A ball is propelled from the ground straight upward with initial velocity 64 ft/s. Ignoring air resistance, find an equation for the height of the ball at any time \( t \). Also, determine the maximum height and the amount of time the ball spends in the air.

**Solution**

With gravity as the only force, the height \( h(t) \) satisfies

\[ h''(t) = -32. \]

The initial conditions are \( h'(0) = 64 \) and \( h(0) = 0 \). We then have

\[ \int h''(t) \, dt = \int -32 \, dt \]

or

\[ h'(t) = -32t + c. \]

From the initial velocity, we have

\[ 64 = h'(0) = -32(0) + c = c \]

and so,

\[ h'(t) = 64 - 32t. \]

Integrating one more time gives us

\[ \int h'(t) \, dt = \int (64 - 32t) \, dt \]

or

\[ h(t) = 64t - 16t^2 + c. \]

From the initial altitude we have

\[ 0 = h(0) = 64(0) - 16(0)^2 + c = c, \]

and so,

\[ h(t) = 64t - 16t^2. \]

Since the height function is quadratic, its maximum occurs at the one time when \( h'(t) = 0 \). [You should also consider the physics of the situation: what happens physically when \( h'(t) = 0 \)?] Solving \( 64 - 32t = 0 \) gives \( t = 2 \) (the time at the maximum altitude) and the corresponding height is \( h(2) = 64(2) - 16(2)^2 = 64 \) feet. Again, the ball lands when \( h(t) = 0 \). Solving

\[ 0 = h(t) = 64t - 16t^2 = 16t(4-t) \]

gives \( t = 0 \) (launch time) and \( t = 4 \) (landing time). The time of flight is thus 4 seconds.

You can observe an interesting property of projectile motion by graphing the height function from example 5.2 along with the line \( y = 48 \) (see Figure 5.43). Notice that the graphs intersect at \( t = 1 \) and \( t = 3 \). Further, the time interval \([1, 3]\) corresponds to exactly half the total time in the air. Notice that this says that the ball is in the top one-fourth of its height for half of its time in the air. You may have marveled at how some athletes jump so high that they seem to “hang in the air.” As this calculation suggests, all objects tend to hang in the air.
It has been reported that basketball star Michael Jordan has a vertical leap of 54". Ignoring air resistance, what is the initial velocity required to jump this high?

**Solution**

Once again, Newton’s second law leads us to the equation \( h''(t) = -32 \) for the height \( h(t) \). Further, the initial velocity and initial position are given by \( h'(0) = v_0 \) and \( h(0) = 0 \), respectively. Our task is to determine the value of \( v_0 \) that will give a maximum altitude of 54". As before, we integrate to get

\[
h'(t) = v_0 - 32t.
\]

Integrating once again and using the initial position \( h(0) = 0 \), we get

\[
h(t) = v_0 t - 16t^2.
\]

The maximum height occurs when \( h'(t) = 0 \). (Why?) Setting

\[
0 = h'(t) = v_0 - 32t,
\]

gives us \( t = \frac{v_0}{32} \). The height at this time (i.e., the maximum altitude) is then

\[
h \left( \frac{v_0}{32} \right) = v_0 \left( \frac{v_0}{32} \right) - 16 \left( \frac{v_0}{32} \right)^2 = \frac{v_0^2}{32} - \frac{v_0^2}{64} = \frac{v_0^2}{64}.
\]

A jump of 54" = 4.5' requires \( \frac{v_0^2}{64} = 4.5 \) or \( v_0^2 = 288 \), so that \( v_0 = \sqrt{288} \approx 17 \) ft/s. Note that this initial velocity is equivalent to roughly 11.6 mph.

---

**So far, our projectiles have only traveled in the vertical direction. Most applications of projectile motion must also consider movement in the horizontal direction. If we ignore air resistance, these calculations are also relatively straightforward. The basic principle is that we can apply Newton’s second law separately for the horizontal and vertical components of the motion. If \( y(t) \) represents the vertical position, then we have \( y''(t) = -g \), as before. Ignoring air resistance, there are no forces acting horizontally on the projectile. So, if \( x(t) \) represents the horizontal position, Newton’s second law gives us \( x''(t) = 0 \).

The initial conditions are slightly more complicated here. In general, we want to consider projectiles that are launched with an initial speed \( v_0 \) at an angle \( \theta \) from the horizontal. Figure 5.44a shows a projectile fired with \( \theta > 0 \). Notice that an initial angle of \( \theta < 0 \) would mean a downward initial velocity.

As shown in Figure 5.44b, the initial velocity can be separated into horizontal and vertical components. From elementary trigonometry, the horizontal component of the initial velocity is \( v_x = v_0 \cos \theta \) and the vertical component is \( v_y = v_0 \sin \theta \).**
Chapter 5  Applications of the Definite Integral

Example 5.4  The Motion of a Projectile in Two Dimensions

An object is launched at angle $\theta = \pi/6$ from the horizontal with initial speed $v_0 = 98$ m/s. Determine the time of flight and the (horizontal) range of the projectile.

Solution  Starting with the vertical component of the motion (and again ignoring air resistance), we have $y''(t) = -9.8$ (since the initial speed is given in terms of meters per second). Referring to Figure 5.44b, notice that the vertical component of the initial velocity is $y'(0) = 98 \sin \pi/6 = 49$ and the initial altitude is $y(0) = 0$. A pair of simple integrations give us the velocity function $y'(t) = -9.8t + 49$ and the position function $y(t) = -4.9t^2 + 49t$. The object hits the ground when $y(t) = 0$ (i.e., when its height above the ground is 0). Solving

$$0 = y(t) = -4.9t^2 + 49t = 49t(1 - 0.1t)$$

gives $t = 0$ (launch time) and $t = 10$ (landing time). The time of flight is then 10 seconds. The horizontal component of motion is determined from the equation $x''(t) = 0$ with initial velocity $x'(0) = 98 \cos \pi/6 = 49\sqrt{3}$ and initial position $x(0) = 0$. Integration gives us $x'(t) = 49\sqrt{3}$ and $x(t) = (49\sqrt{3})t$. In Figure 5.45, we give a plot of the path of the ball. [You can easily do the same by using the parametric plot mode on your graphing calculator or CAS. Simply enter the separate equations for $x(t)$ and $y(t)$ and set the range of $t$-values to be $0 \leq t \leq 10$. Alternatively, you could easily solve for $t$ in terms of $x$ and replace $t$ by $\frac{1}{49\sqrt{3}} x$, to see that the curve is simply a parabola.] Notice that the horizontal location of the ball when it lands is the value of $x(t)$ at $t = 10$ (the landing time). The range is then

$$x(10) = (49\sqrt{3})(10) = 490\sqrt{3} \approx 849 \text{ meters}.$$  

Remark 5.1

We urge you to resist the temptation to reduce this section to a few memorized formulas. It is true that if you ignore air resistance, the vertical component of position will always turn out to be $y(t) = -\frac{1}{2}gt^2 + (v_0 \sin \theta)t + y(0)$. (We’ll pursue this further in the exercises.) However, your understanding of the process and your chances of finding the correct answer will improve dramatically if you start each problem with Newton’s second law and work through the integrations (which are not difficult).

Example 5.5  The Motion of a Tennis Serve

Suppose that a tennis player hits a serve from a height of 8 feet at an initial speed of 120 mph and at an angle of $5^\circ$ below the horizontal. The serve is “in” if the ball clears a 3’-high net that is 39’ away and hits the ground in front of the service line 60’ away. (We illustrate this situation in Figure 5.46.) Determine whether the serve is in or out.

Solution  As in example 5.4, we start with the vertical motion of the ball. Since distance is given in feet, the equation of motion is $y''(t) = -32$. The initial speed must be converted to feet per second: $120 \text{ mph} = \frac{120 \times 5280}{3600} \text{ ft/s} = 176 \text{ ft/s}$. The vertical component of the initial velocity is then $y'(0) = 176 \sin(-5^\circ) \approx -15.34 \text{ ft/s}$. Integration then gives us

$$y'(t) = -32t - 15.34.$$
The initial height is \( y(0) = 8 \text{ ft} \), so another integration gives us

\[
y(t) = -16t^2 - 15.34t + 8 \text{ ft}.\]

The horizontal component of motion is determined from \( x''(t) = 0 \) with initial velocity \( x'(0) = 176 \cos(-5^\circ) \approx 175.33 \text{ ft/s} \) and initial position \( x(0) = 0 \). Integrations give us \( x'(t) = 175.33 \text{ ft/s} \) and \( x(t) = 175.33t \text{ ft}. \) Summarizing, we have

\[
x(t) = 175.33t, \\
y(t) = -16t^2 - 15.34t + 8. \\
\]

For the ball to clear the net, \( y \) must be at least 3 when \( x = 39 \). We have \( x(t) = 39 \) when \( 175.33t = 39 \) or \( t \approx 0.2224 \). At this time, \( y(0.2224) \approx 3.8 \), showing that the ball is high enough to clear the net. The second requirement is that we need to have \( x \leq 60 \) when the ball lands \( (y = 0) \). We have \( y(t) = 0 \) when \( -16t^2 - 15.34t + 8 = 0 \). From the quadratic formula, we get \( t \approx -1.33 \) and \( t \approx 0.3749 \). Ignoring the negative solution, we compute \( x(0.3749) \approx 65.7 \), so that the serve lands beyond the service line. The serve is not in.

One reason for you to start each problem with Newton’s second law is to force yourself to consider the forces that are (and are not) being considered. For example, we have thus far ignored air resistance. It is important to realize that this is a simplification of reality. Some calculations using such simplified equations are reasonably accurate. Others, such as in the following example, are not.

**Example 5.6**  
An Example Where Air Resistance Can’t Be Ignored

Suppose a raindrop falls from a cloud 3000 feet above the ground. Ignoring air resistance, how fast would the raindrop be falling when it hits the ground?

**Solution**  
If the height of the raindrop at time \( t \) is given by \( y(t) \), Newton’s second law of motion tells us that \( y''(t) = -32 \). Further, we have the initial velocity \( y'(0) = 0 \) (since the drop falls as opposed to being thrown down) and the initial altitude \( y(0) = 3000 \). Integrating and using the initial conditions gives us \( y'(t) = -32t \) and \( y(t) = 3000 - 16t^2 \). The raindrop hits the ground when \( y(t) = 0 \). Setting

\[
0 = y(t) = 3000 - 16t^2, \\
\]

we find

\[
t = \sqrt{\frac{3000}{16}} = 25 \sqrt{2} \approx 35.36 \text{ s}. \\
\]

Thus, the raindrop would be falling at a speed of \( 32 \times 25 \sqrt{2} \approx 353.6 \text{ ft/s} \) when it hits the ground.
gives us \( t = \sqrt{\frac{3000}{16}} \approx 13.693 \) seconds. The velocity at this time is then
\[
y'(\sqrt{\frac{3000}{16}}) = -32\sqrt{\frac{3000}{16}} \approx -438.18 \text{ ft/s}.
\]
This corresponds to nearly 300 mph! Fortunately, air resistance does play a significant role in the fall of a raindrop, which has an actual landing speed of about 10 mph.

The obvious lesson from example 5.6 is that it is not always reasonable to ignore air resistance. Some of the mathematical tools needed to more fully analyze projectile motion with air resistance are developed in Chapter 6.

The air resistance (more precisely, air drag) that slows the raindrop down is only one of the ways in which air can affect the motion of an object. The Magnus force, produced by the spinning of an object or asymmetries (i.e., lack of symmetry) in the shape of an object, can cause the object to change directions and curve. Perhaps the most common example of a Magnus force occurs on an airplane. One side of an airplane wing is curved and the other side is comparatively flat (see Figure 5.47). The lack of symmetry causes the air to move over the top of the wing faster than it moves over the bottom. This produces a Magnus force in the upward direction (lift), lifting the airplane into the air.

A more down-to-earth example of a Magnus force occurs in an unusual baseball pitch called the knuckleball. To throw this pitch, the pitcher grips the ball with all five fingers and tries to throw the ball with as little spin as possible (despite the name, the knuckles are not used). Baseball players claim that the knuckleball “dances around” unpredictably and is exceptionally hard to hit or catch. There still is no complete agreement on why the knuckleball moves so much, but we will present one current theory due to physicists Robert Watts and Terry Bahill.

The cover of the baseball is sewn on with stitches that are raised up slightly from the rest of the ball. These curved stitches act much like an airplane wing, creating a Magnus force that affects the ball. The direction of the Magnus force depends on the exact orientation of the ball’s stitches. Measurements by Watts and Bahill indicate that the lateral force (left/right from the pitcher’s perspective) is approximately \( F_m = -0.1 \sin(4\theta) \text{ lb} \), where \( \theta \) is the angle in radians of the ball’s position rotated from a particular starting position.

Since gravity does not affect the lateral motion of the ball, the only force acting on the ball laterally is the Magnus force. Newton’s second law applied to the lateral motion of the knuckleball gives \( m \ddot{x}(t) = -0.1 \sin(4\theta) \). The mass of a baseball is about 0.01 slugs. (If you are unfamiliar with slugs, these are the units of measurement of mass in the English system. To get the more familiar weight in pounds, simply multiply the mass by \( g = 32 \).) We now have
\[
\ddot{x}(t) = -10 \sin(4\theta).
\]
If the ball is spinning at the rate of \( \omega \) radians per second, then \( 4\theta = 4\omega t + \theta_0 \), where the initial angle \( \theta_0 \) depends on where the pitcher grips the ball. Our model is then
\[
\ddot{x}(t) = -10 \sin(4\omega t + \theta_0),
\tag{5.1}
\]
with initial conditions \( x'(0) = 0 \) and \( x(0) = 0 \). For a typical knuckleball speed of 60 mph, it takes about 0.68 second for the pitch to reach home plate.

\textbf{Example 5.7} \hspace{1cm} \textbf{An Equation for the Motion of a Knuckleball}

For a spin rate of \( \omega = 2 \) radians per second and \( \theta_0 = 0 \), find an equation for the motion of the knuckleball and graph it for \( 0 \leq t \leq 0.68 \). Repeat this for \( \theta_0 = \pi/2 \).
3. In example 5.4, we derived separate equations for the horizontal and vertical components of position. To discover one consequence of this separation, consider the following situation. Two people are standing next to each other with arms raised to the same height. One person fires a bullet horizontally from a gun. At the same time, the other person drops a bullet. Explain why the bullets will hit the ground at the same time.

4. For the falling raindrop in example 5.6, a more accurate model would be $y''(t) = -32 + f(t)$, where $f(t)$ represents the force due to air resistance (divided by the mass). If $v(t)$ is the downward velocity of the raindrop, explain why this equation is equivalent to $v'(t) = 32 - f(t)$. Explain in physical terms why the larger $v(t)$ is, the larger $f(t)$ is. Thus, a model such as $f(t) = v(t)$ or $f(t) = [v(t)]^2$ would be reasonable. (In most situations, it turns out that $[v(t)]^2$ matches the experimental data better.)
In exercises 5–8, identify the initial conditions \( y(0) \) and \( y'(0) \).

5. An object is dropped from a height of 80 feet.
6. An object is dropped from a height of 100 feet.
7. An object is released from a height of 60 feet with an upward velocity of 10 ft/s.
8. An object is released from a height of 20 feet with a downward velocity of 4 ft/s.

In exercises 9–57, ignore air resistance.

9. A diver starts 30 feet above the water (about the height of an Olympic platform diver). What is the diver’s velocity at impact?
10. A diver starts 120 feet above the water (about the height of divers at the Acapulco Cliff Diving competition). What is the diver’s velocity at impact?
11. For the diver of example 5.1, suppose that \(-32\) ft/s is the impact velocity of the diver’s hands. If the diver measures 6 feet from hands to feet (see the figure), the feet would fall 6 feet more before hitting the water. Find the impact velocity of an object falling from a height of 6 feet with initial velocity \(-32\) ft/s.

12. Give at least one reason why the impact velocity of exercise 11 would not actually be the impact velocity of the diver’s feet.
13. Compare the impact velocities of objects falling from 30 feet (exercise 9), 120 feet (exercise 10) and 3000 feet (example 5.5). If height is increased by a factor of \(h\), by what factor does the impact velocity increase?
14. The Washington monument is 555 feet, \(5\frac{1}{4}\) inches high. In a famous experiment, a baseball was dropped from the top of the monument to see if a player could catch it. How high would the ball be going?
15. A certain not-so-wily coyote discovers that he just stepped off the edge of a cliff. Four seconds later, he hits the ground in a puff of dust. How high was the cliff?
16. A large boulder dislodged by the falling coyote in exercise 15 falls for 3 seconds before landing on the coyote. How far did the boulder fall? What was its velocity when it flattened the coyote?
17. The coyote’s next scheme involves launching himself into the air with an Acme catapult. If the coyote is propelled vertically from the ground with initial velocity 64 ft/s, find an equation for the height of the coyote at any time \(t\). Find his maximum height, the amount of time spent in the air and his velocity when he smacks back into the catapult.
18. On the rebound, the coyote in exercise 13 is propelled to a height of 256 feet. What is the initial velocity required to reach this height?
19. One of the authors has a vertical “jump” of 20 inches. What is the initial velocity required to jump this high? How does this compare to Michael Jordan’s velocity, found in example 5.3?
20. If the author underwent an exercise program and increased his initial velocity by 10%, by what percentage would he increase his vertical jump?
21. A typical flea jumps vertically with an initial velocity of about 5 ft/s. How high does the flea jump?
22. Compute the ratio of the vertical jump of the flea in exercise 21 to its height (about \(\frac{1}{12}\) inch). If the (6-foot-tall) author in exercise 19 could maintain the same ratio, how high would he jump?
23. Show that an object dropped from a height of \(H\) feet will hit the ground at time \(T = \frac{1}{4}\sqrt{H}\) seconds with impact velocity \(V = -8\sqrt{H}\) ft/s.
24. Show that an object propelled from the ground with initial velocity \(v_0\) ft/s will reach a maximum height of \(v_0^2/64\) ft.

In exercises 25–42, sketch the parametric graphs as in example 5.4 to indicate the flight path.

25. An object is launched at angle \(\theta = \pi/3\) radians from the horizontal with an initial speed of 98 m/s. Determine the time of flight and the horizontal range.

26. Compare the launch angles, initial speeds and ranges of the objects in exercise 25 and example 5.4.

27. Find the time of flight and horizontal range of an object launched at angle 30° with initial speed 120 ft/s.
28. Find the time of flight and horizontal range of an object launched at angle 60° with initial speed 120 ft/s. Compare the ranges in exercises 27 and 28.

29. Repeat example 5.5 with an initial angle of 6°.

30. By trial and error find the smallest and largest angles for which the serve of example 5.5 will be in.

31. Repeat example 5.5 with an initial speed of 170 ft/s.

32. By trial and error find the smallest and largest initial speeds for which the serve of example 5.5 will be in.

33. A baseball pitcher releases the ball horizontally from a height of 6 ft with an initial speed of 130 ft/s. Find the height of the ball when it reaches home plate 60 feet away. (Hint: Determine the time of flight from the x-equation, then use the y-equation to determine the height.)

34. Repeat exercise 33 with an initial speed of 80 ft/s. (Hint: Carefully interpret the negative answer.)

35. A baseball player throws a ball toward first base 120 feet away. The ball is released from a height of 5 feet with an initial speed of 120 ft/s at an angle of 5° above the horizontal. Find the height of the ball when it reaches first base.

36. Repeat exercise 35 with an initial angle of 10° above the horizontal.

37. By trial and error, find the angle at which the ball in exercises 35 and 36 will reach first base at the catchable height of 5 feet. At this angle, how far above the first baseman’s head would the thrower be aiming?

38. In exercise 35, if the ball is released horizontally, how far from first base would the ball first bounce?

39. A daredevil plans to jump over 25 cars. If the cars are all compact cars with a width of 5 feet and the ramp angle is 30°, determine the initial velocity required to complete the jump successfully.

40. Repeat exercise 39 with a takeoff angle of 45°. In spite of the reduced initial velocity requirement, why might the daredevil prefer an angle of 30° to 45°?

41. A plane at an altitude of 256 feet wants to drop supplies to a specific location on the ground. If the plane has a horizontal velocity of 100 ft/s, how far away from the target should the plane release the supplies in order to hit the target location? (Hint: Use the y-equation to determine the time of flight, then use the x-equation to determine how far the supplies will drift.)

42. Repeat exercise 41 for an altitude of 144 feet.

43. Consider a knuckleball (see example 5.7) with lateral motion satisfying the initial value problem \( x''(t) = -25 \sin(4 \cot \theta_0) \), \( x'(0) = x(0) = 0 \). With \( \theta_0 = 0 \) and \( \omega = 1 \), find an equation for \( x(t) \) and graph the solution for \( 0 \leq t \leq 0.68 \).

44. Repeat exercise 43 for \( \theta_0 = \pi/2 \) and \( \omega = 1 \).

45. Repeat exercise 43 for \( \theta_0 = \pi/4 \) and \( \omega = 2 \).

46. Repeat exercise 43 for \( \theta_0 = \pi/4 \) and \( \omega = 1 \).

47. For the Olympic diver in exercise 9, what would be the average angular velocity (measured in radians per second) necessary to complete 2 1/2 somersaults?

48. In the Flying Zucchini Circus’ human cannonball act, a performer is shot out of a cannon from a height of 10 feet at an angle of 45° with an initial speed of 160 ft/s. If the safety net stands 5 feet above the ground, how far should the safety net be placed from the cannon? If the safety net can only withstand an impact velocity of 160 ft/s, will the Flying Zucchini land safely or come down squash?

49. In a basketball free throw, a ball is shot from a height of \( h \) feet toward a basket 10 feet above the ground at a horizontal distance of 15 feet. If \( h = 6 \), \( \theta = 52° \) and \( v_0 = 25 \) ft/s, show that the free throw is good. Since the basket is larger than the ball, a free throw has a margin of error of several inches. If any shot that passes through height 10 ft with \( 14.65 \leq x \leq 15.35 \) is good, show that, for the given speed \( v_0 \), the margin of error is \( 48° \leq \theta \leq 57° \). Sketch parametric graphs to show several of these free throws.

50. For the basketball shot in exercise 49, fix \( h = 6 \) and \( \theta = 52° \). Find the range of speeds \( v_0 \) for which the free throw will be good. Given that it requires a force of \( F = 0.01v_0^2 \) pounds to launch a free throw with speed \( v_0 \), how much margin of error is there in the applied force?
51. Soccer player Roberto Carlos of Brazil is known for his curving kicks. Suppose that he has a free kick from 30 yards out. Orienting the x- and y-axes as shown below, suppose the kick has initial speed 100 ft/s at an angle of 5° from the positive y-axis. Assume that the only force on the ball is a Magnus force to the left caused by the spinning of the ball. With \( x''(t) = -20 \) and \( y''(t) = 0 \), determine if the ball goes in the goal at \( y = 90 \) and \(-24 \leq x \leq 0 \).

![Diagram of soccer kick](image)

52. For the kick in exercise 51, how far to the right of the goal would the kick have gone without spin?

53. To train astronauts to operate in a weightless environment, NASA sends them up in a special plane (nicknamed the Vomit Comet). To allow the passengers to experience weightlessness, the vertical acceleration of the plane must exactly match the acceleration due to gravity. If \( y''(t) \) is the vertical acceleration of the plane, then \( y''(t) = -g \). Show that, for a constant horizontal velocity, the plane follows a parabolic path. NASA’s plane flies parabolic paths of approximately 2500 feet in height (2500 feet up and 2500 down). The time to complete such a path is the amount of weightless time for the passengers. Compute this time.

54. In a typical golf tee shot, the ball is launched at an angle of 9.3° at an initial speed of 220 ft/s. In the absence of air resistance, how far (horizontally) would this shot carry? The actual carry on such a shot is about 240 yards (720 feet)! In this case, a backspin of 4000 rpm gives the ball a huge upward Magnus force, which offsets most of the air resistance and gravity.

55. In the 1992 Summer Olympics in Barcelona, Spain, an archer lit the Olympic flame by shooting an arrow towards a cauldron at a distance of about 70 meters horizontally and 30 meters vertically. If the arrow reached the cauldron at the peak of its trajectory, determine the initial speed and angle of the arrow. (Hint: Show that \( y'(t) = 0 \) if \( t = (v_0 \sin \theta)/9.8 \). For this \( t \), show that \( x(t) = 2\cot \theta = \frac{7}{3} \) and solve for \( \theta \). Then solve for \( v_0 \).)

56. Professional jugglers generally agree that 10 is the maximum number of balls that a human being can successfully maintain. To get an idea why, suppose that it takes 1-second to catch and toss a ball up in the air (in other words, the juggler can process 2 balls per second). To juggle 10 balls, then, each ball would have to be in the air for 5 seconds. How high would the ball have to be tossed to stay in the air this long? Given this height, does 1-second per ball sound like much time? How much higher would the balls have to be tossed to juggle 11 balls?

57. In the text and exercises 43–46, we discussed the differential equation \( x''(t) = -25 \sin(4 \omega t + \theta_0) \) for the lateral motion of a knuckleball. Integrate and apply the initial conditions \( x'(0) = 0 \) and \( x(0) = 0 \) to derive the general equation \( x(t) = \frac{25}{16\omega^2} \sin(4 \omega t + \theta_0) - \left(\frac{25}{4 \omega}\cos \theta_0\right)t - \frac{25}{16\omega^2} \sin \theta_0 \). If you have access to three-dimensional graphics, graph \( x(t, \omega) \) for \( \theta_0 = 0 \) with \( 0 \leq t \leq 0.68 \) and \( 0.01 \leq \omega \leq 10 \). (Note: Some plotters will have trouble with \( \omega = 0 \).) Repeat with \( \theta_0 = \pi/4, \theta_0 = \pi/2 \) and two choices of your own for \( \theta_0 \). A pitcher wants the ball to move as much as possible back and forth but end up near home plate \( (x = 0) \). Based on these criteria, pick the combinations of \( \theta_0 \) and \( \omega \) that produce the four best pitches. Graph these pitches in two dimensions with \( x = x(t) \) as in Figures 5.48a and 5.48b.

58. Although we have commented on some inadequacies of the gravity-only model of projectile motion, we have not presented any alternatives. Such models tend to be somewhat more mathematically complex. The model explored in this exercise takes into account air resistance in a way that is mathematically tractable but still not very realistic. Assume that the force of air resistance is proportional to the speed and acts in the opposite direction of velocity. For a horizontal motion (no gravity), we have \( a(t) = F(t)/m = -cv(t) \) for some constant \( c \). Explain what the minus sign indicates. Since \( a(t) = v'(t) \), the model is \( v'(t) = -cv(t) \). Show that the function \( v(t) = v_0 e^{-ct} \) satisfies the equation \( v'(t) = -cv(t) \) and the initial condition \( v(0) = v_0 \). If an object starts at \( x(0) = a \), integrate \( v(t) = v_0 e^{-ct} \) to find its position at any time \( t \). Show that the amount of time needed to reach \( x = b \) (for \( a < b \)) is given by \( T = \frac{1}{c} \ln \left(1 - \frac{b-a}{v_0} \right) \). For a baseball \( (c = 0.15) \) thrown at 125 ft/s from \( a = 0 \), determine how long it takes to reach \( b = 60 \) and compute its velocity at that point. By what percentage has the velocity decreased? In baseball, two different types of radar guns are used to measure the velocity of a pitch. One measures the speed of the ball right as it leaves the pitcher’s hand. The other measures the speed of the ball part way to home plate. If the first gun registers 94 mph and the second registers 89 mph, how far from the plate does the second gun make its measurement?
5.6 WORK, MOMENTS AND HYDROSTATIC FORCE

In this section, we explore several applications of integration in physics. In each case, we will define a basic concept to help solve a specific problem. We’ll then use the definite integral to generalize the concept to solve a much wider range of problems. This use of integration is an excellent example of how the mathematical theory helps you find solutions of practical problems.

Imagine that you are at the bottom of a snow-covered hill with a sled. To get a good ride, you want to push the sled as far up the hill as you can. A physicist would say that the higher up you are, the more potential energy you have. Sliding down the hill converts the potential energy into kinetic energy. (This is the fun part!) But pushing the sled up the hill requires you to do some work: you must exert a force over a long distance.

Physicists have studied how things like force and speed are related. While it’s certainly clear that increasing the force in the direction of motion will increase the speed, there is much more to it than that. One way in which physicists simplify the process is by identifying a small number of key quantities to keep track of. In this section, we will discuss three of these: work, impulse and moments.

Our first task is to quantify work. If you push a sled up a hill, you’re doing work, but can you give a measure of how much? Certainly, if you push twice the weight (i.e., exert twice the force), you’re doing twice the work. So, it should seem reasonable that work is directly proportional to the force you’re exerting. Further, if you push the sled twice as far, you’ve done twice the work. That is, work is directly proportional to the distance over which you exert the force. In view of these observations, for any constant force \( F \) applied over a distance \( d \), we define the work \( W \) done as

\[
W = Fd.
\]

Unfortunately, forces are generally not constant. For instance, think of a rocket that burns its fuel in flight. Since its mass is not constant, neither is the force exerted on the rocket by gravity. We can extend this notion of work to the case of a nonconstant force \( F(x) \) applied on the interval \([a, b]\) as follows. First, we divide the interval \([a, b]\) into \( n \) equal subintervals, each of width \( \Delta x = \frac{b-a}{n} \) and consider the work done on each subinterval.

Notice that if \( \Delta x \) is small, then the force \( F(x) \) applied on the interval \([x_{i-1}, x_i]\) can be approximated by the constant force \( F(c_i) \) for some point \( c_i \in [x_{i-1}, x_i] \). The work done moving the object along the subinterval is then approximately \( F(c_i) \Delta x \). The total work done \( W \) is then approximately

\[
W \approx \sum_{i=1}^{n} F(c_i) \Delta x.
\]

You should recognize that this is a Riemann sum and that as \( n \) gets larger, the Riemann sums should approach the actual work. Taking the limit as \( n \to \infty \) gives us

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} F(c_i) \Delta x = \int_{a}^{b} F(x) \, dx.
\]

We take (6.1) as our definition of work.

You’ve probably noticed that the further a spring is compressed (or stretched) from its natural resting position, the more force is required to further compress (or stretch) the spring. According to **Hooke’s Law**, the force required to maintain a spring in a given position is proportional to the distance it’s compressed (or stretched). That is, if \( x \) is the distance
a spring is compressed (or stretched) from its natural length, the force $F(x)$ exerted by the spring is given by

$$F(x) = kx,$$

for some constant $k$ (the spring constant).

### Example 6.1  Computing the Work Done Stretching a Spring

A force of 3 pounds is required to hold a spring that has been stretched $\frac{1}{4}$ foot from its natural length (see Figure 5.49). Find the work done in stretching the spring 6 inches beyond its natural length.

**Solution**  First, we have from Hooke’s Law (6.2) that

$$3 = F\left(\frac{1}{4}\right) = k\left(\frac{1}{4}\right),$$

so that $k = 12$ and $F(x) = 12x$. From (6.1), the work done in stretching the spring 6 inches (1/2 foot) is then

$$W = \int_{0}^{1/2} F(x) \, dx = \int_{0}^{1/2} 12x \, dx = \frac{3}{2} \text{ foot-pounds}.$$

In this case, notice that stretching the spring transfers potential energy to the spring (if the spring is later released, it springs back toward its resting position and converts the potential energy to kinetic energy).

### Example 6.2  Computing the Work Done by a Weightlifter

A weightlifter lifts a 200-pound barbell a distance of 3 feet. How much work was done? Also, determine the work done by the weightlifter if the weight is raised 4 feet above the ground and then lowered back into place.

**Solution**  Since the force (the weight) is constant here, we simply have

$$W = Fd = 200 \times 3 = 600 \text{ foot-pounds}.$$

On the other hand, find the amount of work done if the weightlifter lifts the same weight 4 feet from the ground and then lowers it back into place. It may seem strange, but since the barbell ends up in the same place as it started, the net distance covered is zero and the work done is zero. Of course, it would feel like work to the weightlifter, but this is where the mathematical notion of work differs from the usual use of the word. As we have defined it, work accounts for the energy change in the object having work done on it. Since the barbell has the same kinetic and potential energy that it started with, the total work done on it is zero.

In the following example, both the force and the distance are nonconstant. This presents some unique challenges and we’ll need to first approximate the work and then recognize the definite integral that this approximation process generates.
A spherical tank of radius 10 feet is filled with water. Find the work done in pumping all of the water out through the top of the tank.

Solution

You should observe that the basic formula $W = Fd$ does not directly apply here, for several reasons. The most obvious reason is that the distance traveled by the water in each part of the tank is different, as the water toward the bottom of the tank must be pumped all the way to the top, while the water near the top of the tank must only be pumped a short distance. To envision this, we partition the water tank into $n$ thin layers, each of depth $\Delta x = \frac{20}{n}$. Let $x$ represent distance as measured from the bottom of the tank, as indicated in Figure 5.50a. Then, the entire tank corresponds to the interval $0 \leq x \leq 20$, which we partition into

$$0 = x_0 < x_1 < \cdots < x_n = 20,$$

where $x_i - x_{i-1} = \Delta x$, for each $i = 1, 2, \ldots, n$. Notice that this partitions the tank into $n$ layers, each corresponding to an interval $[x_{i-1}, x_i]$ (see Figure 5.50b). You can think of the water in the layer corresponding to $[x_{i-1}, x_i]$ as being approximately cylindrical, of height $\Delta x$. This layer must be pumped a distance of approximately $20 - c_i$, for some $c_i \in [x_{i-1}, x_i]$. (You might think of taking $c_i$ as the midpoint of the interval, although the choice of $c_i$ doesn’t really matter.) Notice from Figure 5.50b that the radius of the $i$th layer depends on the value of $x$. From Figure 5.50c, (where we show a cross section of the tank) the radius $r_i$ corresponding to a depth of $x = c_i$ is the base of a right triangle with hypotenuse 10 and height $|10 - c_i|$. From the Pythagorean Theorem, we now have

$$(10 - c_i)^2 + r_i^2 = 10^2.$$

Solving this for $r_i^2$, we have

$$r_i^2 = 10^2 - (10 - c_i)^2 = 100 - (100 - 20c_i + c_i^2) = 20c_i - c_i^2.$$

The force $F_i$ required to move the $i$th layer is then simply the force exerted on the water by gravity (i.e., its weight). For this we will need to know the weight density of
water: 62.4 lb/ft³. We now have
\[ F_i \approx (\text{Volume of cylindrical slice})(\text{weight of water per unit volume}) \]
\[ = (\pi r_i^2 h)(62.4 \text{ lb/ft}^3) \]
\[ = 62.4 \pi (20c_i - c_i^2) \Delta x. \]
The work required to pump out the \( i \)th slice is then given approximately by
\[ W_i \approx (\text{Force})(\text{distance}) \]
\[ = 62.4 \pi (20c_i - c_i^2) \Delta x (20 - c_i) \]
\[ = 62.4 \pi c_i (20 - c_i)^2 \Delta x. \]
The work required to pump out all of the water is then the sum of the work required for each of the \( n \) slices:
\[ W \approx \sum_{i=1}^{n} 62.4 \pi c_i (20 - c_i)^2 \Delta x. \]
Finally, taking the limit as \( n \to \infty \) gives the exact work, which you should recognize as a definite integral:
\[ W = \lim_{n \to \infty} \sum_{i=1}^{n} 62.4 \pi c_i (20 - c_i)^2 \Delta x = \int_0^{20} 62.4 \pi x (20 - x)^2 \, dx \]
\[ = 62.4 \pi \int_0^{20} (400x - 40x^2 + x^3) \, dx \]
\[ = 62.4 \pi \left[ 400 \frac{x^2}{2} - 40 \frac{x^3}{3} + \frac{x^4}{4} \right]_0^{20} \]
\[ = 62.4 \pi \left( \frac{40,000}{3} \right) \approx 2.61 \times 10^6 \text{ foot-pounds.} \]

**Impulse** is a physical quantity closely related to work. Instead of relating force and distance to account for changes in energy, impulse relates force and time to account for changes in velocity. First, suppose that a constant force \( F \) is applied to an object from time \( t = 0 \) to time \( t = T \). If the position of the object at time \( t \) is given by \( x(t) \), then Newton’s second law says that \( F = ma = m\ddot{x}(t) \). Integrating this equation once with respect to \( t \) gives us
\[ \int_0^T F \, dt = m \int_0^T \dddot{x}(t) \, dt, \]
or
\[ F(T - 0) = m[x'(T) - x'(0)]. \]
Recall that \( x'(t) \) is the velocity \( v(t) \), so that
\[ FT = m[v(T) - v(0)] \]
or \( FT = m\Delta v \), where \( \Delta v = v(T) - v(0) \) is the change in velocity. The quantity \( FT \) is called the **impulse**, \( mv(T) \) is the **momentum** at time \( t \) and the equation relating the impulse to the change in velocity is called the **impulse-momentum equation**.
Since we defined the impulse for a constant force, we must now generalize the notion to the case of a nonconstant force. Think about this some and try to guess what the definition should be.

We define the impulse $J$ of a force $F(t)$ applied over the time interval $[a, b]$ to be

$$J = \int_a^b F(t) \, dt.$$ 

Note the similarities and differences between work and impulse. We leave the derivation of the impulse integral for the case of a nonconstant force as an exercise. The impulse-momentum equation also generalizes to the case of a nonconstant force:

$$J = m[v(b) - v(a)].$$

### Example 6.4  Estimating the Impulse for a Baseball

Suppose that a baseball traveling at 130 ft/s (about 90 mph) collides with a bat. The following data (adapted from *The Physics of Baseball* by Robert Adair) shows the force exerted by the bat on the ball at 0.0001 second intervals.

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>0</th>
<th>0.0001</th>
<th>0.0002</th>
<th>0.0003</th>
<th>0.0004</th>
<th>0.0005</th>
<th>0.0006</th>
<th>0.0007</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(t)$ (lb)</td>
<td>0</td>
<td>1250</td>
<td>4250</td>
<td>7500</td>
<td>9000</td>
<td>5500</td>
<td>1250</td>
<td>0</td>
</tr>
</tbody>
</table>

Estimate the impulse of the bat on the ball and (using $m = 0.01$ slugs) the speed of the ball after impact.

**Solution**  In this case, the impulse $J$ is given by $\int_0^{0.0007} F(t) \, dt$. However, we can’t evaluate this integral, as we don’t know the force function $F(t)$. Since we’re given only a fixed number of measurements of $F(t)$, the best we can do is approximate the integral numerically (e.g., using Simpson’s Rule). Recall that Simpson’s Rule requires an odd value for $n + 1$, the number of points in the partition, which means that you need an even number $n$ of subintervals. (Think about this some!) Using $n = 8$ and adding a 0 function value at $t = 0.0008$ (why is it fair to do this?), Simpson’s Rule gives us

$$J \approx [0 + 4(1250) + 2(4250) + 4(7500) + 2(9000) + 4(5500) + 2(1250) + 4(0)] \frac{0.0001}{3}$$

$$\approx 2.866.$$ 

In this case, the impulse-momentum equation $J = m \Delta v$ becomes $2.866 = 0.01 \Delta v$ or $\Delta v = 286.6$ ft/s. Since the ball started out with a speed of 130 ft/s in one direction and it ended up traveling in the opposite direction, it ended up traveling at a speed of 156.6 ft/s.

The concept of the first moment, like work, involves force and distance. Moments are used to solve problems of balance and rotation. We start with a simple problem involving two children on a playground seesaw (or teeter-totter). Suppose that the child on the left in Figure 5.51a is heavier (i.e., has larger mass) than the child on the right. If the children sit an equal distance from the pivot point, you know what will happen: the left side will be pulled down and the lighter child will be raised into the air. You may also know that the

![Figure 5.51a](Balancing two masses.)
children can balance each other if the heavier child moves closer to the pivot point. That is, the balance is determined both by weight (force) and distance from the pivot point. It turns out that balance is easy to describe mathematically. If the children have masses $m_1$ and $m_2$ and are sitting at distances $d_1$ and $d_2$, respectively, from the pivot point, then they balance each other if and only if

$$m_1 d_1 = m_2 d_2.$$  

(6.3)

Let’s turn the problem around slightly. Suppose there are two objects, of mass $m_1$ and $m_2$, located at different places, say $x_1$ and $x_2$, respectively, with $x_1 < x_2$. We consider the objects to be point-masses. That is, each is treated as a single point, with all of the mass concentrated at that point (see Figure 5.51b).

Suppose that you want to find the center of mass $\bar{x}$, that is, the location at which you could place the pivot of a seesaw and have the objects balance. From the balance equation (6.3), you’ll need $m_1(x - x_1) = m_2(x_2 - x)$. Solving this equation for $x$ gives us

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$  

Notice that the denominator in this equation for the center of mass is the total mass of the “system” (i.e., the total mass of the two objects). The numerator of this expression is called the first moment of the system.

More generally, for a system of $n$ masses $m_1, m_2, \ldots, m_n$, located at $x = x_1, x_2, \ldots, x_n$, respectively, the center of mass $\bar{x}$ is given by the first moment divided by the total mass, that is,

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.$$  

Rather than compute the mass of a system of discrete objects, suppose instead that we wish to find the mass and center of mass of an object of variable density that extends from $x = a$ to $x = b$. Here, we assume that the density function $\rho(x)$ (measured in units of mass per unit length) is known. In the simplest possible case, note that if the density is a constant $\rho$, the mass of the object is simply given by $m = \rho L$, where $L$ is the length of the object ($L = b - a$). On the other hand, if the density varies throughout the object, it is not immediately clear how to compute its mass. Since we don’t know how to find this exactly, we first settle for finding it approximately. As we have done many times now, we begin by dividing the interval $[a, b]$ into $n$ pieces of equal width $\Delta x = \frac{b - a}{n}$. On each subinterval $[x_{i-1}, x_i]$, notice that the mass is approximately $\rho(c_i) \Delta x$, where $c_i$ is a point in the subinterval and $\Delta x$ is the length of the subinterval. The total mass is then approximately

$$m \approx \sum_{i=1}^{n} \rho(c_i) \Delta x.$$  

You should observe that this is a Riemann sum that approaches the total mass as $n \to \infty$. Taking the limit as $n \to \infty$, we get that

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(c_i) \Delta x = \int_{a}^{b} \rho(x) \, dx.$$  

(6.4)
A 30-inch baseball bat can be represented approximately by an object extending from $x = 0$ to $x = 30$ inches, with density $ho(x) = \left(\frac{1}{46} + \frac{x}{690}\right)^2$ slugs per inch. The density takes into account the fact that a baseball bat is similar to an elongated cone. Find the mass of the object.

Solution

From (6.4), the mass is given by

$$m = \int_0^{30} \left(\frac{1}{46} + \frac{x}{690}\right)^2 dx$$

$$= \frac{690}{3} \left[\left(\frac{1}{46} + \frac{30}{690}\right)^3 - \left(\frac{1}{46}\right)^3\right]$$

$$\approx 6.144 \times 10^{-2} \text{ slugs.}$$

To make the result more meaningful, you might compute the weight (in ounces). To do this, multiply the mass by $32 \cdot 16$. You should find that the bat weighs roughly 31.5 ounces.

To compute the first moment for an object of nonconstant density $\rho(x)$ extending from $x = a$ to $x = b$, we again divide the interval into $n$ equal pieces. From our earlier argument, for each $i = 1, 2, \ldots, n$, the mass of the $i$th slice of the object is approximately $\rho(c_i) \Delta x$, for any choice of $c_i \in [x_{i-1}, x_i]$. We then represent the $i$th slice of the object with a particle of mass $m_i = \rho(c_i) \Delta x$ located at $x = c_i$. We can now think of the original object as having been approximated by $n$ distinct point-masses, as indicated in Figure 5.52.

.notice that the first moment $M_n$ of this approximate system is

$$M_n = [\rho(c_1) \Delta x]c_1 + [\rho(c_2) \Delta x]c_2 + \cdots + [\rho(c_n) \Delta x]c_n$$

$$= [c_1 \rho(c_1) + c_2 \rho(c_2) + \cdots + c_n \rho(c_n)] \Delta x = \sum_{i=1}^{n} c_i \rho(c_i) \Delta x.$$

Taking the limit as $n \to \infty$, the sum approaches the first moment

$$M = \lim_{n \to \infty} \sum_{i=1}^{n} c_i \rho(c_i) \Delta x = \int_a^b x \rho(x) \, dx. \quad (6.5)$$

The center of mass of the object is then given by

$$\bar{x} = \frac{M}{m} = \frac{\int_a^b x \rho(x) \, dx}{\int_a^b \rho(x) \, dx}. \quad (6.6)$$
Find the center of mass of the baseball bat from example 6.5.

**Solution**  From (6.5), the first moment is given by

\[ M = \int_{0}^{30} x \left( \frac{1}{46} + \frac{x}{690} \right)^2 \, dx = \left[ \frac{x^2}{4232} + \frac{x^3}{47,610} + \frac{x^4}{1,904,400} \right]_{0}^{30} \approx 1.205. \]

Recall that we had already found the mass to be \( m \approx 6.144 \times 10^{-2} \) slugs and so, from (6.6), the center of mass of the bat is

\[ \bar{x} = \frac{M}{m} \approx \frac{1.205}{6.144 \times 10^{-2}} \approx 19.6 \text{ inches}. \]

Note that for a baseball bat, the center of mass is one candidate for the so-called “sweet spot” of the bat, the best place to hit the ball.

For our final application of integration in this section, we consider hydrostatic force. Imagine a dam holding back a lake full of water. What force must the dam be built to withstand?

As usual, we solve a simpler problem first. If you have a flat rectangular plate oriented horizontally underwater, notice that the force exerted on the plate by the water (the hydrostatic force) is simply the weight of the water lying above the plate. To find this, we must only find the volume of the water lying above the plate and multiply this by the weight density of water (62.4 lb/ft³). If the area of the plate is \( A \) ft² and it lies \( d \) ft below the surface (see Figure 5.53), then the force on the plate is

\[ F = 62.4Ad. \]

According to Pascal’s Principle, the pressure at a given depth \( d \) in a fluid is the same in all directions. This says that if a flat plate is submerged in a fluid, then the pressure on one side of the plate at any given point is \( \rho \cdot d \), where \( \rho \) is the weight density of the fluid and \( d \) is the depth. In particular, this says that it’s irrelevant whether the plate is submerged vertically, horizontally or otherwise (see Figure 5.54).

Consider now a vertically oriented wall (a dam) holding back a lake. It is convenient to orient the \( x \)-axis vertically with \( x = 0 \) located at the surface of the water and the bottom of the wall at \( x = a > 0 \) (see Figure 5.55). In this way, notice that \( x \) measures the depth of a section of the dam. Suppose \( w(x) \) is the width of the wall at depth \( x \) (where all distances are measured in feet).

Partition the interval \([0, a]\) into \( n \) subintervals of equal width \( \Delta x = \frac{a}{n} \). Observe that this has the effect of slicing the dam into \( n \) slices, each of width \( \Delta x \). We now consider the force acting on each slice of the dam. For each \( i = 1, 2, \ldots, n \), observe that the area of the \( i \)th slice is approximately \( w(c_i) \Delta x \), where \( c_i \) is some point in the subinterval \([x_{i-1}, x_i]\). Further, the depth at every point on this slice is approximately \( c_i \). We can then approximate the force \( F_i \) acting on this slice of the dam by the weight of the water lying above a plate the size of this portion but which is oriented horizontally:

\[ F_i \approx 62.4 \, w(c_i) \Delta x \, c_i = 62.4 \, c_i \, w(c_i) \Delta x. \]
Adding together the forces acting on each slice, notice that the total force \( F \) on the dam is approximately
\[
F \approx \sum_{i=1}^{n} 62.4 c_i w(c_i) \Delta x.
\]
Taking the limit as \( n \to \infty \), the Riemann sums approach the total hydrostatic force on the dam,
\[
F = \lim_{n \to \infty} \sum_{i=1}^{n} 62.4 c_i w(c_i) \Delta x = \int_{0}^{60} 62.4 x w(x) \, dx,
\] (6.7)
where you should recognize the integral as the limit of a Riemann sum.

**Example 6.7** Finding the Hydrostatic Force on a Dam

A dam is shaped like a trapezoid with height 60 ft. The width at the top is 100 ft and the width at the bottom is 40 ft (see Figure 5.56). Find the maximum hydrostatic force that the wall will need to withstand. Find the hydrostatic force if a drought lowers the water level by 10 ft.

**Solution** Notice that the width function is a linear function of depth with \( w(0) = 100 \) and \( w(60) = 40 \). The slope is then \( \frac{60}{100} = -1 \) and so, \( w(x) = 100 - x \). From (6.7),
Chapter 5 Applications of the Definite Integral

1. For each of work, impulse and the first moment: identify the quantities in the definition (e.g., force and distance) and the calculations for which it is used (e.g., change in velocity).

2. The center of mass is not always the location at which half the mass is on one side and half the mass is on the other side. Give an example where more than half the mass is on one side (see examples 6.5 and 6.6) and explain why the object balances at the center of mass.

3. People who play catch have a seemingly instinctive method of pulling their hand back as they catch the ball. To catch a ball, you must apply an impulse equal to the mass times velocity of the ball. By pulling your hand back, you increase the amount of time in which you decelerate the ball. Use the impulse-momentum equation to explain why this reduces the average force on your hand.

4. A tennis ball comes toward you at 100 mph. After you hit the ball, it is moving away from you at 100 mph. Work measures changes in energy. Explain why work has been done by the tennis racket on the ball even though the ball has the same speed before and after the hit.

5. A force of 5 pounds stretches a spring 4 inches. Find the work done in stretching this spring 6 inches beyond its natural length.

6. A force of 10 pounds stretches a spring 2 inches. Find the work done in stretching this spring 3 inches beyond its natural length.

7. A force of 20 pounds stretches a spring \( \frac{1}{3} \) foot. Find the work done in stretching this spring 1 foot beyond its natural length.

8. A force of 12 pounds stretches a spring \( \frac{1}{3} \) foot. Find the work done in stretching this spring 4 inches beyond its natural length.

9. A weightlifter lifts 250 pounds a distance of 20 inches. Find the work done (as measured in foot-pounds).

10. A wrestler lifts his 300-pound opponent overhead, a height of 6 feet. Find the work done (as measured in foot-pounds).

11. A person lifts a 100-pound couch up 3 feet onto a truck. Find the work done.

13. A rocket full of fuel weighs 10,000 pounds at launch. After launch, the rocket gains altitude and loses weight as the fuel burns. Assume that the rocket loses 1 pound of fuel for every 15 feet of altitude gained. Explain why the work done raising the rocket to an altitude of 30,000 feet is \( \int_{0}^{30,000} (10,000 - x/15) \, dx \) and compute the integral.

14. Referring to exercise 13, suppose that a rocket weighs 8000 pounds at launch and loses 1 pound of fuel for every 10 feet of altitude gained. Find the work needed to raise the rocket to a height of 10,000 feet.

15. Suppose that a car engine exerts a force of \( 800x(1 - x) \) pounds when the car is at position \( x \) miles, \( 0 \leq x \leq 1 \). Compute the work done.

16. Horsepower measures the rate of work done as a function of time. For the situation in exercise 15, explain why this is not equal to \( 800x(1 - x) \). If the car in exercise 15 takes 80 seconds to travel the mile, compute the average horsepower (1 hp = 550 ft-lb/s).

17. A water tower is spherical in shape with radius 50 feet, extending from 200 feet to 300 feet above ground. Compute the work done in filling the tank.

18. Compute the work done in filling the tank of exercise 17 half way.

19. Compute the work done in pumping half of the water out of the top of the tank in example 6.3.

20. A water tank is in the shape of a right circular cone of altitude 10 feet and base radius 5 feet, with its vertex at the ground. (Think of an ice cream cone with its point facing down.) If the tank is full, find the work done in pumping all of the water out the top of the tank.

21. In example 6.4, suppose that the baseball was traveling at 100 ft/s. The force exerted by the bat on the ball would change to the values in the table. Estimate the impulse and the speed of the ball after impact.

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>0.0001</th>
<th>0.0002</th>
<th>0.0003</th>
<th>0.0004</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F ) (lb)</td>
<td>0</td>
<td>1000</td>
<td>2100</td>
<td>4000</td>
<td>5000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t ) (s)</th>
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<th>0.0006</th>
<th>0.0007</th>
<th>0.0008</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F ) (lb)</td>
<td>5200</td>
<td>2500</td>
<td>1000</td>
<td>0</td>
</tr>
</tbody>
</table>

22. In exercise 21, suppose that the baseball was traveling at 85 ft/s. The force exerted by the bat on the ball would change to the values in the table. Estimate the impulse and the speed of the ball after impact.

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>0.0001</th>
<th>0.0002</th>
<th>0.0003</th>
<th>0.0004</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F ) (lb)</td>
<td>0</td>
<td>600</td>
<td>1200</td>
<td>2000</td>
<td>2500</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0.0005</th>
<th>0.0006</th>
<th>0.0007</th>
<th>0.0008</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F ) (lb)</td>
<td>3000</td>
<td>2500</td>
<td>1100</td>
<td>300</td>
</tr>
</tbody>
</table>

23. Suppose that a crash test is performed on a vehicle. The force of the wall on the front bumper is shown below. Estimate the impulse and the speed of the vehicle (assume \( m = 200 \)).

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F ) (lb)</td>
<td>0</td>
<td>8000</td>
<td>16000</td>
<td>24000</td>
<td>15000</td>
<td>9000</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F ) (lb)</td>
<td>100</td>
<td>100</td>
<td>80</td>
<td>40</td>
<td>0</td>
</tr>
</tbody>
</table>

24. Two football players collide. The force of the defensive player on the offensive player is given below. Estimate the impulse. If the offensive player has mass \( m = 7 \) and velocity of 29 ft/s before the collision, does the defensive player stop the offensive player?

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F ) (lb)</td>
<td>0</td>
<td>300</td>
<td>500</td>
<td>400</td>
<td>250</td>
<td>150</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F ) (lb)</td>
<td>100</td>
<td>100</td>
<td>80</td>
<td>40</td>
<td>0</td>
</tr>
</tbody>
</table>

25. Compute the mass and center of mass of an object with density \( \rho(x) = \frac{x}{6} + 2 \) kg/m, \( 0 \leq x \leq 6 \). Briefly explain in terms of the density function why the center of mass is not at \( x = 3 \).

26. Compute the mass and center of mass of an object with density \( \rho(x) = 3 - \frac{x}{6} \) kg/m, \( 0 \leq x \leq 6 \). Briefly explain in terms of the density function why the center of mass is not at \( x = 3 \).

27. Compute the mass and center of mass of an object with density \( \rho(x) = 4 + \frac{x^2}{4} \) kg/m, \( -2 \leq x \leq 2 \).

28. Compute the mass and center of mass of an object with density \( \rho(x) = 4 - \sin \frac{\pi x}{4} \) kg/m, \( 0 \leq x \leq 4 \).

29. Compute the weight in ounces of an object extending from \( x = -3 \) to \( x = 27 \) with density \( \rho(x) = \left( \frac{1}{46} + \frac{x + 3}{690} \right)^2 \) slugs/in.

30. Compute the weight in ounces of an object extending from \( x = 0 \) to \( x = 32 \) with density \( \rho(x) = \left( \frac{1}{46} + \frac{x + 3}{690} \right)^2 \) slugs/in.
31. Compute the center of mass of the object in exercise 29. This object models the baseball bat of example 6.5 “choked up” (held 3 inches up the handle). Compare the masses and centers of mass of the two bats.

32. Compute the center of mass of the object in exercise 30. This object models a baseball bat that is 2 inches longer than the bat of example 6.5. Compare the masses and centers of mass of the two bats.

33. Compute the mass and weight in ounces and center of mass of an object extending from \( x = 0 \) to \( x = 30 \) with density \( \rho(x) = 0.00468 \left( \frac{1}{x^2} + \frac{1}{x^3} \right) \) slugs/in.

34. The object in exercise 33 models an aluminum baseball bat (hollow and \( \frac{1}{4} \)-inch thick). Compare the mass and center of mass of the wooden bat of example 6.5. Baseball experts claim that it is easier to hit an inside pitch (small \( x \) value) with an aluminum bat. Explain why your calculations indicate that this is true.

35. The figure above shows the outline of a model rocket. Assume that the vertical scale is 3 units high and the horizontal scale is 6 units wide. Use basic geometry to compute the area of each of the three regions of the rocket outline. Assuming a constant density \( \rho \), locate the x-coordinate of the center of mass of each region. (Hint: The first region can be thought of as extending from \( x = 0 \) to \( x = 1 \) with density \( \rho(3-2x) \). The third region extends from \( x = 5 \) to \( x = 6 \) with density \( \rho(6-x) \)).

36. For the model rocket in exercise 35, replace the rocket with 3 particles, one for each region. Assume that the mass of each particle equals the area of the region, and the location of the particle on the x-axis equals the center of mass of the region. Find the center of mass of the 3-particle system. [Rockets are designed with bottom fins large enough that the center of mass is shifted near the bottom (or, in the figure above, left) of the rocket. This improves the flight stability of the rocket.]

37. A dam is in the shape of a trapezoid with height 60 feet. The width at the top is 40 feet and the width at the bottom is 100 feet. Find the maximum hydrostatic force the wall would need to withstand. Explain why the force is so much greater than the force in example 6.7.

38. Find the maximum hydrostatic force in exercise 37 if a drought lowers the water level by 10 feet.

39. An underwater viewing window is installed at an aquarium. The window is circular with radius 5 feet. The center of the window is 40 feet below the surface of the water. Find the hydrostatic force on the window.

40. An underwater viewing window is rectangular with width 40 feet. The window extends from the surface of the water to a depth of 10 feet. Find the hydrostatic force on the window.

41. The camera’s window on a robotic submarine is circular with radius 3 inches. How much hydrostatic force would the window need to withstand to descend to a depth of 1000 feet?

42. A diver wears a watch to a depth of 60 feet. The face of the watch is circular with a radius of 1 inch. How much hydrostatic force will the face need to withstand if the watch is to keep on ticking?

43. In the text, we mentioned that work calculates changes in energy. For example, a 200-pound pole vaulter is propelled by a pole to a height of 20 feet. The work done by the pole, equal to 4000 ft-lb, gives the vaulter a large potential energy. To see what this means, compute the speed \( v \) of the vaulter when he reaches the ground. Show that the kinetic energy at impact, given by \( \frac{1}{2}mv^2 \) \((m = 200/32)\), also equals 4000 ft-lb. This illustrates the concept of conservation of energy.

44. Compute the speed of the falling vaulter in exercise 43 at the 10-foot mark. Show that the sum of the potential energy (32mh) and the kinetic energy \( (\frac{1}{2}mv^2) \) equals 4000 ft-lb.

45. Compute the horsepower needed to lift a 100-ton object such as a blue whale at 20 mph \((1 \text{ hp} = 550 \text{ ft-lbs})\). (Note that blue whales swim so efficiently that they can maintain this speed with an output of 60–70 hp.)

46. For a constant force \( F \) exerted over a length of time \( t \), impulse is defined by \( F \cdot t \). For a variable force \( F(t) \), derive the impulse formula \( J = \int_{0}^{t} F(t) \, dt \).

47. The first moment of a solid of density \( \rho(x) \) is \( \int_{a}^{b} x \rho(x) \, dx \). The second moment about the y-axis, defined by \( \int_{a}^{b} x^2 \rho(x) \, dx \), is also important in applications. The larger this number is, the more difficult it is to rotate the solid about the y-axis. Compute the second moments of the baseball bats in example 6.5 and exercise 29. Choking up on a bat makes it easier to swing (and control). Compute the percentage by which the second moment is reduced by choking up 3 inches.

48. Occasionally, baseball players illegally “cork” their bats by drilling out a portion of wood from the end of the bats and filling the hole with light substance such as cork. The advantage of this procedure is that the second moment is significantly reduced. To model this, take the bat of example 6.5 and change the density to

\[
\rho(x) = \begin{cases} 
\left( \frac{1}{x^2} + \frac{1}{x^3} \right)^2 & \text{if } 0 \leq x \leq 28 \\
\left( \frac{1}{x^2} + \frac{1}{x^3} \right)^2 & \text{if } 28 < x \leq 30,
\end{cases}
\]

representing a hole of radius \( \frac{1}{2} " \) and length 2". Compute the mass and second moment of the corked bat, and compare to the original bat.
49. The second moment (see exercise 47) of a disk of density $\rho$ in the shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by

$$\int_{-a}^{a} 2\rho b x^2 \sqrt{1 - \frac{x^2}{a^2}} \, dx.$$ Use your CAS to evaluate this integral.

50. Use the result from exercise 49 to show that the second moment of the tennis racket head in the diagram is $M = \rho \frac{\pi}{4} [ba^3 - (b - w)(a - w)^3]$.

51. For tennis rackets, a large second moment (see exercises 49 and 50) means less twisting of the racket on off-center shots. Compare the second moment of a wooden racket ($a = 9$, $b = 12$, $w = 0.5$), a midsize racket ($a = 10$, $b = 13$, $w = 0.5$) and an oversized racket ($a = 11$, $b = 14$, $w = 0.5$).

52. Let $M$ be the second moment found in exercise 50. Show that $\frac{dM}{da} > 0$ and conclude that larger rackets have larger second moments. Also, show that $\frac{dM}{dw} > 0$ and interpret this result.

53. As equipment has improved, heights cleared in the pole vault have increased. A crude estimate of the maximum pole vault possible can be derived from conservation of energy principles. Assume that the maximum speed a pole-vaulter could run carrying a long pole is 25 mph. Convert this speed to ft/s. The kinetic energy of this vaulter would be $\frac{1}{2} m v^2$ (leave $m$ as an unknown for the time being). This initial kinetic energy would equal the potential energy at the top of the vault minus whatever energy is absorbed by the pole (which we will ignore). Set the potential energy, $32 mh$, equal to the kinetic energy and solve for $h$. This represents the maximum amount the vaulter’s center of mass could be raised. Add 3 feet for the height of the vaulter’s center of mass and you have an estimate of the maximum vault possible. Compare this to Sergei Bubka’s 1994 world record vault of $20'1\frac{1}{4}''$.

### 5.7 PROBABILITY

You may have suspected this for some time, but in real-world applications, you are rarely simply handed a function to analyze. More than likely, you will need to go out and take numerous measurements and/or develop a detailed theory about the process you’re studying. It is often quite challenging to identify an appropriate function(s) to study, especially if the measurements are subject to random or unpredictable errors. The analysis of random processes is the focus of the mathematical fields of probability and statistics. In this section, we give a brief introduction to the use of calculus in probability theory. It may surprise you to learn that calculus provides insight into random processes, but this is in fact, a very important application of integrals. We urge you to continue your studies by taking a course(s) in probability and statistics.

We begin with a simple example involving coin-tossing. Suppose that you toss two coins, each of which has a 50% chance of coming up heads. Because of the randomness involved, you cannot calculate exactly how many heads you will get on a given number of tosses. But you can calculate the likelihood of each of the possible outcomes. If we denote heads by H and tails by T, then the four possible outcomes from tossing two coins are HH, HT, TH and TT. Each of these four outcomes is equally likely, so we can say that each has probability $\frac{1}{4}$. This means that, on average, each of these events will occur in one-fourth of your tries. Said a different way, the relative frequency with which each event occurs in a large number of trials will be approximately $\frac{1}{4}$. 

---

**Exercise 49**

The second moment of a disk of density $\rho$ in the shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by the integral

$$\int_{-a}^{a} 2\rho b x^2 \sqrt{1 - \frac{x^2}{a^2}} \, dx.$$ Use your CAS to evaluate this integral.

**Exercise 50**

Use the result from exercise 49 to show that the second moment of a tennis racket head in the diagram is $M = \rho \frac{\pi}{4} [ba^3 - (b - w)(a - w)^3]$.

**Exercise 51**

For tennis rackets, a large second moment means less twisting of the racket on off-center shots. Compare the second moment of a wooden racket ($a = 9$, $b = 12$, $w = 0.5$), a midsize racket ($a = 10$, $b = 13$, $w = 0.5$) and an oversized racket ($a = 11$, $b = 14$, $w = 0.5$).

**Exercise 52**

Let $M$ be the second moment found in exercise 50. Show that $\frac{dM}{da} > 0$ and conclude that larger rackets have larger second moments. Also, show that $\frac{dM}{dw} > 0$ and interpret this result.

**Exercise 53**

As equipment has improved, heights cleared in the pole vault have increased. A crude estimate of the maximum pole vault possible can be derived from conservation of energy principles. Assume that the maximum speed a pole-vaulter could run carrying a long pole is 25 mph. Convert this speed to ft/s. The kinetic energy of this vaulter would be $\frac{1}{2} m v^2$ (leave $m$ as an unknown for the time being). This initial kinetic energy would equal the potential energy at the top of the vault minus whatever energy is absorbed by the pole (which we will ignore). Set the potential energy, $32 mh$, equal to the kinetic energy and solve for $h$. This represents the maximum amount the vaulter’s center of mass could be raised. Add 3 feet for the height of the vaulter’s center of mass and you have an estimate of the maximum vault possible. Compare this to Sergei Bubka’s 1994 world record vault of $20'1\frac{1}{4}''$.
Suppose that we are primarily interested in recording the number of heads. Based on our calculations above, the probability of getting two heads is $\frac{1}{4}$, the probability of getting one head is $\frac{2}{4}$ (since there are two ways for this to happen: HT and TH) and the probability of getting zero heads is $\frac{1}{4}$. We often summarize such information by displaying it in a histogram or bar graph (see Figure 5.57).

Suppose that we instead toss eight coins. The probabilities for getting a given number of heads are given in the accompanying table and the corresponding histogram is shown in Figure 5.58. You should notice that the sum of all the probabilities is 1 (or 100%, since it’s certain that one of the possible outcomes will occur on a given try). This is one of the defining properties of probability theory. Another basic property is called the addition principle: to compute the probability of getting 6, 7 or 8 heads (or any other mutually exclusive outcomes), simply add together the individual probabilities:

$$P(6, 7 \text{ or } 8 \text{ heads}) = \frac{28}{256} + \frac{8}{256} + \frac{1}{256} = \frac{37}{256} \approx 0.144.$$  

A graphical interpretation of this calculation is very revealing. In the histogram in Figure 5.58, notice that each bar is a rectangle of width 1. Then the probability associated with each bar equals the area of the rectangle. In graphical terms,

- The total area in such a histogram is 1.
- The probability of getting between 6 and 8 heads (inclusive) equals the sum of the areas of the rectangles located between 6 and 8 (inclusive).

Not all probability events have the nice theoretical structure of coin-tossing. For instance, the situation is somewhat different if we want to find the probability that a randomly chosen person will have a height of 5’9” or 5’10”. There is no easy theory we can use here to compute the probabilities (since not all heights are equally likely). In this case, we would need to use the correspondence between probability and relative frequency. If we collected information about the heights of a large number of adults (for instance, from driver’s license data), we might find the following.

<table>
<thead>
<tr>
<th>Height</th>
<th>&lt;64”</th>
<th>64”</th>
<th>65”</th>
<th>66”</th>
<th>67”</th>
<th>68”</th>
<th>69”</th>
<th>70”</th>
<th>71”</th>
<th>72”</th>
<th>73”</th>
<th>&gt;73”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of people</td>
<td>23</td>
<td>32</td>
<td>61</td>
<td>94</td>
<td>133</td>
<td>153</td>
<td>155</td>
<td>134</td>
<td>96</td>
<td>62</td>
<td>31</td>
<td>26</td>
</tr>
</tbody>
</table>

Since the total number of people in the survey is 1000, the relative frequency of the height 5’9” (69”) is $\frac{135}{1000} = 0.135$ and the relative frequency of the height 5’10” (70”) is $\frac{134}{1000} = 0.134$. An estimate of the probability of being 5’9” or 5’10” is then $0.135 + 0.134 = 0.269$. A histogram is shown in Figure 5.59.
Suppose that we want to be more specific: for example, what is the probability that a randomly chosen person is 5'8'\(\frac{1}{2}\)" or 5'9"? To answer this question, we would need to have our data broken down further, as in the following partial table.

<table>
<thead>
<tr>
<th>66(\frac{1}{2})&quot;</th>
<th>67&quot;</th>
<th>67(\frac{1}{2})&quot;</th>
<th>68&quot;</th>
<th>68(\frac{1}{2})&quot;</th>
<th>69&quot;</th>
<th>69(\frac{1}{2})&quot;</th>
<th>70&quot;</th>
<th>70(\frac{1}{2})&quot;</th>
<th>71&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>52</td>
<td>61</td>
<td>72</td>
<td>71</td>
<td>82</td>
<td>81</td>
<td>74</td>
<td>69</td>
<td>65</td>
<td>58</td>
</tr>
</tbody>
</table>

The probability that a person is 5'9" can be estimated by the relative frequency of 5'9" people in our survey, which is \(\frac{81}{1000} = 0.081\). Similarly, the probability that a person is 5'8'\(\frac{1}{2}\)" is approximately \(\frac{82}{1000} = 0.082\). The probability of being 5'8'\(\frac{1}{2}\)" or 5'9" is then approximately \(0.081 + 0.082 = 0.163\). A histogram for this portion of the data is shown in Figure 5.60a.

![Histogram for relative frequency of heights.](image1)

![Histogram showing double the relative frequency.](image2)

Notice that since each bar of the histogram now represents a half-inch range of height, we can no longer interpret area in the histogram as the probability (we had assumed each bar has width 1). We will modify the histogram to make the area connection clearer. In Figure 5.60b, we have labeled the horizontal axis with the height in inches, and the vertical axis shows twice the relative frequency. The bar at 69" has height 0.162 and width \(\frac{1}{2}\). Its area, \(\frac{1}{2}(0.162) = 0.081\), corresponds to the relative frequency (or probability) of the height 5'9".

Of course, we could continue subdividing the height intervals into smaller and smaller pieces. Think of doing this while modifying the scale on the vertical axis so that the area of each rectangle (length times width of interval) always gives the relative frequency (probability) of that height interval. For example, suppose that there are \(n\) height intervals between 5'8" and 5'9". Let \(x\) represent height in inches and \(f(x)\) equal the height of the histogram bar for the interval containing \(x\). Let \(x_1 = 68 + \frac{1}{2}\), \(x_2 = 68 + \frac{1}{2}\) and so on, so that \(x_i = 68 + \frac{i}{n}\), for \(1 \leq i \leq n\) and let \(\Delta x = \frac{1}{n}\). For a randomly selected person, the probability that their height is between 5'8" and 5'9" is estimated by the sum of the areas of the corresponding histogram rectangles, given by

\[
P(68 \leq x \leq 69) \approx f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x = \sum_{i=1}^{n} f(x_i) \Delta x. \tag{7.1}
\]

Observe that as \(n\) increases, the histogram of Figure 5.61 (on the following page) will “smooth out,” approaching a curve like the one shown in Figure 5.62 (on the following page).
We call this limiting function \( f(x) \), the **probability density function (pdf)** for heights. Notice that for any given \( i = 1, 2, \ldots, n \), \( f(x_i) \) does not give the probability that a person’s height equals \( x_i \). Instead, for small values of \( \Delta x \), the quantity \( f(x_i) \Delta x \) is an approximation of the probability that the height is in the range \( [x_{i-1}, x_i] \).

Look carefully at (7.1) and think about what will happen as \( n \) increases. The Riemann sum on the right should approach an integral \( \int_a^b f(x) \, dx \). In this particular case, the limits of integration are 68 (5′8″) and 69 (5′9″). We have

\[
\lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x = \int_{68}^{69} f(x) \, dx.
\]

Notice that by adjusting the function values so that probability corresponds to area (given by an integral), we have found a familiar and direct technique for computing probabilities. We now summarize our discussion with some definitions. The preceding examples are of **discrete probability distributions** (the term *discrete* indicates that the quantity being measured can only assume values from a certain finite set or from a fixed sequence of values). For instance, in coin-tossing, the number of heads must be an integer. By contrast, many distributions are **continuous**. That is, the quantity of interest (the random variable) assumes values from a continuous range of numbers (an interval). For instance, suppose you are measuring the height of people in a certain group. Although height is normally rounded off to the nearest integer number of inches, a person’s actual height can be any number. (For instance, one of the authors lists his height as 6′0″, while his actual height is closer to 5′11.8″, depending on how good his posture is.)

For continuous distributions, the graph corresponding to a histogram is the graph of a probability density function (pdf). We now give a precise definition of a pdf.

**Definition 7.1**

Suppose that \( X \) is a random variable that may assume any value \( x \) with \( a \leq x \leq b \). A probability density function for \( X \) is a function \( f(x) \) satisfying

(i) \( f(x) \geq 0 \) for \( a \leq x \leq b \). **Probability density functions are never negative.**

(ii) \( \int_a^b f(x) \, dx = 1 \) and **The total probability is 1.**

(iii) The probability that the (observed) value of \( X \) falls between \( c \) and \( d \) is given by the area under the graph of the pdf on that interval. That is,

\[
P(c \leq X \leq d) = \int_c^d f(x) \, dx. \quad \text{Probability corresponds to area under the curve.}
\]
To verify that a function defines a pdf for some (unknown) random variable, we must show that it satisfies properties (i) and (ii) of Definition 7.1.

### Example 7.1  Verifying That a Function Is a pdf on an Interval

Show that \( f(x) = 3x^2 \) defines a pdf on the interval \([0, 1]\) by verifying properties (i) and (ii) of Definition 7.1.

**Solution**

Clearly, \( f(x) \geq 0 \). For property (ii), we integrate the pdf over its domain. We have

\[
\int_0^1 3x^2 \, dx = x^3 \bigg|_0^1 = 1.
\]

### Example 7.2  Using a pdf to Estimate Probabilities

Suppose that \( f(x) = \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} \) is the probability density function for the heights in inches of adult American males. Find the probability that a randomly selected adult American male will be between 5’8” and 5’9”. Also, find the probability that a randomly selected adult American male will be between 6’2” and 6’4”.

**Solution**

To compute the probabilities, you first need to convert the specified heights into inches. The probability of being between 68 and 69 inches tall is

\[
P(68 \leq X \leq 69) = \int_{68}^{69} \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} \, dx \approx 0.15542.
\]

Here, we approximated the value of the integral numerically, since we don’t know an antiderivative for the integrand. (You can use Simpson’s Rule or the numerical integration method built into your calculator or CAS.) Similarly, the probability of being between 74 and 76 inches is

\[
P(74 \leq X \leq 76) = \int_{74}^{76} \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} \, dx \approx 0.00751,
\]

where we have again approximated the value of the integral numerically.

### Example 7.3  Computing Probability with an Exponential pdf

Suppose that the lifetime in years of a certain brand of lightbulb is exponentially distributed with pdf \( f(x) = 4e^{-4x} \). Find the probability that a given lightbulb lasts 3 months or less.

According to data in Gyles Brandreth’s *Your Vital Statistics*, the pdf for the heights of adult males in the United States looks like the graph of \( f(x) = \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} \) shown in Figure 5.63 and used in example 7.2. You probably have seen bell-shaped curves like this before. This distribution is referred to as a normal distribution. Besides the normal distribution, there are many other probability distributions that are important in applications.
Solution  First, since the random variable measures lifetime in years, convert 3 months to \( \frac{1}{4} \) year. The probability is then
\[
P\left( 0 \leq X \leq \frac{1}{4} \right) = \int_{0}^{1/4} 4e^{-4x} \, dx = 4 \left( \frac{-1}{4} \right) e^{-4x} \bigg|_{0}^{1/4}
\]
\[= -e^{-1} + e^{0} = 1 - e^{-1} \approx 0.63212.
\]

In some cases, there may be theoretical reasons for assuming that a pdf has a certain form. In this event, the first task is to determine the values of any constants to achieve the properties of a pdf.

**Example 7.4**  Determining the Coefficient of a pdf

Suppose that the pdf for a random variable has the form \( f(x) = ce^{-3x} \) for some constant \( c \), with \( 0 \leq x \leq 1 \). Find the value of \( c \) that makes this a pdf.

**Solution**  To be a pdf, we first need that \( f(x) = ce^{-3x} \geq 0 \), for all \( x \in [0, 1] \). (This will be the case as long as \( c \geq 0 \).) Also, the integral over the domain must equal 1. So, we set
\[
1 = \int_{0}^{1} ce^{-3x} \, dx = c \left( -\frac{1}{3} \right) e^{-3x} \bigg|_{0}^{1} = -\frac{c}{3} e^{-3} + \frac{c}{3} = \frac{c}{3}(1 - e^{-3}).
\]
It now follows that \( c = \frac{3}{1 - e^{-3}} \approx 3.1572 \).

Given a pdf, it is possible to compute various statistics to summarize the properties of the random variable. The most common statistic is the mean, the best-known measure of average value. If you wanted to average test scores of 85, 89, 93 and 93, you would probably compute the mean, given by \( \frac{85 + 89 + 93 + 93}{4} = 90 \).

Notice here that there were three different test scores recorded: 85, which has a relative frequency of \( \frac{1}{4} \), 89, also with a relative frequency of \( \frac{1}{4} \), and 93, with a relative frequency of \( \frac{2}{4} \). We can also compute the mean by multiplying each value by its relative frequency and then summing: \((85)\frac{1}{4} + (89)\frac{1}{4} + (93)\frac{2}{4} = 90 \).

Now, suppose we wanted to compute the mean height of the people in the following table.

<table>
<thead>
<tr>
<th>Height</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>63”</td>
<td>23</td>
</tr>
<tr>
<td>64”</td>
<td>32</td>
</tr>
<tr>
<td>65”</td>
<td>61</td>
</tr>
<tr>
<td>66”</td>
<td>94</td>
</tr>
<tr>
<td>67”</td>
<td>133</td>
</tr>
<tr>
<td>68”</td>
<td>153</td>
</tr>
<tr>
<td>69”</td>
<td>155</td>
</tr>
<tr>
<td>70”</td>
<td>134</td>
</tr>
<tr>
<td>71”</td>
<td>96</td>
</tr>
<tr>
<td>72”</td>
<td>62</td>
</tr>
<tr>
<td>73”</td>
<td>31</td>
</tr>
<tr>
<td>74”</td>
<td>26</td>
</tr>
</tbody>
</table>

It would be silly to write out the heights of all 1000 people, add and divide by 1000. It is much easier to multiply each value by its relative frequency and add the results. Following this route, the mean \( m \) is given by
\[
m = \left( \frac{63}{1000} \right) \frac{23}{1000} + \left( \frac{64}{1000} \right) \frac{32}{1000} + \left( \frac{65}{1000} \right) \frac{61}{1000} + \left( \frac{66}{1000} \right) \frac{94}{1000} + \left( \frac{67}{1000} \right) \frac{133}{1000} + \cdots + \left( \frac{74}{1000} \right) \frac{26}{1000}
\]
\[= 68.523.
\]
If we denote the heights by $x_1, x_2, \ldots, x_n$ and let $f(x_i)$ be the corresponding relative frequencies or probabilities, the mean then has the form

$$m = x_1f(x_1) + x_2f(x_2) + x_3f(x_3) + \cdots + x_{12}f(x_{12}).$$

If the heights in our data set were given for every half-inch or tenth-of-an-inch, we would compute the mean by multiplying each $x_i$ by the corresponding probability $f(x_i)/\Delta x$, where $\Delta x$ is the fraction of an inch between data points. The mean now has the form

$$m = \left[ x_1f(x_1) + x_2f(x_2) + x_3f(x_3) + \cdots + x_nf(x_n) \right] \Delta x = \sum_{i=1}^{n} x_i f(x_i) \Delta x,$$

where $n$ is the number of data points. Notice that, as $n$ increases and $\Delta x$ approaches 0, the Riemann sum approaches the integral $\int_a^b xf(x) \, dx$. This gives us the following definition.

**Definition 7.2**
The mean $\mu$ of a random variable with pdf $f(x)$ on the interval $[a, b]$ is given by

$$\mu = \int_a^b xf(x) \, dx. \quad (7.2)$$

Although the mean is commonly used to report the average value of a random variable, it is important to realize that it is not the only measure of average used by statisticians. An alternative measurement of average is the median, the $x$-value that divides the probability in half. (That is, half of all values of the random variable lie at or below the median and half lie at or above the median.) In the following example and in the exercises, you will explore situations in which each measure provides a better or worse indication about the average of a random variable.

**Example 7.5**

Finding the Mean Age and Median Age of a Group of Cells

Suppose that the age in days of a type of single-celled organism has pdf $f(x) = (\ln 2)e^{-kx}$, where $k = \frac{1}{2} \ln 2$. The domain is $0 \leq x \leq 2$ (the assumption here is that upon reaching an age of 2 days, each cell divides into two daughter cells). Find (a) the mean age of the cells, (b) the proportion of cells that are younger than the mean and (c) the median age of the cells.

**Solution**

For part (a), we have from (7.2) that the mean is given by

$$\mu = \int_0^2 x \ln 2 e^{-(\ln 2)x/2} \, dx \approx 0.88539 \text{ days},$$

where we have approximated the value of the integral numerically, since at present we do not know how to find an antiderivative for this integrand. Notice that even though the cells range in age from 0 to 2 days, the mean is not 1. Look at the graph of the pdf in Figure 5.64 to see why the mean should be less than 1. The graph shows that younger ages are more likely than older ages and this causes the mean to be less than 1.

For part (b), notice that the proportion of cells younger than the mean is the same as the probability that a randomly selected cell is younger than the mean. This probability is given by

$$P(0 \leq X \leq \mu) = \int_0^{0.88539} \ln 2 e^{-(\ln 2)x/2} \, dx \approx 0.52848,$$
where we have again approximated the value of the integral numerically. Therefore, the proportion of cells younger than the mean is about 53%. Notice that in this case the mean does not represent the 50% mark for probabilities. In other words, the mean is not the same as the median.

To find the median in part (c), we must solve for the constant \( c \) such that

\[
0.5 = \int_{0}^{c} \ln 2 e^{-(\ln 2)x/2} \, dx.
\]

Since an antiderivative of \( e^{-(\ln 2)x/2} \) is \( -\frac{2}{\ln 2} e^{-(\ln 2)x/2} \), we have

\[
0.5 = \int_{0}^{c} \ln 2 e^{-(\ln 2)x/2} \, dx = \ln 2 \left[ -\frac{2}{\ln 2} e^{-(\ln 2)x/2} \right]_{0}^{c} = -2e^{-(\ln 2)c/2} + 2.
\]

Subtracting 2 from both sides, we have

\[
-1.5 = -2e^{-(\ln 2)c/2},
\]

so that dividing by \(-2\) yields

\[
0.75 = e^{-(\ln 2)c/2}.
\]

Taking the natural log of both sides, gives us

\[
\ln 0.75 = -(\ln 2)c/2.
\]

Finally, solving for \( c \) gives us

\[
c = \frac{-2 \ln 0.75}{\ln 2},
\]

so that the median is \(-2 \ln 0.75/\ln 2 \approx 0.83\). We can now conclude that half of the cells are younger than 0.83 days and half the cells are older than 0.83 days.

**EXERCISES 5.7**

1. In the text, we stated that the probability of tossing two fair coins and getting two heads is \( \frac{1}{4} \). If you try this experiment four times, explain why you will not always get two heads exactly one out of four times. If probability doesn’t give precise predictions, what is its usefulness? To answer this question, discuss the information conveyed by knowing that in the above experiment the probability of getting one head and one tail is \( \frac{1}{4} \) (twice as big as \( \frac{1}{8} \)).

2. Suppose you toss two coins numerous times (or simulate this on your calculator or computer). Theoretically, the probability of getting two heads is \( \frac{1}{4} \). In the long run (as the coins are tossed more and more often), what proportion of the time should two heads occur? Try this and discuss how your results compare to the theoretical calculation.

3. Based on Figures 5.57 and 5.58, describe what you expect the histogram to look like for larger numbers of coins. Compare to Figure 5.63.

4. The height of a person is determined by numerous factors, both hereditary and environmental (e.g., diet). Explain why this might produce a histogram similar to that produced by tossing a large number of coins.
In exercises 5–12, show that the given function is a pdf on the indicated interval.

5. \( f(x) = 4x^3, [0, 1] \)  
6. \( f(x) = \frac{3}{2}x^2, [0, 2] \)

7. \( f(x) = x + 2x^3, [0, 1] \)  
8. \( f(x) = \frac{3}{2}x^2, [-1, 1] \)

9. \( f(x) = \frac{1}{2} \sin x, [0, \pi] \)  
10. \( f(x) = \cos x, [0, \pi/2] \)

11. \( f(x) = e^{-x/2}, [0, \ln 4] \)  
12. \( f(x) = \frac{1}{4 - 8/x} e^{-x/2}, [0, 2] \)

In exercises 13–18, find a value of \( c \) for which \( f(x) \) is a pdf on the indicated interval.

13. \( f(x) = cx^3, [0, 1] \)  
14. \( f(x) = cx + x^2, [0, 1] \)

15. \( f(x) = ce^{-4x}, [0, 1] \)  
16. \( f(x) = ce^{-x/2}, [0, 2] \)

17. \( f(x) = 2ce^{-ex}, [0, 2] \)  
18. \( f(x) = 2ce^{-ex}, [0, 4] \)

In exercises 19–22, use the pdf in example 7.2 to find the probability that a randomly selected American male has height in the indicated range.

19. Between 5’10” and 6’.  
20. Between 6’6” and 6’10”.

21. Between 7’ and 10’.  
22. Between 2’ and 5’.

In exercises 23–26, find the indicated probabilities, given that the lifetime of a lightbulb is exponentially distributed with pdf \( f(x) = 6e^{-6x} \) (with \( x \) measured in years).

23. The lightbulb lasts less than 3 months.

24. The lightbulb lasts less than 6 months.

25. The lightbulb lasts between 1 and 2 years.

26. The lightbulb lasts between 3 and 10 years.

In exercises 27–30, find the indicated probabilities, given that the lifetime of a lightbulb is exponentially distributed with pdf \( f(x) = 8e^{-8x} \) (with \( x \) measured in years).

27. The lightbulb lasts between 1 and 2 months.

28. The lightbulb lasts between 5 and 6 months.

29. The lightbulb lasts less than 6 months.

30. The lightbulb lasts more than 6 months.

In exercises 31–34, suppose the lifetime of an organism has pdf \( f(x) = 4xe^{-2x} \) (with \( x \) measured in years).

31. Find the probability that the organism lives less than 1 year.

32. Find the probability that the organism lives between 1 and 2 years.

33. Find the mean lifetime (0 ≤ \( x \) ≤ 10).

34. Graph the pdf and compare the maximum value of the pdf to the mean.

In exercises 35–42, find (a) the mean and (b) the median of the random variable with the given pdf.

35. \( f(x) = 3x^2, 0 ≤ x ≤ 1 \)  
36. \( f(x) = 4x^3, 0 ≤ x ≤ 1 \)

37. \( f(x) = \frac{1}{2} \sin x, 0 ≤ x ≤ \pi \)

38. \( f(x) = \cos x, 0 ≤ x ≤ \pi/2 \)

39. \( f(x) = \frac{1}{2} (\ln 3)e^{-kx}, k = \frac{1}{2} \ln 3, 0 ≤ x ≤ 3 \)

40. \( f(x) = \frac{1}{2} (\ln 4)e^{-kx}, k = \frac{1}{2} \ln 4, 0 ≤ x ≤ 4 \)

41. \( f(x) = \frac{4}{1 - e^{-4}} e^{-4x}, 0 ≤ x ≤ 1 \)

42. \( f(x) = \frac{6}{1 - e^{-6}} e^{-6x}, 0 ≤ x ≤ 1 \)

43. For \( f(x) = ce^{-4x} \), find \( c \) so that \( f(x) \) is a pdf on the interval \([0, b]\) for \( b > 0 \). What happens to \( c \) as \( b \to \infty \)?

44. For the pdf of exercise 43, find the mean exactly (use a CAS for the antiderivative). As \( b \) increases, what happens to the mean?

45. Repeat exercises 43 and 44 for \( f(x) = ce^{-6x} \).

46. Based on the results of exercises 43–45, conjecture the values for \( c \) and the mean as \( b \to \infty \), for \( f(x) = ce^{-ax}, a > 0 \).

47. For eight coins being tossed, the probabilities of getting a given number of heads is shown in the table. Use the addition principle to find the probability of each event indicated below.

<table>
<thead>
<tr>
<th>Number of heads</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( \frac{1}{256} )</td>
<td>( \frac{8}{256} )</td>
<td>( \frac{28}{256} )</td>
<td>( \frac{56}{256} )</td>
<td>( \frac{70}{256} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of heads</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( \frac{56}{256} )</td>
<td>( \frac{28}{256} )</td>
<td>( \frac{8}{256} )</td>
<td>( \frac{1}{256} )</td>
</tr>
</tbody>
</table>

(a) three or fewer heads  
(b) more heads than tails  
(c) all heads or all tails  
(d) an odd number of heads
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48. In one version of the game of keno, you choose 10 numbers between 1 and 80. A random drawing selects 20 numbers between 1 and 80. Your payoff depends on how many of your numbers are selected. Use the given probabilities (rounded to 4 digits) to find the probability of each event indicated below. (To win, at least 5 of your numbers must be selected. On a $2 bet, you win $40 or more if 6 or more of your numbers are selected.)

<table>
<thead>
<tr>
<th>Number selected</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.0458</td>
<td>0.1796</td>
<td>0.2953</td>
<td>0.2674</td>
<td>0.1473</td>
</tr>
<tr>
<td>Number selected</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>Probability</td>
<td>0.0514</td>
<td>0.0115</td>
<td>0.0016</td>
<td>0.0001</td>
<td>0.0</td>
</tr>
</tbody>
</table>

(a) winning (at least 5 selected)  
(b) losing (4 or fewer selected)  
(c) winning big (6 or more)  
(d) 3 or 4 numbers selected

49. In the baseball World Series, two teams play games until one team or the other wins four times. Suppose team A should win each game with probability 0.6. The probabilities for team A’s record (given as wins/losses) in the World Series are shown. Find the probability of each event indicated below.

<table>
<thead>
<tr>
<th>Wins/losses</th>
<th>0/4</th>
<th>1/4</th>
<th>2/4</th>
<th>3/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.0256</td>
<td>0.0615</td>
<td>0.0922</td>
<td>0.1106</td>
</tr>
<tr>
<td>Wins/losses</td>
<td>4/3</td>
<td>4/2</td>
<td>4/1</td>
<td>4/0</td>
</tr>
<tr>
<td>Probability</td>
<td>0.1659</td>
<td>0.2073</td>
<td>0.2073</td>
<td>0.1296</td>
</tr>
</tbody>
</table>

(a) Team A wins the World Series  
(b) Team B wins the World Series  
(c) One team wins all four games  
(d) The teams play six or seven games

50. Suppose a basketball player makes 70% of her free throws. If she shoots three free throws and the probability of making each one is 0.7, the probabilities for the total number made are as shown. Find the probability of each event indicated below.

<table>
<thead>
<tr>
<th>Number made</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.027</td>
<td>0.189</td>
<td>0.441</td>
<td>0.343</td>
</tr>
</tbody>
</table>

(a) She makes 2 or 3  
(b) She makes at least 1

51. In any given time period, some species become extinct. Mass extinctions (such as that of the dinosaurs) are relatively rare. Fossil evidence indicates that the probability that p percent (1 < p < 100) of the species become extinct in a 4-million-year period is approximately \( e(p) = cp^{-2} \) for some constant c. Find c to make \( e(p) \) a pdf and estimate the probability that in a 4-million-year period 60–70% of the species will become extinct.

52. In example 7.5, we found the median (also called the second quartile). Now find the first and third quartiles, the ages such that the probability of being older are 0.25 and 0.75, respectively.

53. The pdf in example 7.2 is the pdf for a normally distributed random variable. The mean is easily read off of \( f(x) \); in example 7.2, the mean is 68. The mean and a number called the standard deviation characterize normal distributions. As Figure 5.63 indicates, the graph of the pdf has a maximum at the mean and has two inflection points located on opposite sides of the mean. The standard deviation equals the distance from the mean to an inflection point. Find the standard deviation in example 7.2.

54. In exercise 53, you found the standard deviation for the pdf in example 7.2. Denoting the mean as \( \mu \) and the standard deviation as \( \sigma \), find the probability that a given height is between \( \mu - \sigma \) and \( \mu + \sigma \) (that is, within one standard deviation of the mean). Find the probability that a given height is within two standard deviations of the mean (\( \mu - 2\sigma \) to \( \mu + 2\sigma \)), and within three standard deviations of the mean. These probabilities are the same for any normal distribution. If you know the mean and standard deviation of a normally distributed random variable, you automatically know these probabilities.

55. When a set of numbers is “curved” to achieve a predetermined mean, it can be difficult to evaluate the quality of the process producing those numbers. For example, suppose a math class consists of students whose scores on the final exam are 400, 480, 550, 620 and 690. Suppose the middle score (median) is always given a C. Argue that if 550 is a C, then 690 should be an A. Now, suppose the students score 680, 680, 685, 690 and 690. If 685 is a C, can 690 be an A? Explain why, under these rules, an absence of A’s doesn’t necessarily mean that the class performed poorly.

56. Renowned biologist Stephen Jay Gould has used the argument in exercise 55 to explain a baseball fact (see his book *Full House*). A baseball player’s batting average records the proportion of times he makes a hit. Traditionally written with three decimal places, a batting average of .300 (hits 30% of the time) is an indicator of a good batter. In the early 1900s, several batters had averages over .400, but this mark has not been achieved since 1941. The rules of baseball are constantly being adjusted, with one of the goals being to maintain an overall batting average of about .270. If all players today are almost as good as the best players of the early 1900s, explain why it
would be more difficult to average .400 today than it was in the past.

57. The pdf for inter-spike intervals of neurons firing in the cochlear nucleus of a cat is \( f(t) = k t^{-3/2} e^{-b t} \), where \( a = 100, \ b = 0.38 \) and \( t \) is measured in microseconds (see Mackey and Glass, *From Clocks to Chaos*). Use your CAS to find the value of \( k \) that makes \( f \) a pdf on the interval \([0, 40]\). Then find the probability that neurons fire between 20 and 30 microseconds apart.

58. The Maxwell-Boltzmann pdf for molecular speeds in a gas at equilibrium is \( f(x) = a x^2 e^{-b x^2} \) for parameters \( a \) and \( b \). Find the most common speed [i.e., find \( x \) to maximize \( f(x) \)]. If the most common speed is \( m \), find \( a \) in terms of \( m \) to make \( f \) a pdf on the interval \([0, 4m]\).

59. The mathematical theory of chaos indicates that numbers generated by very simple algorithms can look random. Chaos researchers look at a variety of graphs to try to distinguish randomness from deterministic chaos. For example, iterate the function \( f(x) = 4x(1-x) \) starting at \( x = 0.1 \). That is, compute \( f(0.1) = 0.36, f(0.36) = 0.9216, f(0.9216) = 0.289 \) and so on. Iterate 50 times and record how many times each first digit occurs (so far, we’ve got a 1, a 3, a 9 and a 2). If the process were truly random, the digits would occur about the same number of times. Does this seem to be happening? To unmask this process as nonrandom, you can draw a phase portrait. To do this, take consecutive iterates as coordinates of a point \((x, y)\) and plot the points. The first three points are \((0.1, 0.36), (0.36, 0.9216) \) and \((0.9216, 0.289)\). Describe the (nonrandom) pattern that appears, identifying it as precisely as possible.

60. Suppose that a spring is oscillating up and down with vertical position given by \( u(t) = \sin t \). If you pick a random time and look at the position of the spring, would you be more likely to find the spring near an extreme \((u = 1 \text{ or } u = -1)\) or near the middle \((u = 0)\)? The pdf is inversely proportional to speed (why is this reasonable?). Show that speed is given by \( |\cos t| = \sqrt{1 - u^2} \), so the pdf is \( f(u) = c/\sqrt{1 - u^2}, \) for some constant \( c \). Show that \( c = 1/\pi, \) then graph \( f(x) \) and describe what positions the spring is likely to be found in.

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**CHAPTER REVIEW EXERCISES**

In exercises 1–8, find the indicated area exactly if possible (estimate if necessary).

1. The area between \( y = x^2 + 2 \) and \( y = \sin x \) for \( 0 \leq x \leq \pi \)
2. The area between \( y = e^x \) and \( y = e^{-x} \) for \( 0 \leq x \leq 1 \)
3. The area between \( y = x^3 \) and \( y = 2x^2 - x \)
4. The area between \( y = x^2 - 3 \) and \( y = -x^2 + 5 \)
5. The area between \( y = e^{-x} \) and \( y = 2 - x^2 \)
6. The area between \( x = y^2 \) and \( y = 1 - x \)
7. The area of the region bounded by \( y = x^2, y = 2 - x \) and \( y = 0 \)
8. The area of the region bounded by \( y = x^2, y = 0 \) and \( x = 2 \)

9. A town has a population of 10,000 with a birth rate of \( 10 + 2t \) people per year and a death rate of \( 4 + t \) people per year. Compute the town’s population after 6 years.

10. From the given data, estimate the area between the curves for \( 0 \leq x \leq 2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0.0 )</th>
<th>( 0.2 )</th>
<th>( 0.4 )</th>
<th>( 0.6 )</th>
<th>( 0.8 )</th>
<th>( 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3.2</td>
<td>3.6</td>
<td>3.8</td>
<td>3.7</td>
<td>3.2</td>
<td>3.4</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>1.2</td>
<td>1.5</td>
<td>1.6</td>
<td>2.2</td>
<td>2.0</td>
<td>2.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3.0</td>
<td>2.8</td>
<td>2.4</td>
<td>2.9</td>
<td>3.4</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>2.2</td>
<td>2.1</td>
<td>2.3</td>
<td>2.8</td>
<td>2.4</td>
</tr>
</tbody>
</table>

11. Find the volume of the solid with cross-sectional area \( A(x) = \pi (3 + x)^2 \) for \( 0 \leq x \leq 2 \).
12. A swimming pool viewed from above has an outline given by \( y = \pm (5 + x) \) for \( 0 \leq x \leq 2 \). The depth is given by \( 4 + x \) (all measurements in feet). Compute the volume.

13. The cross-sectional areas of an underwater object are given. Estimate the volume.

| \( x \) | 0   | 0.4 | 0.8 | 1.2 | 1.6 | 2.0 | 2.4 | 2.8 \\ 
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(x) )</td>
<td>0.4</td>
<td>1.4</td>
<td>1.8</td>
<td>2.0</td>
<td>2.1</td>
<td>1.8</td>
<td>1.1</td>
<td>0.4</td>
</tr>
</tbody>
</table>

In exercises 14–18, find the volume of the indicated solid of revolution.

14. The region bounded by \( y = x^2, y = 0 \) and \( x = 1 \) revolved about (a) the \( x \)-axis; (b) the \( y \)-axis; (c) \( x = 2 \); (d) \( y = -2 \)

15. The region bounded by \( y = x^2 \) and \( y = 4 \) revolved about (a) the \( x \)-axis; (b) the \( y \)-axis; (c) \( x = 2 \); (d) \( y = -2 \)

16. The region bounded by \( y = x, y = 2x \) and \( x = 2 \) revolved about (a) the \( x \)-axis; (b) the \( y \)-axis; (c) \( x = -1 \); (d) \( y = 4 \)

17. The region bounded by \( y = x, y = 2 - x \) and \( x = 0 \) revolved about (a) the \( x \)-axis; (b) the \( y \)-axis; (c) \( x = -1 \); (d) \( y = 4 \)

18. The region bounded by \( x = 4 - y^2 \) and \( x = y^2 - 4 \) revolved about (a) the \( x \)-axis; (b) the \( y \)-axis; (c) \( x = 4 \); (d) \( y = 4 \)

In exercises 19–22, set up an integral for the arc length and numerically estimate the integral.

19. The portion of \( y = x^2 \) for \(-1 \leq x \leq 1 \)

20. The portion of \( y = x^2 + x \) for \(-1 \leq x \leq 0 \)

21. The portion of \( y = e^x/2 \) for \(-2 \leq x \leq 2 \)

22. The portion of \( y = \sin 2x \) for \(0 \leq x \leq \pi \)

In exercises 23 and 24, set up an integral for the surface area and numerically estimate the integral.

23. The surface generated by revolving \( y = 1 - x^2, 0 \leq x \leq 1 \), about the \( x \)-axis

24. The surface generated by revolving \( y = x^3, 0 \leq x \leq 1 \), about the \( x \)-axis

In exercises 25–32, ignore air resistance.

25. A diver drops from a height of 64 feet. What is the velocity at impact?

26. If the diver in exercise 25 has an initial upward velocity of 4 ft/s, what will be the impact velocity?

27. An object is launched from the ground at an angle of 20° with an initial speed of 48 ft/s. Find the time of flight and the horizontal range.

28. Repeat exercise 27 for an object launched from a height of 6 feet.

29. A football is thrown from a height of 6 feet with initial speed 80 ft/s at an angle of 8°. A person stands 40 yards downfield in the direction of the throw. Is it possible to catch the ball?

30. Repeat exercise 29 with a launch angle of 24°. By trial and error, find the range of angles (rounded to the nearest degree) that produce a catchable throw.

31. Find the initial velocity needed to propel an object to a height of 128 feet. Find the object’s velocity at impact.

32. A plane at an altitude of 120 ft drops supplies to a location on the ground. If the plane has a horizontal velocity of 100 ft/s, how far from the target should the supplies be released.

33. A force of 60 pounds stretches a spring 1 foot. Find the work done to stretch the spring 8 inches beyond its natural length.

34. A car engine exerts a force of \( 800 + 2x \) pounds when the car is at position \( x \) miles. Find the work done as the car moves from \( x = 0 \) to \( x = 8 \).

35. Compute the mass and center of mass of an object with density \( \rho(x) = x^2 - 2x + 8 \) for \( 0 \leq x \leq 4 \). Explain why the center of mass is not at \( x = 2 \).

36. Compute the mass and center of mass of an object with density \( \rho(x) = x^2 - 2x + 8 \) for \( 0 \leq x \leq 2 \). Explain why the center of mass is at \( x = 1 \).

37. A dam has the shape of a trapezoid with height 80 feet. The width at the top of the dam is 60 feet and the width at the bottom of the dam is 140 feet. Find the maximum hydrostatic force that the dam will need to withstand.

38. An underwater viewing window is a rectangle with width 20 feet extending from 5 feet below the surface to 10 feet below the surface. Find the maximum hydrostatic force that the window will need to withstand.

39. The force exerted by a bat on a ball over time is shown in the table. Use the data to estimate the impulse. If the ball (mass
Chapter Review Exercises

47. Find the (a) mean and (b) median of a random variable with pdf \( f(x) = x + 2x^3 \) on the interval \([0, 1]\).

48. Find the (a) mean and (b) median of a random variable with pdf \( f(x) = \frac{5}{3}e^{-2x} \) on the interval \([0, \ln 2]\).

49. As indicated in section 5.5, general formulas can be derived for many important quantities in projectile motion. For an object launched from the ground at angle \( \theta_0 \) with initial speed \( v_0 \) ft/s, find the horizontal range \( R \) ft and use the trig identity \( \sin(2\theta_0) = 2\sin\theta_0\cos\theta_0 \) to show that \( R = \frac{v_0^2 \sin(2\theta_0)}{32} \). Conclude that the maximum range is achieved with angle \( \theta_0 = \pi/4 \) (45°).

50. To follow up on exercise 49, suppose that the ground makes an angle of \( A' \) with the horizontal. If \( A > 0 \) (i.e., the projectile is being launched uphill), explain why the maximum range would be achieved with an angle larger than 45°. If \( A < 0 \) (launching downhill), explain why the maximum range would be achieved with an angle less than 45°. To determine the exact value of the optimal angle, first argue that the ground can be represented by the line \( y = (\tan A)x \). Show that the projectile reaches the ground at time \( t = v_0\sin \theta_0 - \tan A \cos \theta_0 \). Compute \( x(t) \) for this value of \( t \) and use a trig identity to replace the quantity \( \sin \theta_0 \cos A - \sin A \cos \theta_0 \) with \( \sin(\theta_0 - A) \). Then use another trig identity to replace \( \cos \theta_0 \sin(\theta_0 - A) \) with \( \sin(2\theta_0 - A) \sin A \). At this stage, the only term involving \( \theta_0 \) will be \( \sin(2\theta_0 - A) \). To maximize the range, maximize this term by taking \( \theta_0 = \frac{\pi}{4} + \frac{1}{2}A \).

\[
\begin{array}{c|cccc}
\text{t (s)} & 0 & 0.0001 & 0.0002 & 0.0003 & 0.0004 \\
\text{F(t) (lb)} & 0 & 800 & 1600 & 2400 & 3000 \\
\hline
\text{t (s)} & 0.0005 & 0.0006 & 0.0007 & 0.0008 \\
\text{F(t) (lb)} & 3600 & 2200 & 1200 & 0 \\
\end{array}
\]

40. If a wall applies a force of \( f(t) = 3000t(2 - t) \) pounds to a car for \( 0 \leq t \leq 2 \), find the impulse. If the car (mass \( m = 100 \) slugs) is motionless after the collision, compute its speed before the collision.

41. Show that \( f(x) = x + 2x^3 \) is a pdf on the interval \([0, 1]\).

42. Show that \( f(x) = \frac{5}{3}e^{-2x} \) is a pdf on the interval \([0, \ln 2]\).

43. Find the value of \( c \) such that \( f(x) = \frac{c}{x^2} \) is a pdf on the interval \([1, 2]\).

44. Find the value of \( c \) such that \( f(x) = ce^{-2x} \) is a pdf on the interval \([0, 4]\).

45. The lifetime of a lightbulb has pdf \( f(x) = 4e^{-4x} \) (x in years). Find the probability that the lightbulb lasts (a) less than 6 months; (b) between 6 months and 1 year.

46. The lifetime of an organism has pdf \( f(x) = 9xe^{-3x} \) (x in years). Find the probability that the organism lasts (a) less than 2 months; (b) between 3 months and 1 year.

\[ m = 0.01 \text{ slugs} \] had speed 120 ft/s before the collision, estimate its speed after the collision.