
   $C(50) = 15$; $C(100) = 27.50$

5. $f(x) = \begin{cases} 
2 - x & \text{if } x < 0 \\
-2 - x & \text{if } x > 0 
\end{cases}$
   Domain: $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$  
   Range: $(-\infty, -2) \cup (2, \infty)$

6. (A) 6  
   (B) 2  
   (C) 3  
   (D) $-4$  
   (E) $-3$;  
   $f$ rounds decimal fractions to the nearest integer.
Problems 1–6 refer to functions $f$, $g$, $h$, $k$, $p$, and $q$ given by the following graphs. (Assume the graphs continue as indicated beyond the parts shown.)

1. For the function $f$, find:
   - (A) Domain
   - (B) Range
   - (C) $x$ intercepts
   - (D) $y$ intercept
   - (E) Intervals over which $f$ is increasing
   - (F) Intervals over which $f$ is decreasing
   - (G) Intervals over which $f$ is constant
   - (H) Any points of discontinuity

2. Repeat Problem 1 for the function $g$.
3. Repeat Problem 1 for the function $h$.
4. Repeat Problem 1 for the function $k$.
5. Repeat Problem 1 for the function $p$.
6. Repeat Problem 1 for the function $q$.

Problems 7–12 describe the graph of a continuous function $f$ over the interval $[-5, 5]$. Sketch the graph of a function that is consistent with the given information.

7. The function $f$ is increasing on $[-5, -2]$, constant on $[-2, 2]$, and decreasing on $[2, 5]$.
8. The function $f$ is decreasing on $[-5, -2]$, constant on $[-2, 2]$, and increasing on $[2, 5]$.
10. The function $f$ is increasing on $[-5, -2]$, constant on $[-2, 2]$, and increasing on $[2, 5]$.
11. The function $f$ is decreasing on $[-5, -2]$, increasing on $[-2, 2]$, and decreasing on $[2, 5]$.
12. The function $f$ is increasing on $[-5, -2]$, decreasing on $[-2, 2]$, and increasing on $[2, 5]$.

In Problems 13–16, find the slope and intercepts, and then sketch the graph.

13. $f(x) = 2x + 4$
14. $f(x) = 3x - 3$
15. $f(x) = -\frac{x}{2} - \frac{5}{3}$
16. $f(x) = -\frac{x}{3} + \frac{2}{3}$

In Problems 17 and 18, find a linear function $f$ satisfying the given conditions.

17. $f(-2) = 7$ and $f(4) = -2$
18. $f(-3) = -2$ and $f(5) = 4$

In Problems 19–22, graph, finding the axis, vertex, maximum or minimum, and range.

19. $f(x) = (x - 3)^2 + 2$
20. $f(x) = \frac{1}{2}(x + 2)^2 - 4$
21. $f(x) = -(x + 3)^2 - 2$
22. $f(x) = -(x - 2)^2 + 4$

In Problems 23–26, graph, finding the axis, vertex, $x$ intercepts, and $y$ intercept.

23. $f(x) = x^2 - 4x - 5$
24. $f(x) = x^2 - 6x + 5$
25. $f(x) = -x^2 + 6x$
26. $f(x) = -x^2 + 2x + 8$

In Problems 27–30, graph, finding the axis, vertex, intervals over which $f$ is increasing, and intervals over which $f$ is decreasing.

27. $f(x) = x^2 + 6x + 11$
28. $f(x) = x^2 - 8x + 14$
29. $f(x) = -x^2 + 6x - 6$
30. $f(x) = -x^2 - 10x - 24$
In Problems 31–38, graph, finding the domain, range, and any points of discontinuity.

31. \( f(x) = \begin{cases} x + 1 & \text{if } -1 \leq x < 0 \\ -x + 1 & \text{if } 0 \leq x \leq 1 \end{cases} \)

32. \( f(x) = \begin{cases} x & \text{if } -2 \leq x < 1 \\ -x + 2 & \text{if } 1 \leq x \leq 2 \end{cases} \)

33. \( f(x) = \begin{cases} -2 & \text{if } -3 \leq x < -1 \\ 4 & \text{if } -1 < x \leq 1 \end{cases} \)

34. \( f(x) = \begin{cases} 1 & \text{if } -2 \leq x < 2 \\ -3 & \text{if } 2 < x \leq 5 \end{cases} \)

35. \( f(x) = \begin{cases} x + 2 & \text{if } x < -1 \\ x - 2 & \text{if } x \geq -1 \end{cases} \)

36. \( f(x) = \begin{cases} -1 - x & \text{if } x \leq 2 \\ 5 - x & \text{if } x > 2 \end{cases} \)

37. \( g(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ -x^2 - 1 & \text{if } x > 0 \end{cases} \)

38. \( h(x) = \begin{cases} -x^2 - 2 & \text{if } x < 0 \\ x^2 + 2 & \text{if } x > 0 \end{cases} \)

C

In Problems 39–44, graph, finding the axis, vertex, maximum or minimum of \( f(x) \), range, intercepts, intervals over which \( f \) is increasing, and intervals over which \( f \) is decreasing.

39. \( f(x) = \frac{1}{2}x^2 + 2x + 3 \)
40. \( f(x) = 2x^2 - 12x + 14 \)
41. \( f(x) = 4x^2 - 12x + 9 \)
42. \( f(x) = -\frac{1}{2}x^2 + 4x - 10 \)
43. \( f(x) = -2x^2 - 8x - 2 \)
44. \( f(x) = -4x^2 - 4x - 1 \)

In Problems 45–50, find a piecewise definition of \( f \) that does not involve the absolute value function (see Example 5). Sketch the graph, and find the domain, range, and any points of discontinuity.

45. \( f(x) = \frac{|x|}{x} \)
46. \( f(x) = x|x| \)
47. \( f(x) = x + \frac{|x - 1|}{x - 1} \)
48. \( f(x) = x + 2\frac{|x + 1|}{x + 1} \)
49. \( f(x) = |x| + |x - 2| \)
50. \( f(x) = |x| - |x - 3| \)

In Problems 51–56, write a piecewise definition for \( f \) (see the discussion of Fig. 14 in this section) and sketch the graph of \( f \). Include sufficient intervals to clearly illustrate both the definition and the graph. Find the domain, range, and any points of discontinuity.

51. \( f(x) = \lfloor x/2 \rfloor \)
52. \( f(x) = \lfloor x/3 \rfloor \)
53. \( f(x) = \lfloor 3x \rfloor \)
54. \( f(x) = \lfloor 2x \rfloor \)
55. \( f(x) = x - \lfloor x \rfloor \)
56. \( f(x) = \lfloor x \rfloor - x \)

57. Given that \( f \) is a quadratic function with min \( f(x) = f(2) = 4 \), find the axis, vertex, range, and \( x \) intercepts.

58. Given that \( f \) is a quadratic function with max \( f(x) = f(-3) = -5 \), find the axis, vertex, range, and \( x \) intercepts.

59. The function \( f \) is continuous and increasing on the interval \([1, 9]\) with \( f(1) = -5 \) and \( f(9) = 4 \).
   (A) Sketch a graph of \( f \) that is consistent with the given information.
   (B) How many times does your graph cross the \( x \) axis? Could the graph cross more times? Fewer times? Support your conclusions with additional sketches and/or verbal arguments.

60. Repeat Problem 59 if the function does not have to be continuous.

61. The function \( f \) is continuous on the interval \([-5, 5]\) with \( f(-5) = -4 \), \( f(1) = 3 \), and \( f(5) = -2 \).
   (A) Sketch a graph of \( f \) that is consistent with the given information.
   (B) How many times does your graph cross the \( x \) axis? Could the graph cross more times? Fewer times? Support your conclusions with additional sketches and/or verbal arguments.

62. Repeat Problem 61 if \( f \) is continuous on \([-8, 8]\) with \( f(-8) = -6 \), \( f(-4) = 3 \), \( f(3) = -2 \), and \( f(8) = 5 \).

Problems 63–66 are calculus-related. In geometry, a line that intersects a circle in two distinct points is called a secant line, as shown in figure (a). In calculus, the line through the points \((x_1, f(x_1))\) and \((x_2, f(x_2))\) is called a secant line for the graph of the function \( f \), as shown in figure (b).

Problems 63 and 64, find the equation of the secant line through the indicated points on the graph of \( f \). Graph \( f \) and the secant line on the same coordinate system.

63. \( f(x) = x^2 - 4; (-1, -3), (3, 5) \)
64. \( f(x) = 9 - x^2; (-2, 5), (4, -7) \)

65. Let \( f(x) = x^3 - 3x + 5 \). If \( h \) is a nonzero real number, then \((2, f(2))\) and \((2 + h, f(2 + h))\) are two distinct points on the graph of \( f \).
   (A) Find the slope of the secant line through these two points.
   (B) Evaluate the slope of the secant line for \( h = 1, h = 0.1, h = 0.01, \) and \( h = 0.001 \). What value does the slope seem to be approaching?

66. Repeat Problem 65 for \( f(x) = x^3 + 2x - 6 \).

Problems 67–74 require the use of a graphing utility.

In problems 67–72, first graph functions \( f \) and \( g \) in the same viewing window, then graph \( m(x) \) and \( n(x) \) in their own viewing windows:

\[
m(x) = 0.5[f(x) + g(x) + |f(x) - g(x)|]
\]

\[
n(x) = 0.5[f(x) + g(x) - |f(x) - g(x)|]
\]

67. \( f(x) = -2x, g(x) = 0.5x \)

68. \( f(x) = 3x + 1, g(x) = -0.5x - 4 \)

69. \( f(x) = 5 - 0.2x^2, g(x) = 0.3x^2 - 4 \)

70. \( f(x) = 0.15x^2 - 5, g(x) = 5 - 1.5|x| \)

71. \( f(x) = 0.2x^2 - 0.4x - 5, g(x) = 0.3x - 3 \)

72. \( f(x) = 8 + 1.5x - 0.4x^2, g(x) = -0.2x + 5 \)

73. How would you characterize the relationship between \( f, g, \) and \( m \) in Problems 67–72? [Hint: See Problem 89 in Exercise 1-4.]

74. How would you characterize the relationship between \( f, g, \) and \( n \) in Problems 67–72? [Hint: See Problem 90 in Exercise 1-4.]

APPLICATIONS

75. Tire Mileage. An automobile tire manufacturer collected the data in Table 1 relating tire pressure \( x \), in pounds per square inch (lb/in.\(^2\)) and mileage, in thousands of miles.

<table>
<thead>
<tr>
<th>TABLE 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
</tr>
<tr>
<td>Mileage</td>
</tr>
</tbody>
</table>

A mathematical model for this data is given by

\( f(x) = -0.518x^2 + 33.3x - 481 \)

(A) Complete Table 2. Round values of \( f(x) \) to one decimal place.

<table>
<thead>
<tr>
<th>TABLE 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
</tr>
<tr>
<td>Mileage</td>
</tr>
<tr>
<td>( f(x) )</td>
</tr>
</tbody>
</table>

(B) Sketch the graph of \( f \) and the mileage data on the same axes.

(C) Use values of the modeling function \( f \) rounded to two decimal places to estimate the mileage for a tire pressure of 31 lb/in\(^2\). For 35 lb/in\(^2\).

(D) Write a brief description of the relationship between tire pressure and mileage.

76. Automobile Production. Table 3 lists General Motors’ total U.S. vehicle production in millions of units from 1989 to 1993.

<table>
<thead>
<tr>
<th>TABLE 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
</tr>
<tr>
<td>Production</td>
</tr>
</tbody>
</table>

A mathematical model for GM’s production data is given by

\( f(x) = 0.33x^2 - 1.3x + 4.8 \)

where \( x = 0 \) corresponds to 1989.

(A) Complete Table 4. Round values of \( x \) to one decimal place.

<table>
<thead>
<tr>
<th>TABLE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
</tr>
<tr>
<td>Production</td>
</tr>
<tr>
<td>( f(x) )</td>
</tr>
</tbody>
</table>

(B) Sketch the graph of \( f \) and the production data on the same axes.

(C) Use values of the modeling function \( f \) rounded to two decimal places to estimate the production in 1994. In 1995.

(D) Write a brief verbal description of GM’s production from 1989 to 1993.
77. **Physics—Spring Stretch.** Hooke’s law states that the relationship between the stretch $s$ of a spring and the weight $w$ causing the stretch is linear (a principle upon which all spring scales are constructed). A 10-pound weight stretches a spring 1 inch, while with no weight the stretch of the spring is 0.

(A) Find a linear function $f: s = f(w) = mw + b$ that represents this relationship. [Hint: Both points (10, 1) and (0, 0) are on the graph of $f$.]

(B) Find $f(15)$ and $f(30)$—that is, the stretch of the spring for 15-pound and 30-pound weights.

(C) What is the slope of the graph of $f$? (The slope indicates the increase in stretch for each pound increase in weight.)

(D) Graph $f$ for $0 \leq w \leq 40$.

78. **Business—Depreciation.** An electronic computer was purchased by a company for $20,000 and is assumed to have a salvage value of $2,000 after 10 years. Its value is depreciated linearly from $20,000 to $2,000.

(A) Find the linear function $f: V = f(t)$ that relates value $V$ in dollars to time $t$ in years.

(B) Find $f(4)$ and $f(8)$, the values of the computer after 4 and 8 years, respectively.

(C) Find the slope of the graph of $f$. (The slope indicates the decrease in value per year.)

(D) Graph $f$ for $0 \leq t \leq 10$.

79. **Sales Commissions.** An appliance salesperson receives a base salary of $200 a week and a commission of 4% on all sales over $3,000 during the week. In addition, if the weekly sales are $8,000 or more, the salesperson receives a $100 bonus. If $x$ represents weekly sales in dollars, express the weekly earnings $E(x)$ as a function of $x$, and sketch its graph. Identify any points of discontinuity. Find $E(5,750)$ and $E(9,200)$.

80. **Service Charges.** On weekends and holidays, an emergency plumbing repair service charges $2.00 per minute for the first 30 minutes of a service call and $1.00 per minute for each additional minute. If $x$ represents the duration of a service call in minutes, express the total service charge $S(x)$ as a function of $x$, and sketch its graph. Identify any points of discontinuity. Find $S(25)$ and $S(45)$.

81. **Construction.** A rectangular dog pen is to be made with 100 feet of fence wire.

(A) If $x$ represents the width of the pen, express its area $A(x)$ in terms of $x$.

(B) Considering the physical limitations, what is the domain of the function $A$?

(C) Graph the function for this domain.

(D) Determine the dimensions of the rectangle that will make the area maximum.

82. **Construction.** Rework Problem 81 with the added assumption that an existing property fence will be used for one side of the pen. (Let $x = $ Width; see the figure.)

83. **Computer Science.** Let $f(x) = 10[0.5 + x/10]$. Evaluate $f$ at $4, -4, 6, -6, 24, 25, 247, -243, -245, and -246$. What operation does this function perform?

84. **Computer Science.** Let $f(x) = 100[0.5 + x/100]$. Evaluate $f$ at $40, -40, 60, -60, 740, 750, 7551, -601, -649, and -651$. What operation does this function perform?

85. **Computer Science.** Use the greatest integer function to define a function that rounds real numbers to the nearest hundredth.

86. **Computer Science.** Use the greatest integer function to define a function that rounds real numbers to the nearest thousandth.

87. **Delivery Charges.** A nationwide package delivery service charges $15 for overnight delivery of packages weighing 1 pound or less. Each additional pound (or fraction thereof) costs an additional $3. Let $C(x)$ be the charge for overnight delivery of a package weighing $x$ pounds.

(A) Write a piecewise definition of $C$ for $0 < x \leq 6$, and sketch the graph of $C$ by hand.

(B) Can the function $f$ defined by $f(x) = 15 + 3[x]$ be used to compute the delivery charges for all $x$, $0 < x \leq 6$? Justify your answer.

88. **Telephone Charges.** Calls to 900 numbers are charged to the caller. A 900 number hot line for tips and hints for video games charges $4 for the first minute of the call and $2 for each additional minute (or fraction thereof). Let $C(x)$ be the charge for a call lasting $x$ minutes.

(A) Write a piecewise definition of $C$ for $0 < x \leq 6$, and sketch the graph of $C$ by hand.

(B) Can the function $f$ defined by $f(x) = 4 + 2[x]$ be used to compute the charges for all $x$, $0 < x \leq 6$? Justify your answer.

89. **Car Rental.** A car rental agency rents 300 cars a day at a rate of $40 per day. For each $1 increase in rate, five fewer cars are rented. At what rate should the cars be rented to produce the maximum income? What is the maximum income?

90. **Rental Income.** A 400-room hotel in Las Vegas is filled to capacity every night at $70 a room. For each $1 increase in rent, four fewer rooms are rented. If each rented room costs $10 to service per day, how much should the management charge for each room to maximize profit? What is the maximum profit?
**91. Physics.** A stunt driver is planning to jump a motorcycle from one ramp to another as illustrated in the figure. The ramps are 10 feet high, and the distance between the ramps is 80 feet. The trajectory of the cycle through the air is given by the graph of

\[ f(x) = \frac{1}{4}x - \left( \frac{16}{v^2} \right)x^2 \]

where \( v \) is the velocity of the cycle in feet per second as it leaves the ramp.

(A) How fast must the cycle be traveling when it leaves the ramp in order to follow the trajectory illustrated in the figure?  
(B) What is the maximum height of the cycle above the ground as it follows this trajectory?

**92. Physics.** The trajectory of a circus performer shot from a cannon is given by the graph of the function

\[ f(x) = x - \frac{1}{100}x^2 \]

Both the cannon and the net are 10 feet high (see the figure).

(A) How far from the muzzle of the cannon should the center of the net be placed so that the performer lands in the center of the net?  
(B) What is the maximum height of the performer above the ground?

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**SECTION 2-5 Combining Functions**

- Operations on Functions
- Composition
- Elementary Functions
- Vertical and Horizontal Shifts
- Reflections, Expansions, and Contractions

If two functions \( f \) and \( g \) are both defined at a real number \( x \), and if \( f(x) \) and \( g(x) \) are both real numbers, then it is possible to perform real number operations such as addition, subtraction, multiplication, or division with \( f(x) \) and \( g(x) \). Furthermore, if \( g(x) \) is a number in the domain of \( f \), then it is also possible to evaluate \( f \) at \( g(x) \). In this section we see how operations on the values of functions can be used to define operations on the functions themselves. We also investigate the graphic implications of some of these operations.

**Operations on Functions**

The functions \( f \) and \( g \) given by

\[ f(x) = 2x + 3 \quad \text{and} \quad g(x) = x^2 - 4 \]

are defined for all real numbers. Thus, for any real \( x \) we can perform the following operations:

\[
\begin{align*}
    f(x) + g(x) &= 2x + 3 + x^2 - 4 = x^2 + 2x - 1 \\
    f(x) - g(x) &= 2x + 3 - (x^2 - 4) = -x^2 + 2x + 7 \\
    f(x)g(x) &= (2x + 3)(x^2 - 4) = 2x^3 + 3x^2 - 8x - 12
\end{align*}
\]