at the same time and follow the same route on the 75-mile trip across the English Channel to Cherbourg, France. The average speed of boat $A$ is 5 miles per hour greater than the average speed of boat $B$. Consequently, boat $A$ arrives at Cherbourg 30 minutes before boat $B$. Find the average speed of each boat.

**58. Transportation.** Bus $A$ leaves Milwaukee at noon and travels west on Interstate 94. Bus $B$ leaves Milwaukee 30 minutes later, travels the same route, and overtakes bus $A$ at a point 210 miles west of Milwaukee. If the average speed of bus $B$ is 10 miles per hour greater than the average speed of bus $A$, at what time did bus $B$ overtake bus $A$?

## SECTION 8-4 Systems of Linear Inequalities in Two Variables

- Graphing Linear Inequalities in Two Variables
- Solving Systems of Linear Inequalities Graphically
- Application

Many applications of mathematics involve systems of inequalities rather than systems of equations. A graph is often the most convenient way to represent the solutions of a system of inequalities in two variables. In this section, we discuss techniques for graphing both a single linear inequality in two variables and a system of linear inequalities in two variables.

**Graphing Linear Inequalities in Two Variables**

We know how to graph first-degree equations such as

$$y = 2x - 3 \quad \text{and} \quad 2x - 3y = 5$$

but how do we graph first-degree inequalities such as

$$y \leq 2x - 3 \quad \text{and} \quad 2x - 3y > 5$$

Actually, graphing these inequalities is almost as easy as graphing the equations. But before we begin, we must discuss some important subsets of a plane in a rectangular coordinate system.

A line divides a plane into two halves called **half-planes**. A vertical line divides a plane into **left** and **right half-planes** [Fig. 1(a)]; a nonvertical line divides a plane into **upper** and **lower half-planes** [Fig. 1(b)].

**FIGURE 1** Half-planes.
EXPLORE-DISCUSS 1  Consider the following linear equation and related linear inequalities:

(1) 2x − 3y = 12  (2) 2x − 3y < 12  (3) 2x − 3y > 12

(A) Graph the line with equation (1).

(B) Find the point on this line with x coordinate 3 and draw a vertical line through this point. Discuss the relationship between the y coordinates of the points on this line and statements (1), (2), and (3).

(C) Repeat part B for x = −3. For x = 9.

(D) Based on your observations in parts B and C, write a verbal description of all the points in the plane that satisfy equation (1), those that satisfy inequality (2), and those that satisfy inequality (3).

Now let’s investigate the half-planes determined by the linear equation y = 2x − 3. We start by graphing y = 2x − 3 (Fig. 2). For any given value of x, there is exactly one value for y such that (x, y) lies on the line. For the same x, if the point (x, y) is below the line, then y < 2x − 3. Thus, the lower half-plane corresponds to the solution of the inequality y < 2x − 3. Similarly, the upper half-plane corresponds to the solution of the inequality y > 2x − 3, as shown in Figure 2.

The four inequalities formed from y = 2x − 3 by replacing the = sign by ≥, >, ≤, and <, respectively, are

\[ y \geq 2x - 3 \quad y > 2x - 3 \quad y \leq 2x - 3 \quad y < 2x - 3 \]

The graph of each is a half-plane. The line y = 2x − 3, called the boundary line for the half-plane, is included for ≥ and ≤ and excluded for > and <. In Figure 3, the half-planes are indicated with small arrows on the graph of y = 2x − 3 and then graphed as shaded regions. Included boundary lines are shown as solid lines, and excluded boundary lines are shown as dashed lines.
Theorem 1  
**Graphs of Linear Inequalities in Two Variables**

The graph of a linear inequality

\[ Ax + By < C \quad \text{or} \quad Ax + By > C \]

with \( B \neq 0 \), is either the upper half-plane or the lower half-plane (but not both) determined by the line \( Ax + By = C \).

If \( B = 0 \), then the graph of

\[ Ax < C \quad \text{or} \quad Ax > C \]

is either the left half-plane or the right half-plane (but not both) determined by the line \( Ax = C \).

As a consequence of Theorem 1, we state a simple and fast mechanical procedure for graphing linear inequalities.

---

**Procedure for Graphing Linear Inequalities in Two Variables**

**Step 1.** Graph \( Ax + By = C \) as a dashed line if equality is not included in the original statement or as a solid line if equality is included.

**Step 2.** Choose a test point anywhere in the plane not on the line and substitute the coordinates into the inequality. The origin \((0, 0)\) often requires the least computation.

**Step 3.** The graph of the original inequality includes the half-plane containing the test point if the inequality is satisfied by that point, or the half-plane not containing that point if the inequality is not satisfied by that point.

---

**EXAMPLE 1  Graphing a Linear Inequality**

Graph: \( 3x - 4y \leq 12 \)
Solution

**Step 1.** Graph $3x - 4y = 12$ as a solid line, since equality is included in the original statement (Fig. 4).

**Step 2.** Pick a convenient test point above or below the line. The origin $(0, 0)$ requires the least computation. Substituting $(0, 0)$ into the inequality

$$3x - 4y \leq 12$$

produces a true statement; therefore, $(0, 0)$ is in the solution set.

**Step 3.** The line $3x - 4y = 12$ and the half-plane containing the origin form the graph of $3x - 4y \leq 12$ (Fig. 5).

**Matched Problem 1**
Graph: $2x + 3y < 6$

**Example 2**

**Graphing a Linear Inequality**

Graph: (A) $y > -3$ (B) $2x \leq 5$

**Solutions**

(A) The graph of $y > -3$ is shown in Figure 6.

(B) The graph of $2x \leq 5$ is shown in Figure 7.

**Matched Problem 2**
Graph: (A) $y \leq 2$ (B) $3x > -8$
We now consider systems of linear inequalities such as
\[
\begin{align*}
  x + y &\geq 6 \\
  2x - y &\geq 0 \\
  2x + y &\leq 22 \\
  x + y &\leq 13 \\
  2x + 5y &\leq 50 \\
  x &\geq 0 \\
  y &\geq 0
\end{align*}
\]
We wish to solve such systems graphically—that is, to find the graph of all ordered pairs of real numbers \((x, y)\) that simultaneously satisfy all the inequalities in the system. The graph is called the solution region for the system. To find the solution region, we graph each inequality in the system and then take the intersection of all the graphs. To simplify the discussion that follows, we will consider only systems of linear inequalities where equality is included in each statement in the system.

**EXAMPLE 3** Solving a System of Linear Inequalities Graphically

Solve the following system of linear inequalities graphically:
\[
\begin{align*}
  x + y &\geq 6 \\
  2x - y &\geq 0
\end{align*}
\]

**Solution** First, graph the line \(x + y = 6\) and shade the region that satisfies the inequality \(x + y \geq 6\). This region is shaded in blue in Figure 8(a). Next, graph the line \(2x - y = 0\) and shade the region that satisfies the inequality \(2x - y \geq 0\). This region is shaded in red in Figure 8(a). The solution region for the system of inequalities is the intersection of these two regions. This is the region shaded in both red and blue in Figure 8(a), which is redrawn in Figure 8(b) with only the solution region shaded for clarity. The coordinates of any point in the shaded region of Figure 8(b) specify a solution to the system. For example, the points (2, 4), (6, 3), and (7.43, 8.56) are three of infinitely many solutions, as can be easily checked. The intersection point (2, 4) can be obtained by solving the equations \(x + y = 6\) and \(2x - y = 0\) simultaneously.
**Matched Problem 3** Solve the following system of linear inequalities graphically:

\[ \begin{align*}
3x + y &\leq 21 \\
x - 2y &\leq 0
\end{align*} \]

---

**EXPLORE-DISCUSS 2**

Refer to Example 3. Graph each boundary line and shade the regions obtained by reversing each inequality. That is, shade the region of the plane that corresponds to the inequality \( x + y < 6 \) and then shade the region that corresponds to the inequality \( 2x - y < 0 \). What portion of the plane is left unshaded? Compare this method with the one used in the solution to Example 3.

The points of intersection of the lines that form the boundary of a solution region play a fundamental role in the solution of linear programming problems, which are discussed in the next section.

---

**Definition 1** \( \text{Corner Point} \)

A **corner point** of a solution region is a point in the solution region that is the intersection of two boundary lines.

The point \((2, 4)\) is the only corner point of the solution region in Example 3; see figure (b).

---

**EXAMPLE 4** \( \text{Solving a System of Linear Inequalities Graphically} \)

Solve the following system of linear inequalities graphically, and find the corner points.

\[ \begin{align*}
2x + y &\leq 22 \\
x + y &\leq 13 \\
2x + 5y &\leq 50 \\
x &\geq 0 \\
y &\geq 0
\end{align*} \]

**Solution**

The inequalities \( x \geq 0 \) and \( y \geq 0 \), called **nonnegative restrictions**, occur frequently in applications involving systems of inequalities since \( x \) and \( y \) often represent quantities that can’t be negative—number of units produced, number of hours worked, etc. The solution region lies in the first quadrant, and we can restrict our attention to that portion of the plane. First, we graph the lines

\[ \begin{align*}
2x + y = 22 \\
x + y = 13 \\
2x + 5y = 50
\end{align*} \]
Next, choosing \((0, 0)\) as a test point, we see that the graph of each of the first three inequalities in the system consists of its corresponding line and the half-plane lying below the line, as indicated by the arrows in Figure 9. Thus, the solution region of the system consists of the points in the first quadrant that simultaneously lie on or below all three of these lines—see Figure 9.

The corner points \((0, 0)\), \((0, 10)\), and \((11, 0)\) can be determined from the graph. The other two corner points are determined as follows:

Solve the system

\[
\begin{align*}
2x + 5y &= 50 \\
y + y &= 13
\end{align*}
\]

to obtain \((5, 8)\).

Solve the system

\[
\begin{align*}
2x + y &= 22 \\
x + y &= 13
\end{align*}
\]

to obtain \((9, 4)\).

Note that the lines \(2x + 5y = 50\) and \(2x + y = 22\) also intersect, but the intersection point is not part of the solution region, and hence, is not a corner point.

**Matched Problem 4**

Solve the following system of linear inequalities graphically, and find the corner points:

\[
\begin{align*}
5x + y &\geq 20 \\
x + y &\geq 12 \\
x + 3y &\geq 18 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]
As we saw in Section 8-1, a graphing utility is a useful tool for graphing lines and finding intersection points. Most graphing utilities also have some limited capabilities for shading regions that satisfy inequalities. Figure 10(a) shows a solution for Example 3 that could be produced on most graphing utilities. On newer models (for example, the TI-83 and TI-96), it is a simple operation to shade the region above or below a graph. Using this option to shade the points that do not satisfy a given inequality, as discussed in Explore-Discuss 2, clearly displays the solution to a system of inequalities. Figure 10(b) shows the result of applying this method to Example 4.

If we compare the solution regions of Examples 3 and 4, we see that there is a fundamental difference between these two regions. We can draw a circle around the solution region in Example 4. However, it is impossible to include all the points in the solution region in Example 3 in any circle, no matter how large we draw it. This leads to the following definition.

**DEFINITION 2**

**Bounded and Unbounded Solution Regions**

A solution region of a system of linear inequalities is **bounded** if it can be enclosed within a circle. If it cannot be enclosed within a circle, then it is **unbounded**.

Thus, the solution region for Example 4 is bounded and the solution region for Example 3 is unbounded. This definition will be important in the next section.

**Application**

**EXAMPLE 5**  
**Production Scheduling**

A manufacturer of surfboards makes a standard model and a competition model. Each standard board requires 6 labor-hours for fabricating and 1 labor-hour for finishing. Each competition board requires 8 labor-hours for fabricating and 3 labor-hours for finishing. The maximum labor-hours available per week in the fabricating and finishing departments are 120 and 30, respectively. What combinations of boards can be produced each week so as not to exceed the number of labor-hours available in each department per week?
Solution To clarify relationships, we summarize the information in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Standard model (labor-hours per board)</th>
<th>Competition model (labor-hours per board)</th>
<th>Maximum labor-hours available per week</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fabricating</td>
<td>6</td>
<td>8</td>
<td>120</td>
</tr>
<tr>
<td>Finishing</td>
<td>1</td>
<td>3</td>
<td>30</td>
</tr>
</tbody>
</table>

Let

\[ x = \text{Number of standard boards produced per week} \]
\[ y = \text{Number of competition boards produced per week} \]

These variables are restricted as follows:

**Fabricating department restriction:**

\[
6x + 8y \leq 120
\]

**Finishing department restriction:**

\[
1x + 3y \leq 30
\]

Since it is not possible to manufacture a negative number of boards, \( x \) and \( y \) also must satisfy the nonnegative restrictions

\[ x \geq 0 \]
\[ y \geq 0 \]

Thus, \( x \) and \( y \) must satisfy the following system of linear inequalities:

\[
6x + 8y \leq 120 \quad \text{Fabricating department restriction} \\
1x + 3y \leq 30 \quad \text{Finishing department restriction} \\
x \geq 0 \quad \text{Nonnegative restriction} \\
y \geq 0 \quad \text{Nonnegative restriction}
\]

Graphing this system of linear inequalities, we obtain the set of feasible solutions, or the feasible region, as shown in Figure 11. For problems of this type and for the linear programming problems we consider in the next section, solution regions are often referred to as feasible regions. Any point within the shaded area, including the boundary lines, represents a possible production schedule. Any point outside the shaded area represents an impossible schedule. For example, it would be possible to
produce 12 standard boards and 5 competition boards per week, but it would not be possible to produce 12 standard boards and 7 competition boards per week (see the figure).

![Figure 11](image)

**Matched Problem 5** Repeat Example 5 using 5 hours for fabricating a standard board and a maximum of 27 labor-hours for the finishing department.

**Remark.** Refer to Example 5. How do we interpret a production schedule of 10.5 standard boards and 4.3 competition boards? It is not possible to manufacture a fraction of a board. But it is possible to average 10.5 standard and 4.3 competition boards per week. In general, we will assume that all points in the feasible region represent acceptable solutions, even though noninteger solutions might require special interpretation.

**Answers to Matched Problems**

1. ![Graph](image)

2. (A) ![Graph](image) (B) ![Graph](image)
Graph each inequality in Problems 1–10.

1. $2x - 3y < 6$
2. $3x + 4y < 12$
3. $3x + 2y \geq 18$
4. $3y - 2x \geq 24$
5. $y \leq \frac{2}{3}x + 5$
6. $y \geq \frac{1}{3}x - 2$
7. $y < 8$
8. $x > -5$
9. $-3 \leq y < 2$
10. $-1 < x \leq 3$
In Problems 11–14, match the solution region of each system of linear inequalities with one of the four regions shown in the figure below.

In Problems 15–20, solve each system of linear inequalities graphically.

In Problems 21–24, match the solution region of each system of linear inequalities with one of the four regions shown in the figure below. Identify the corner points of each solution region.

In Problems 25–36, solve the systems graphically, and indicate whether each solution region is bounded or unbounded. Find the coordinates of each corner point.

In Problems 37–44, solve the systems graphically, and indicate whether each solution region is bounded or unbounded. Find the coordinates of each corner point.
43. \(16x + 13y \leq 119\)  
\(12x + 6y \geq 101\)  
\(-4x + 3y \leq 11\)  
44. \(8x + 4y \leq 41\)  
\(-15x + 5y \leq 19\)  
\(2x + 6y \geq 37\)

**APPLICATIONS**

45. **Manufacturing—Resource Allocation.** A manufacturing company makes two types of water skis: a trick ski and a slalom ski. The trick ski requires 6 labor-hours for fabricating and 1 labor-hour for finishing. The slalom ski requires 4 labor-hours for fabricating and 1 labor-hour for finishing. The maximum labor-hours available per day for fabricating and finishing are 108 and 24, respectively. If \(x\) is the number of trick skis and \(y\) is the number of slalom skis produced per day, write a system of inequalities that indicates appropriate restraints on \(x\) and \(y\). Find the set of feasible solutions graphically for the number of each type of ski that can be produced.

46. **Manufacturing—Resource Allocation.** A furniture manufacturing company manufactures dining room tables and chairs. A table requires 8 labor-hours for assembling and 2 labor-hours for finishing. A chair requires 2 labor-hours for assembling and 1 labor-hour for finishing. The maximum labor-hours available per day for assembly and finishing are 400 and 120, respectively. If \(x\) is the number of tables and \(y\) is the number of chairs produced per day, write a system of inequalities that indicates appropriate restraints on \(x\) and \(y\). Find the set of feasible solutions graphically for the number of tables and chairs that can be produced.

**47. Manufacturing—Resource Allocation.** Refer to Problem 45. The company makes a profit of $50 on each trick ski and a profit of $60 on each slalom ski.

(A) If the company makes 10 trick and 10 slalom skis per day, the daily profit will be $1,100. Are there other feasible production schedules that will result in a daily profit of $1,100? How are these schedules related to the graph of the line \(50x + 60y = 1,100\)?

(B) Find a feasible production schedule that will produce a daily profit greater than $1,100 and repeat part A for this schedule.

(C) Discuss methods for using lines like those in parts A and B to find the largest possible daily profit.

**48. Manufacturing—Resource Allocation.** Refer to Problem 46. The company makes a profit of $50 on each table and a profit of $15 on each chair.

(A) If the company makes 20 tables and 20 chairs per day, the daily profit will be $1,300. Are there other feasible production schedules that will result in a daily profit of $1,300? How are these schedules related to the graph of the line \(50x + 15y = 1,300\)?

(B) Find a feasible production schedule that will produce a daily profit greater than $1,300 and repeat part A for this schedule.

(C) Discuss methods for using lines like those in parts A and B to find the largest possible daily profit.

49. **Nutrition—Plants.** A farmer can buy two types of plant food, mix A and mix B. Each cubic yard of mix A contains 20 pounds of phosphoric acid, 30 pounds of nitrogen, and 5 pounds of potash. Each cubic yard of mix B contains 10 pounds of phosphoric acid, 30 pounds of nitrogen, and 10 pounds of potash. The minimum requirements are 460 pounds of phosphoric acid, 960 pounds of nitrogen, and 220 pounds of potash. If \(x\) is the number of cubic yards of mix A used and \(y\) is the number of cubic yards of mix B used, write a system of inequalities that indicates appropriate restraints on \(x\) and \(y\). Find the set of feasible solutions graphically for the amount of mix A and mix B that can be used.

50. **Nutrition.** A dietitian in a hospital is to arrange a special diet using two foods. Each ounce of food \(M\) contains 30 units of calcium, 10 units of iron, and 10 units of vitamin A. Each ounce of food \(N\) contains 10 units of calcium, 10 units of iron, and 30 units of vitamin A. The minimum requirements in the diet are 360 units of calcium, 160 units of iron, and 240 units of vitamin A. If \(x\) is the number of ounces of food \(M\) used and \(y\) is the number of ounces of food \(N\) used, write a system of linear inequalities that reflects the conditions indicated. Find the set of feasible solutions graphically for the amount of each kind of food that can be used.

51. **Sociology.** A city council voted to conduct a study on inner-city community problems. A nearby university was contacted to provide sociologists and research assistants. Each sociologist uses 30 hours per week analyzing data in the field and 360 hours in the research center. Each research assistant will spend 30 hours per week analyzing data in the field and 30 hours per week analyzing data in the research center. The minimum weekly labor-hour requirements are 280 hours in the field and 360 hours in the research center. If \(x\) is the number of sociologists hired for the study and \(y\) is the number of research assistants hired, write a system of linear inequalities that indicates appropriate restrictions on \(x\) and \(y\). Find the set of feasible solutions graphically.

52. **Psychology.** In an experiment on conditioning, a psychologist uses two types of Skinner (conditioning) boxes with mice and rats. Each mouse spends 10 minutes per day in box \(A\) and 20 minutes per day in box \(B\). Each rat spends 20 minutes per day in box \(A\) and 10 minutes per day in box \(B\). The total maximum time available per day is 800 minutes for box \(A\) and 640 minutes for box \(B\). We are interested in the various numbers of mice and rats that can be used in the experiment under the conditions stated. If \(x\) is the number of mice used and \(y\) is the number of rats used, write a system of linear inequalities that indicates appropriate restrictions on \(x\) and \(y\). Find the set of feasible solutions graphically.