of packages weighing 1 pound or less and a surcharge for each additional pound (or fraction thereof). A customer is billed $27.75 for shipping a 5-pound package and $64.50 for shipping a 20-pound package. Find the base price and the surcharge for each additional pound.

70. Delivery Charges. Refer to Problem 69. Federated Shipping, a competing overnight delivery service, informs the customer in Problem 69 that it would ship the 5-pound package for $29.95 and the 20-pound package for $59.20.
   (A) If Federated Shipping computes its cost in the same manner as United Express, find the base price and the surcharge for Federated Shipping.
   (B) Devise a simple rule that the customer can use to choose the cheaper of the two services for each package shipped. Justify your answer.

71. Resource Allocation. A coffee manufacturer uses Colombian and Brazilian coffee beans to produce two blends, robust and mild. A pound of the robust blend requires 12 ounces of Colombian beans and 4 ounces of Brazilian beans. A pound of the mild blend requires 6 ounces of Colombian beans and 10 ounces of Brazilian beans. Coffee is shipped in 132-pound burlap bags. The company has 50 bags of Colombian beans and 40 bags of Brazilian beans on hand. How many pounds of each blend should it produce in order to use all the available beans?

   (A) If the company decides to discontinue production of the robust blend and only produce the mild blend, how many pounds of the mild blend can it produce and how many beans of each type will it use? Are there any beans that are not used?
   (B) Repeat part A if the company decides to discontinue production of the mild blend and only produce the robust blend.

SECTION 8-2 Gauss–Jordan Elimination

- Reduced Matrices
- Solving Systems by Gauss–Jordan Elimination
- Application

Now that you have had some experience with row operations on simple augmented matrices, we will consider systems involving more than two variables. In addition, we will not require that a system have the same number of equations as variables. It turns out that the results for two-variable–two-equation linear systems, stated in Theorem 1 in Section 8-1, actually hold for linear systems of any size.

Possible Solutions to a Linear System

It can be shown that any linear system must have exactly one solution, no solution, or an infinite number of solutions, regardless of the number of equations or the number of variables in the system. The terms unique, consistent, inconsistent, dependent, and independent are used to describe these solutions, just as they are for systems with two variables.

• Reduced Matrices

In the last section we used row operations to transform the augmented coefficient matrix for a system of two equations in two variables

\[
\begin{bmatrix}
  a_{11} & a_{12} & | & k_1 \\
  a_{21} & a_{22} & | & k_2
\end{bmatrix}
\]

\[a_{11}x_1 + a_{12}x_2 = k_1\]

\[a_{21}x_1 + a_{22}x_2 = k_2\]
into one of the following simplified forms:

\[
\begin{align*}
\text{Form 1} & : \begin{bmatrix} 1 & 0 & m \\ 0 & 1 & n \end{bmatrix} \\
\text{Form 2} & : \begin{bmatrix} 1 & m & n \\ 0 & 0 & 0 \end{bmatrix} \\
\text{Form 3} & : \begin{bmatrix} 1 & m & n \\ 0 & 0 & p \end{bmatrix}
\end{align*}
\]  

(1)

where \( m, n, \) and \( p \) are real numbers, \( p \neq 0 \). Each of these reduced forms represents a system that has a different type of solution set, and no two of these forms are row-equivalent. Thus, we consider each of these to be a different simplified form. Now we want to consider larger systems with more variables and more equations.

**EXPLORE-DISCUSS 1** Forms 1, 2, and 3 above represent systems that have, respectively, a unique solution, an infinite number of solutions, and no solution. Discuss the number of solutions for the systems of three equations in three variables represented by the following augmented coefficient matrices.

\[
\begin{align*}
(A) & : \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & (B) & : \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & (C) & : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

Since there is no upper limit on the number of variables or the number of equations in a linear system, it is not feasible to explicitly list all possible “simplified forms” for larger systems, as we did for systems of two equations in two variables. Instead, we state a general definition of a simplified form called a reduced matrix that can be applied to all matrices and systems, regardless of size.

**DEFINITION 1** Reduced Matrix

A matrix is in **reduced form** if:

1. Each row consisting entirely of 0’s is below any row having at least one nonzero element.
2. The leftmost nonzero element in each row is 1.
3. The column containing the leftmost 1 of a given row has 0’s above and below the 1.
4. The leftmost 1 in any row is to the right of the leftmost 1 in the preceding row.

**EXAMPLE 1** Reduced Forms

The matrices below are not in reduced form. Indicate which condition in the definition is violated for each matrix. State the row operation(s) required to transform the matrix to reduced form, and find the reduced form.
Solutions

(A) Condition 4 is violated: The leftmost 1 in row 2 is not to the right of the leftmost 1 in row 1. Perform the row operation $R_1 \leftrightarrow R_2$ to obtain the reduced form:

\[
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & -2
\end{bmatrix}
\]

(B) Condition 3 is violated: The column containing the leftmost 1 in row 2 does not have a zero above the 1. Perform the row operation $2R_2 + R_1 \to R_1$ to obtain the reduced form:

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(C) Condition 1 is violated: The second row contains all zeros, and it is not below any row having at least one nonzero element. Perform the row operation $R_2 \leftrightarrow R_3$ to obtain the reduced form:

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]

(D) Condition 2 is violated: The leftmost nonzero element in row 2 is not a 1. Perform the row operation $\frac{1}{2}R_2 \to R_2$ to obtain the reduced form:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{1}{2} \\
0 & 0 & 1
\end{bmatrix}
\]

Matched Problem 1

The matrices below are not in reduced form. Indicate which condition in the definition is violated for each matrix. State the row operation(s) required to transform the matrix to reduced form and find the reduced form.

(A) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -6 \end{bmatrix}$  
(B) $\begin{bmatrix} 1 & 5 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$  
(D) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$
We are now ready to outline the Gauss–Jordan elimination method for solving systems of linear equations. The method systematically transforms an augmented matrix into a reduced form. The system corresponding to a reduced augmented coefficient matrix is called a **reduced system**. As we will see, reduced systems are easy to solve.

The Gauss–Jordan elimination method is named after the German mathematician Carl Friedrich Gauss (1777–1855) and the German geodesist Wilhelm Jordan (1842–1899). Gauss, one of the greatest mathematicians of all time, used a method of solving systems of equations that was later generalized by Jordan to solve problems in large-scale surveying.

### EXAMPLE 2 Solving a System Using Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination:

\[
\begin{align*}
2x_1 - 2x_2 + x_3 &= 3 \\
3x_1 + x_2 - x_3 &= 7 \\
x_1 - 3x_2 + 2x_3 &= 0
\end{align*}
\]

**Solution** Write the augmented matrix and follow the steps indicated at the right to produce a reduced form.

\[
\begin{bmatrix}
2 & -2 & 1 & | & 3 \\
3 & 1 & -1 & | & 7 \\
1 & -3 & 2 & | & 0
\end{bmatrix}
\]

**Step 1:** Choose the leftmost nonzero column and get a 1 at the top.

\[
\begin{bmatrix}
1 & -3 & 2 & | & 0 \\
3 & 1 & -1 & | & 7 \\
2 & -2 & 1 & | & 3
\end{bmatrix}
\]

**Step 2:** Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

\[
\begin{bmatrix}
1 & -3 & 2 & | & 0 \\
0 & 10 & -7 & | & 7 \\
0 & 4 & -3 & | & 3
\end{bmatrix}
\]

**Step 3:** Repeat step 1 with the submatrix formed by (mentally) deleting the top row.

\[
\begin{bmatrix}
1 & 0 & -0.1 & | & 2.1 \\
0 & 1 & -0.7 & | & 0.7 \\
0 & 0 & -0.2 & | & 0.2
\end{bmatrix}
\]

**Step 4:** Repeat step 2 with the entire matrix.

\[
\begin{bmatrix}
1 & 0 & -0.1 & | & 2.1 \\
0 & 1 & -0.7 & | & 0.7 \\
0 & 0 & 1 & | & -1
\end{bmatrix}
\]
The solution to this system is \(x_1 = 2, x_2 = 0, x_3 = -1\). You should check this solution in the original system.

**Remarks**

1. Even though each matrix has a unique reduced form, the sequence of steps (algorithm) presented here for transforming a matrix into a reduced form is not unique. That is, other sequences of steps (using row operations) can produce a reduced matrix. (For example, it is possible to use row operations in such a way that computations involving fractions are minimized.) But we emphasize again that we are not interested in the most efficient hand methods for transforming small matrices into reduced forms. Our main interest is in giving you a little experience with a method that is suitable for solving large-scale systems on a computer or graphing utility.

2. Most graphing utilities have the ability to find reduced forms, either directly or with some programming. Figure 1 illustrates the solution of Example 2 on a graphing calculator that has a built-in routine for finding reduced forms. Notice that in row 2 and column 4 of the reduced form the graphing calculator has displayed the very small number \(-3.5E-13\) instead of the exact value 0. This is a common occurrence on a graphing calculator and causes no problems. Just replace any very small numbers displayed in scientific notation with 0.
Matched Problem 2  Solve by Gauss–Jordan elimination:  
\[
\begin{align*}
3x_1 + x_2 - 2x_3 &= 2 \\
x_1 - 2x_2 + x_3 &= 3 \\
2x_1 - x_2 - 3x_3 &= 3
\end{align*}
\]

Example 3  Solving a System Using Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination:  
\[
\begin{align*}
2x_1 - 4x_2 + x_3 &= -4 \\
4x_1 - 8x_2 + 7x_3 &= 2 \\
-2x_1 + 4x_2 - 3x_3 &= 5
\end{align*}
\]

Solution
\[
\begin{bmatrix}
2 & -4 & 1 & -4 \\
4 & -8 & 7 & 2 \\
-2 & 4 & -3 & 5
\end{bmatrix}
\]

\[0.5R_1 \rightarrow R_1\]

\[
\begin{bmatrix}
1 & -2 & 0.5 & -2 \\
4 & -8 & 7 & 2 \\
-2 & 4 & -3 & 5
\end{bmatrix}
\]

\[(-4)R_1 + R_2 \rightarrow R_2, 2R_3 + R_3 \rightarrow R_3\]

\[
\begin{bmatrix}
1 & -2 & 0.5 & -2 \\
0 & 0 & 5 & 10 \\
0 & 0 & -2 & 1
\end{bmatrix}
\]

\[0.2R_2 \rightarrow R_2\]  Note that column 3 is the leftmost nonzero column in this submatrix.

\[
\begin{bmatrix}
1 & -2 & 0.5 & -2 \\
0 & 0 & 1 & 2 \\
0 & 0 & -2 & 1
\end{bmatrix}
\]

\[(-0.5)R_2 + R_1 \rightarrow R_1, 2R_2 + R_3 \rightarrow R_3\]

\[
\begin{bmatrix}
1 & -2 & 0 & -3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 5
\end{bmatrix}
\]

We stop the Gauss–Jordan elimination, even though the matrix is not in reduced form, since the last row produces a contradiction.

The system is inconsistent and has no solution.

Matched Problem 3  Solve by Gauss–Jordan elimination:  
\[
\begin{align*}
2x_1 - 4x_2 - x_3 &= -8 \\
4x_1 - 8x_2 + 3x_3 &= 4 \\
-2x_1 + 4x_2 + x_3 &= 11
\end{align*}
\]
EXAMPLE 4 Solving a System Using Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination:

\[
\begin{align*}
3x_1 + 6x_2 - 9x_3 &= 15 \\
2x_1 + 4x_2 - 6x_3 &= 10 \\
-2x_1 - 3x_2 + 4x_3 &= -6
\end{align*}
\]

Solution

\[
\begin{bmatrix}
3 & 6 & -9 & | & 15 \\
2 & 4 & -6 & | & 10 \\
-2 & -3 & 4 & | & -6
\end{bmatrix}
\]

\[
\begin{align*}
&\frac{3}{5}R_1 \rightarrow R_1 \\
&\begin{bmatrix}
1 & 2 & -3 & | & 5 \\
2 & 4 & -6 & | & 10 \\
-2 & -3 & 4 & | & -6
\end{bmatrix} \\
&(-2)R_1 + R_2 \rightarrow R_2 \\
&2R_1 + R_3 \rightarrow R_3 \\
&\begin{bmatrix}
1 & 2 & -3 & | & 5 \\
0 & 0 & 0 & | & 0 \\
0 & 1 & -2 & | & 4
\end{bmatrix} \\
&R_2 \leftrightarrow R_3 \text{ Note that we must interchange rows 2 and 3 to obtain a nonzero entry at the top of the second column of this submatrix.} \\
&\begin{bmatrix}
1 & 2 & -3 & | & 5 \\
0 & 1 & -2 & | & 4 \\
0 & 0 & 0 & | & 0
\end{bmatrix} \\
&(-2)R_2 + R_1 \rightarrow R_1 \\
&\begin{bmatrix}
1 & 0 & 1 & | & -3 \\
0 & 1 & -2 & | & 4 \\
0 & 0 & 0 & | & 0
\end{bmatrix} \text{ This matrix is now in reduced form. Write the corresponding reduced system and solve.}
\end{align*}
\]

\[
\begin{align*}
x_1 + x_3 &= -3 \\
x_2 - 2x_3 &= 4
\end{align*}
\]

We discard the equation corresponding to the third (all 0) row in the reduced form, since it is satisfied by all values of \(x_1, x_2,\) and \(x_3\).

Note that the leftmost variable in each equation appears in one and only one equa-
We solve for the leftmost variables $x_1$ and $x_2$ in terms of the remaining variable $x_3$:

\begin{align*}
x_1 &= -x_3 - 3 \\
x_2 &= 2x_3 + 4
\end{align*}

This dependent system has an infinite number of solutions. We will use a parameter to represent all the solutions. If we let $x_3 = t$, then for any real number $t$,

\begin{align*}
x_1 &= -t - 3 \\
x_2 &= 2t + 4 \\
x_3 &= t
\end{align*}

You should check that $(-t - 3, 2t + 4, t)$ is a solution of the original system for any real number $t$. Some particular solutions are

\begin{align*}
t = 0 & \quad t = -2 & \quad t = 3.5 \\
(-3, 4, 0) & \quad (-1, 0, -2) & \quad (-6.5, 11, 3.5)
\end{align*}

**Matched Problem 4** Solve by Gauss–Jordan elimination:

\begin{align*}
2x_1 - 2x_2 - 4x_3 &= -2 \\
3x_1 - 3x_2 - 6x_3 &= -3 \\
-2x_1 + 3x_2 + x_3 &= 7
\end{align*}

In general,

If the number of leftmost 1’s in a reduced augmented coefficient matrix is less than the number of variables in the system and there are no contradictions, then the system is dependent and has infinitely many solutions.

There are many different ways to use the reduced augmented coefficient matrix to describe the infinite number of solutions of a dependent system. We will always proceed as follows: Solve each equation in a reduced system for its leftmost variable and then introduce a different parameter for each remaining variable. As the solution to Example 4 illustrates, this method produces a concise and useful representation of the solutions to a dependent system. Example 5 illustrates a dependent system where two parameters are required to describe the solution.

**EXPLORE-DISCUS 2** Explain why the definition of reduced form ensures that each leftmost variable in a reduced system appears in one and only one equation and no equation contains more than one leftmost variable. Discuss methods for determining if a consistent system is independent or dependent by examining the reduced form.
**EXAMPLE 5 Solving a System Using Gauss–Jordan Elimination**

Solve by Gauss–Jordan elimination:

\[
\begin{align*}
2x_1 + 4x_2 + 8x_3 + 3x_4 - 4x_5 &= 2 \\
x_1 + 3x_2 + 7x_3 + 3x_4 &= -2
\end{align*}
\]

**Solution**

\[
\begin{bmatrix}
1 & 2 & 4 & 1 & -1 & \mid & 1 \\
2 & 4 & 8 & 3 & -4 & \mid & 2 \\
1 & 3 & 7 & 0 & 3 & \mid & -2
\end{bmatrix}
\]

\(-2)R_1 + R_2 \rightarrow R_2
\]

\(-1)R_1 + R_3 \rightarrow R_3
\]

\[
\begin{bmatrix}
1 & 2 & 4 & 1 & -1 & \mid & 1 \\
0 & 0 & 0 & 1 & -2 & \mid & 0 \\
0 & 1 & 3 & -1 & 4 & \mid & -3
\end{bmatrix}
\]

R_2 \leftrightarrow R_3

\[
\begin{bmatrix}
1 & 2 & 4 & 1 & -1 & \mid & 1 \\
0 & 1 & 3 & -1 & 4 & \mid & -3 \\
0 & 0 & 0 & 1 & -2 & \mid & 0
\end{bmatrix}
\]

\(-2)R_2 + R_1 \rightarrow R_1

\[
\begin{bmatrix}
1 & 0 & -2 & 3 & -9 & \mid & 7 \\
0 & 1 & 3 & -1 & 4 & \mid & -3 \\
0 & 0 & 0 & 1 & -2 & \mid & 0
\end{bmatrix}
\]

R_3 + R_2 \rightarrow R_2

\[
\begin{bmatrix}
1 & 0 & -2 & 0 & -3 & \mid & 7 \\
0 & 1 & 3 & 0 & 2 & \mid & -3 \\
0 & 0 & 0 & 1 & -2 & \mid & 0
\end{bmatrix}
\]

Matrix is in reduced form.

\[
\begin{align*}
x_1 - 2x_3 & - 3x_5 = 7 \\
x_2 + 3x_3 & + 2x_5 = -3 \\
x_4 - 2x_5 & = 0
\end{align*}
\]

Solve for the leftmost variables \(x_1, x_2, \) and \(x_4\) in terms of the remaining variables \(x_3\) and \(x_5\):

\[
\begin{align*}
x_1 &= 2x_3 + 3x_4 + 7 \\
x_2 &= -3x_3 - 2x_4 - 3 \\
x_4 &= 2x_5
\end{align*}
\]

If we let \(x_3 = s\) and \(x_5 = t\), then for any real numbers \(s\) and \(t\),

\[
\begin{align*}
x_1 &= 2s + 3t + 7 \\
x_2 &= -3s - 2t - 3 \\
x_3 &= s \\
x_4 &= 2t \\
x_5 &= t
\end{align*}
\]

is a solution. The check is left for you to perform.
Matched Problem 5

Solve by Gauss–Jordan elimination:
\[
\begin{aligned}
  x_1 - x_2 + 2x_3 - 2x_5 &= 3 \\
  -2x_1 + 2x_2 - 4x_3 - x_4 + x_5 &= -5 \\
  3x_1 - 3x_2 + 7x_3 + x_4 - 4x_5 &= 6
\end{aligned}
\]

Application

We now consider an application that involves a dependent system of equations.

Example 6 Purchasing

A chemical manufacturer wants to purchase a fleet of 24 railroad tank cars with a combined carrying capacity of 250,000 gallons. Tank cars with three different carrying capacities are available: 6,000 gallons, 8,000 gallons, and 18,000 gallons. How many of each type of tank car should be purchased?

Solution

Let
\[
\begin{align*}
  x_1 &= \text{Number of 6,000-gallon tank cars} \\
  x_2 &= \text{Number of 8,000-gallon tank cars} \\
  x_3 &= \text{Number of 18,000-gallon tank cars}
\end{align*}
\]

Then
\[
\begin{aligned}
  x_1 + x_2 + x_3 &= 24 \quad \text{Total number of tank cars} \\
  6,000x_1 + 8,000x_2 + 18,000x_3 &= 250,000 \quad \text{Total carrying capacity}
\end{aligned}
\]

Now we can form the augmented matrix of the system and solve by using Gauss–Jordan elimination:

\[
\begin{bmatrix}
  1 & 1 & 1 & \vdots & 24 \\
  6,000 & 8,000 & 18,000 & \vdots & 250,000
\end{bmatrix}
\]

\[
\begin{aligned}
  &\frac{1}{1000}R_2 \rightarrow R_2 \text{ (simplify } R_2) \\
  \sim &\begin{bmatrix}
  1 & 1 & 1 & \vdots & 24 \\
  6 & 8 & 18 & \vdots & 250
\end{bmatrix} \\
  &(-6)R_1 + R_2 \rightarrow R_2 \\
  \sim &\begin{bmatrix}
  1 & 1 & 1 & \vdots & 24 \\
  6 & 2 & 12 & \vdots & 106
\end{bmatrix} \\
  &\frac{1}{2}R_2 \rightarrow R_2 \\
  \sim &\begin{bmatrix}
  1 & 1 & 1 & \vdots & 24 \\
  0 & 1 & 6 & \vdots & 53
\end{bmatrix} \\
  &(-1)R_2 + R_1 \rightarrow R_1 \\
  \sim &\begin{bmatrix}
  1 & 0 & -5 & \vdots & -29 \\
  0 & 1 & 6 & \vdots & 53
\end{bmatrix} \\
  &\text{Matrix is in reduced form}
\end{aligned}
\]

\[
\begin{aligned}
  x_1 - 5x_3 &= -29 \
  &\quad \text{or } x_1 = 5x_3 - 29 \\
  x_2 + 6x_3 &= 53 \
  &\quad \text{or } x_2 = -6x_3 + 53
\end{aligned}
\]
Let \( x_3 = t \). Then for \( t \) any real number,

\[
\begin{align*}
  x_1 &= 5t - 29 \\
  x_2 &= -6t + 53 \\
  x_3 &= t
\end{align*}
\]

is a solution—or is it? Since the variables in this system represent the number of tank cars purchased, the values of \( x_1, x_2, \) and \( x_3 \) must be nonnegative integers. Thus, the third equation requires that \( t \) must be a nonnegative integer. The first equation requires that \( 5t - 29 \geq 0 \), so \( t \) must be at least 6. The middle equation requires that \( -6t + 53 \geq 0 \), so \( t \) can be no larger than 8. Thus, 6, 7, and 8 are the only possible values for \( t \). There are only three possible combinations that meet the company’s specifications of 24 tank cars with a total carrying capacity of 250,000 gallons, as shown in Table 1:

<table>
<thead>
<tr>
<th></th>
<th>6,000-gallon tank cars</th>
<th>8,000-gallon tank cars</th>
<th>18,000-gallon tank cars</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( x_3 )</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>17</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>11</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

The final choice would probably be influenced by other factors. For example, the company might want to minimize the cost of the 24 tank cars.

**Matched Problem 6**

A commuter airline wants to purchase a fleet of 30 airplanes with a combined carrying capacity of 960 passengers. The three available types of planes carry 18, 24, and 42 passengers, respectively. How many of each type of plane should be purchased?

**Answers to Matched Problems**

1. (A) Condition 2 is violated: The 3 in row 2 and column 2 should be a 1. Perform the operation \( \frac{1}{3}R_2 \rightarrow R_2 \) to obtain:

\[
\begin{bmatrix}
  1 & 0 & 2 \\
  0 & 1 & -2
\end{bmatrix}
\]

(B) Condition 3 is violated: The 5 in row 1 and column 2 should be a 0. Perform the operation \((-5)R_2 + R_1 \rightarrow R_1\) to obtain:

\[
\begin{bmatrix}
  1 & 0 & -6 & 8 \\
  0 & 1 & 2 & -1 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]
In Problems 1–8, indicate whether each matrix is in reduced form.

1. \[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 2 & 6
\end{bmatrix}
\]
2. \[
\begin{bmatrix}
1 & 0 & 5 \\
0 & 1 & -3
\end{bmatrix}
\]
3. \[
\begin{bmatrix}
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
4. \[
\begin{bmatrix}
1 & -1 & 4 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
5. \[
\begin{bmatrix}
0 & 0 & 1 & 2 \\
0 & 1 & 0 & -5 \\
1 & 0 & 0 & 4
\end{bmatrix}
\]
6. \[
\begin{bmatrix}
1 & -2 & 4 & 1 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
7. \[
\begin{bmatrix}
0 & 1 & 6 & 0 & -8 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
8. \[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

In Problems 9–16, write the linear system corresponding to each reduced augmented matrix and solve.

9. \[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0
\end{bmatrix}
\] \[
\begin{bmatrix}
1 & 0 & 0 & 0 & -2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 3
\end{bmatrix}
\]
11. \[
\begin{bmatrix}
1 & 0 & -2 & 3 \\
0 & 1 & 1 & -5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] \[
\begin{bmatrix}
1 & -2 & 0 & -3 \\
0 & 1 & 1 & -5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
13. \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
14. \[
\begin{bmatrix}
1 & 0 & 5 \\
0 & 1 & -3 \\
0 & 0 & 0
\end{bmatrix}
\]
15. \[
\begin{bmatrix}
1 & -2 & 0 & -3 & -5 \\
0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
16. \[
\begin{bmatrix}
1 & 0 & -2 & 3 & 4 \\
0 & 1 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

B

Use row operations to change each matrix in Problems 17–22 to reduced form.

17. \[
\begin{bmatrix}
1 & 2 & -1 \\
0 & 1 & 3
\end{bmatrix}
\]
18. \[
\begin{bmatrix}
1 & 3 & 1 \\
0 & 2 & -4
\end{bmatrix}
\]
19. \[
\begin{bmatrix}
1 & 0 & -3 & 1 \\
0 & 1 & 2 & 0 \\
0 & 0 & 3 & -6
\end{bmatrix}
\]
20. \[
\begin{bmatrix}
1 & 0 & 4 & 0 \\
0 & 1 & -3 & -1 \\
0 & 0 & -2 & 2
\end{bmatrix}
\]
21. \[
\begin{bmatrix}
1 & 2 & -2 & -1 \\
0 & 3 & -6 & 1 \\
0 & -1 & 2 & -1
\end{bmatrix}
\]

22. \[
\begin{bmatrix}
0 & -2 & 8 & 1 \\
2 & -2 & 6 & -4 \\
0 & -1 & 4 & \frac{1}{2}
\end{bmatrix}
\]


23. \(2x_1 + 4x_2 - 10x_3 = -2\)  
\(3x_1 + 9x_2 - 21x_3 = 0\)  
\(x_1 + 5x_2 - 12x_3 = 1\)

24. \(3x_1 + 5x_2 - x_3 = -7\)  
\(x_1 + x_2 + x_3 = -1\)  
\(2x_1 + 11x_2 = 7\)

25. \(3x_1 + 8x_2 - x_3 = -18\)  
\(2x_1 + x_2 + 5x_3 = 8\)  
\(2x_1 + 4x_2 + 2x_3 = -4\)

26. \(2x_1 + 7x_2 + 15x_3 = -12\)  
\(4x_1 + 7x_2 + 13x_3 = -10\)  
\(3x_1 + 6x_2 + 12x_3 = -9\)

27. \(x_1 - x_2 - 3x_3 = 8\)  
\(x_1 - 2x_2 = 7\)

28. \(2x_1 + 4x_2 - 6x_3 = 10\)  
\(3x_1 - 3x_2 - 3x_3 = 6\)

29. \(x_1 - x_2 = 0\)  
\(3x_1 + 2x_2 = 7\)  
\(x_1 - x_2 = -2\)

30. \(2x_1 - x_2 = 0\)  
\(3x_1 + 2x_2 + x_3 = 7\)  
\(x_1 - x_2 - x_3 = -1\)

31. \(3x_1 - 4x_2 - x_3 = 1\)  
\(2x_1 - 3x_2 + x_3 = 1\)  
\(x_1 + 2x_2 + 3x_3 = 2\)

32. \(3x_1 + 7x_2 - x_3 = 11\)  
\(x_1 + 2x_2 - x_3 = 3\)  
\(2x_1 + 4x_2 - 2x_3 = 10\)

33. \(-2x_1 + x_2 + 3x_3 = -7\)  
\(x_1 - 4x_2 + 2x_3 = 0\)  
\(x_1 - 3x_2 + x_3 = -1\)

34. \(2x_1 + 5x_2 + 4x_3 = -7\)  
\(-4x_1 - 5x_2 + 2x_3 = 9\)  
\(-2x_1 - x_2 + 4x_3 = 3\)

35. \(2x_1 - 2x_2 - 4x_3 = -2\)  
\(-3x_1 + 3x_2 + 6x_3 = 3\)

36. \(2x_1 + 8x_2 - 6x_3 = 4\)  
\(-3x_1 - 12x_2 + 9x_3 = -6\)

37. \(4x_1 - x_2 + 2x_3 = 3\)  
\(-4x_1 + x_2 - 3x_3 = -10\)  
\(8x_1 - 2x_2 + 9x_3 = -1\)

38. \(-4x_1 + 2x_2 + 2x_3 = 5\)  
\(-6x_1 + 3x_2 - 3x_3 = -2\)  
\(10x_1 - 5x_2 + 9x_3 = 4\)

39. \(2x_1 - 5x_2 - 3x_3 = 7\)  
\(-4x_1 + 10x_2 + 2x_3 = 6\)  
\(6x_1 - 15x_2 - 9x_3 = -19\)

40. \(-4x_1 + 8x_2 + 10x_3 = -6\)  
\(6x_1 - 12x_2 - 15x_3 = 9\)  
\(-8x_1 + 14x_2 + 19x_3 = -8\)

41. \(5x_1 - 3x_2 + 2x_3 = 13\)  
\(2x_1 - x_2 - 3x_3 = 1\)  
\(-3x_1 - x_2 - 2x_3 = -10\)

42. \(-4x_1 - 2x_2 + 4x_3 = 12\)  
\(2x_1 + 4x_2 - x_3 = -1\)

44. Consider a system of three linear equations in three variables. Give examples of two reduced forms that are not row equivalent if the system is (A) Consistent and dependent (B) Inconsistent


45. \(x_1 + 2x_2 - x_3 = 7\)  
\(x_1 + 5x_2 - 9x_3 = 16\)  
\(x_1 - 5x_2 - 7x_3 - 7x_4 = 13\)

46. \(2x_1 + 4x_2 + 5x_3 + 4x_4 = 8\)  
\(x_1 + 2x_2 + x_4 = 3\)

47. \(-x_1 - x_2 + 3x_3 - 2x_4 = 1\)  
\(-2x_1 + 4x_2 - 3x_3 + x_4 = 0.5\)  
\(3x_1 - x_2 + 10x_3 - 4x_4 = 2.9\)  
\(4x_1 - 3x_2 + 8x_3 - 2x_4 = 0.6\)

48. \(x_1 + x_2 + 4x_3 + x_4 = 1.3\)  
\(-x_1 + x_2 - x_3 = 1.1\)  
\(2x_1 + x_3 + 3x_4 = -4.4\)  
\(2x_1 + 5x_2 + 11x_3 + 3x_4 = 5.6\)

49. \(-x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 2\)  
\(-2x_1 + 4x_2 + 2x_3 + 2x_4 - 2x_5 = 0\)  
\(3x_1 - 6x_2 + x_3 + x_4 + 5x_5 = 4\)  
\(-x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 3\)

50. \(-x_1 - 3x_2 + x_3 + x_4 + 2x_5 = 2\)  
\(-x_1 + 5x_2 + 2x_3 + 2x_4 - 2x_5 = 0\)  
\(2x_1 - 6x_2 + 2x_3 + 2x_4 + 4x_5 = 4\)  
\(-x_1 + 3x_2 - x_3 - x_5 = 3\)

APPLICATIONS


51. Puzzle. A friend of yours came out of the post office after spending $14.00 on 15¢, 20¢, and 35¢ stamps. If she bought 45 stamps in all, how many of each type did she buy?

52. Puzzle. A parking meter accepts only nickels, dimes, and quarters. If the meter contains 32 coins with a total value of $6.80, how many of each type are there?

53. Chemistry. A chemist can purchase a 10% saline solution in 500 cubic centimeter containers, a 20% saline solution in 500 cubic centimeter containers, and a 50% saline solution in 1,000 cubic centimeter containers. He needs 12,000 cubic centimeters of 30% saline solution. How many containers of each type of solution should he purchase in order to form this solution?
54. **Chemistry.** Repeat Problem 53 if the 50% saline solution is available only in 1,500 cubic centimeter containers.

55. **Geometry.** Find $a$, $b$, and $c$ so that the graph of the parabola with equation $y = a + bx + cx^2$ passes through the points $(−2, 3), (−1, 2),$ and $(1, 6)$.

56. **Geometry.** Find $a$, $b$, and $c$ so that the graph of the parabola with equation $y = a + bx + cx^2$ passes through the points $(1, 3), (2, 2),$ and $(3, 5)$.

57. **Geometry.** Find $a$, $b$, and $c$ so that the graph of the circle with equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(6, 2), (4, 6),$ and $(-3, -1)$.

58. **Geometry.** Find $a$, $b$, and $c$ so that the graph of the circle with equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(−4, 1), (−1, 2),$ and $(3, −6)$.

59. **Production Scheduling.** A small manufacturing plant makes three types of inflatable boats: one-person, two-person, and four-person models. Each boat requires the services of three departments, as listed in the table. The cutting, assembly, and packaging departments have available a maximum of 380, 330, and 120 labor-hours per week, respectively. How many boats of each type must be produced each week for the plant to operate at full capacity?

<table>
<thead>
<tr>
<th></th>
<th>One-person boat</th>
<th>Two-person boat</th>
<th>Four-person boat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting department</td>
<td>0.5 h</td>
<td>1.0 h</td>
<td>1.5 h</td>
</tr>
<tr>
<td>Assembly department</td>
<td>0.6 h</td>
<td>0.9 h</td>
<td>1.2 h</td>
</tr>
<tr>
<td>Packaging department</td>
<td>0.2 h</td>
<td>0.3 h</td>
<td>0.5 h</td>
</tr>
</tbody>
</table>

60. **Production Scheduling.** Repeat Problem 59 assuming the cutting, assembly, and packaging departments have available a maximum of 350, 330, and 120 labor-hours per week, respectively.

61. **Production Scheduling.** Rework Problem 59 assuming the packaging department is no longer used.

62. **Production Scheduling.** Rework Problem 60 assuming the packaging department is no longer used.

63. **Production Scheduling.** Rework Problem 59 assuming the four-person boat is no longer produced.

64. **Production Scheduling.** Rework Problem 60 assuming the four-person boat is no longer produced.

65. **Nutrition.** A dietitian in a hospital is to arrange a special diet using three basic foods. The diet is to include exactly 340 units of calcium, 180 units of iron, and 220 units of vitamin A. The number of units per ounce of each special ingredient for each of the foods is indicated in the table. How many ounces of each food must be used to meet the diet requirements?

<table>
<thead>
<tr>
<th>Units per ounce</th>
<th>Food A</th>
<th>Food B</th>
<th>Food C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calcium</td>
<td>30</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Iron</td>
<td>10</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Vitamin A</td>
<td>10</td>
<td>30</td>
<td>20</td>
</tr>
</tbody>
</table>

66. **Nutrition.** Repeat Problem 65 if the diet is to include exactly 400 units of calcium, 160 units of iron, and 240 units of vitamin A.

67. **Nutrition.** Solve Problem 65 with the assumption that food C is no longer available.

68. **Nutrition.** Solve Problem 66 with the assumption that food C is no longer available.

69. **Nutrition.** Solve Problem 65 assuming the vitamin A requirement is deleted.

70. **Nutrition.** Solve Problem 66 assuming the vitamin A requirement is deleted.

71. **Sociology.** Two sociologists have grant money to study school busing in a particular city. They wish to conduct an opinion survey using 600 telephone contacts and 400 house contacts. Survey company A has personnel to do 30 telephone and 10 house contacts per hour; survey company B can handle 20 telephone and 20 house contacts per hour. How many hours should be scheduled for each firm to produce exactly the number of contacts needed?

72. **Sociology.** Repeat Problem 71 if 650 telephone contacts and 350 house contacts are needed.