Motion in Two and Three Dimensions

FIGURE 3.1 Multiple exposure sequence of a bouncing ball.
WHAT WE WILL LEARN

- You will learn to handle motion in two and three dimensions using methods developed for one-dimensional motion.
- You will determine the parabolic path of ideal projectile motion.
- You will be able to calculate the maximum height and maximum range of an ideal projectile trajectory in terms of the initial velocity vector and the initial position.
- You will learn to describe the velocity vector of a projectile at any time during its flight.
- You will appreciate that the realistic trajectories of objects like baseballs are affected by air friction and are not exactly parabolic.
- You will learn to transform velocity vectors from one reference frame to another.

What We Will Learn

Everyone has seen a bouncing ball, but have you ever looked closely at the path it takes? If you could slow the ball down, as in the photo in Figure 3.1, you would see the symmetrical arch of each bounce, which gets smaller until the ball stops. This path is characteristic of a kind of two-dimensional motion known as projectile motion. You can see the same parabolic shape in water fountains, fireworks, basketball shots—any kind of isolated motion where the force of gravity is relatively constant and the moving object is dense enough that air resistance (a force that tends to slow down objects moving though air) can be ignored.

This chapter extends Chapter 2’s discussion of displacement, velocity, and acceleration to two-dimensional motion. The definitions of these vectors in two dimensions are very similar to the one-dimensional definitions, but we can apply them to a greater variety of real-life situations. Two-dimensional motion is still more restricted than general motion in three dimensions, but it applies to a wide range of common and important motions that we will consider throughout this course.

3.1 Three-Dimensional Coordinate Systems

Having studied motion in one dimension, we next tackle more complicated problems in two and three spatial dimensions. To describe this motion, we will work in Cartesian coordinates. In a three-dimensional Cartesian coordinate system, we choose the x- and y-axes to lie in the horizontal plane and the z-axis to point vertically upward (Figure 3.2). The three coordinate axes are at 90° (orthogonal) to one another, as required for a Cartesian coordinate system.

The convention that is followed without exception in this book is that the Cartesian coordinate system is right-handed. This convention means that you can obtain the relative orientation of the three coordinate axes using your right hand. To determine the positive directions of the three axes, hold your right hand with the thumb sticking straight up and the index finger pointing straight out; they will naturally have a 90° angle relative to each other. Then stick out your middle finger so that it is at a right angle with both the index finger and the thumb (Figure 3.3). The three axes are assigned to the fingers as shown in Figure 3.3a: thumb is x, index finger is y, and middle finger is z. You can rotate your right
hand in any direction, but the relative orientation of thumb and fingers stays the same. If you want, you can exchange the letters on the fingers, as shown in Figure 3.3b and Figure 3.3c. However, \( z \) always has to follow \( y \), which always has to follow \( x \). Figure 3.3 shows all possible combinations of the right-handed assignment of the axes to the fingers. You really only have to remember one of them, because your hand can always be orientated in three-dimensional space in such a way that the axes assignments on your fingers can be brought into alignment with the schematic coordinate axes shown in Figure 3.2.

With this set of Cartesian coordinates, a position vector can be written in component form as

\[
\mathbf{r} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}.
\]

A velocity vector is

\[
\mathbf{v} = (v_x, v_y, v_z) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}.
\]

For one-dimensional vectors, the time derivative of the position vector defines the velocity vector. This is also the case for more than one dimension:

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}.
\]

In the last step of this equation, we used the sum and product rules of differentiation, as well as the fact that the unit vectors are constant vectors (fixed directions along the coordinate axes and constant magnitude of 1). Comparing equations 3.2 and 3.3, we see that

\[
v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}.
\]

The same procedure leads us from the velocity vector to the acceleration vector by taking another time derivative:

\[
\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}.
\]

We can therefore write the Cartesian components of the acceleration vector:

\[
a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}.
\]

### 3.2 Velocity and Acceleration in a Plane

The most striking difference between velocity along a line and velocity in two or more dimensions is that the latter can change direction as well as magnitude. Because acceleration is defined as a change in velocity—any change in velocity—divided by a time interval, there can be acceleration even when the magnitude of the velocity does not change.

Consider, for example, a particle moving in two dimensions (that is, in a plane). At time \( t_1 \), the particle has velocity \( \mathbf{v}_1 \), and at a later time \( t_2 \), the particle has velocity \( \mathbf{v}_2 \). The change in velocity of the particle is \( \Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 \). The average acceleration, \( \mathbf{a}_{\text{ave}} \), for the time interval \( \Delta t = t_2 - t_1 \) is given by

\[
\mathbf{a}_{\text{ave}} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1}.
\]

Figure 3.4 shows three different cases for the change in velocity of a particle moving in two dimensions over a given time interval. Figure 3.4a shows the initial and final velocities of the particle having the same direction, but the magnitude of the final velocity is greater than the magnitude of the initial velocity. The resulting change in velocity and the average acceleration are in the same direction as the velocities. Figure 3.4b again shows the initial and final velocities pointing in the same direction, but the magnitude of the final velocity is less than the magnitude of the initial velocity. The resulting change in velocity and the average acceleration are in the opposite direction from the velocities. Figure 3.4c illustrates the
case when the initial and final velocities have the same magnitude but the direction of the final velocity vector is different from the direction of the initial velocity vector. Even though the magnitudes of the initial and final velocity vectors are the same, the change in velocity and the average acceleration are not zero and can be in a direction not obviously related to the initial or final velocity directions.

Thus, in two dimensions, an acceleration vector arises if an object's velocity vector changes in magnitude or direction. Any time an object travels along a curved path, in two or three dimensions, it must have acceleration. We will examine the components of acceleration in more detail in Chapter 9, when we discuss circular motion.

### 3.3 Ideal Projectile Motion

In some special cases of three-dimensional motion, the horizontal projection of the trajectory, or flight path, is a straight line. This situation occurs whenever the accelerations in the horizontal xy-plane are zero, so the object has constant velocity components, \( v_x \) and \( v_y \), in the horizontal plane. Such a case is shown in Figure 3.5 for a baseball tossed in the air. In this case, we can assign new coordinate axes such that the \( x \)-axis points along the horizontal projection of the trajectory and the \( y \)-axis is the vertical axis. In this special case, the motion in three dimensions can in effect be described as a motion in two spatial dimensions. A large class of real-life problems falls into this category, especially problems that involve ideal projectile motion.

An ideal projectile is any object that is released with some initial velocity and then moves only under the influence of gravitational acceleration, which is assumed to be constant and in the vertical downward direction. A basketball free throw (Figure 3.6) is a good example of ideal projectile motion, as is the flight of a bullet or the trajectory of a car that becomes airborne. Ideal projectile motion neglects air resistance and wind speed, spin of the projectile, and other effects influencing the flight of real-life projectiles. For realistic situations in which a golf ball, tennis ball, or baseball moves in air, the actual trajectory is not well described by ideal projectile motion and requires a more sophisticated analysis. We will discuss these effects in Section 3.5, but will not go into quantitative detail.

Let’s begin with ideal projectile motion, with no effects due to air resistance or any other forces besides gravity. We work with two Cartesian components: \( x \) in the horizontal direction and \( y \) in the vertical (upward) direction. Therefore, the position vector for projectile motion is

\[
\mathbf{r} = (x, y) = x\mathbf{\hat{x}} + y\mathbf{\hat{y}},
\]

and the velocity vector is

\[
\mathbf{v} = (v_x, v_y) = v_x\mathbf{\hat{x}} + v_y\mathbf{\hat{y}} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \frac{dx}{dt}\mathbf{\hat{x}} + \frac{dy}{dt}\mathbf{\hat{y}}.
\]

Given our choice of coordinate system, with a vertical \( y \)-axis, the acceleration due to gravity acts downward, in the negative \( y \)-direction; there is no acceleration in the horizontal direction:

\[
\mathbf{a} = (0, -g) = -g\mathbf{\hat{y}}.
\]

For this special case of a constant acceleration only in the \( y \)-direction and with zero acceleration in the \( x \)-direction, we have a free-fall problem in the vertical direction and motion with
constant velocity in the horizontal direction. The kinematical equations for the $x$-direction are those for an object moving with constant velocity:

\begin{equation}
    x = x_0 + v_{x0}t \\ (3.11)
\end{equation}

\begin{equation}
    v_x = v_{x0}. \\
\end{equation} (3.12)

Just as in Chapter 2, we use the notation $v_{x0} = v_x(t = 0)$ for the initial value of the $x$-component of the velocity. The kinematical equations for the $y$-direction are those for free-fall motion in one dimension:

\begin{equation}
    y = y_0 + v_{y0}t - \frac{1}{2}gt^2 \\
\end{equation} (3.13)

\begin{equation}
    y = y_0 + \bar{v}_y t \\
\end{equation} (3.14)

\begin{equation}
    v_y = v_{y0} - gt \\
\end{equation} (3.15)

\begin{equation}
    \bar{v}_y = \frac{1}{2}(v_y + v_{y0}) \\
\end{equation} (3.16)

\begin{equation}
    v_y^2 = v_{y0}^2 - 2g(y - y_0). \\
\end{equation} (3.17)

For consistency, we write $v_{y0} = v_y(t = 0)$. With these seven equations for the $x$- and $y$-components, we can solve any problem involving an ideal projectile. Notice that since two-dimensional motion can be split into separate one-dimensional motions, these equations are written in component form, without the use of unit vectors.

### Example 3.1 Shoot the Monkey

Many lecture demonstrations illustrate that motion in the $x$-direction and motion in the $y$-direction are indeed independent of each other, as assumed in the derivation of the equations for projectile motion. One popular demonstration, called “shoot the monkey,” is shown in Figure 3.7. The demonstration is motivated by a story. A monkey has escaped from the zoo and has climbed a tree. The zookeeper wants to shoot the monkey with a tranquilizer dart in order to recapture it, but she knows that the monkey will let go of the branch it is holding onto at the sound of the gun firing. Her challenge is therefore to hit the monkey in the air as it is falling.

**Figure 3.7** The shoot-the-monkey lecture demonstration. On the right are some of the individual frames of the video, with information on their timing in the upper-left corners. On the left, these frames have been combined into a single image with a superimposed yellow line indicating the initial aim of the projectile launcher.

Continued—
PROBLEM
Where does the zookeeper need to aim to hit the falling monkey?

SOLUTION
The zookeeper must aim directly at the monkey, as shown in Figure 3.7, assuming that the time for the sound of the gun firing to reach the monkey is negligible and the speed of the dart is fast enough to cover the horizontal distance to the tree. As soon as the dart leaves the gun, it is in free fall, just like the monkey. Because both the monkey and the dart are in free fall, they fall with the same acceleration, independent of the dart's motion in the x-direction and of the dart's initial velocity. The dart and the monkey will meet at a point directly below the point from which the monkey dropped.

DISCUSSION
Any sharpshooter can tell you that, for a fixed target, you need to correct your gun sight for the free-fall motion of the projectile on the way to the target. As you can infer from Figure 3.7, even a bullet fired from a high-powered rifle will not fly in a straight line but will drop under the influence of gravitational acceleration. Only in a situation like the shoot-the-monkey demonstration, where the target is in free fall as soon as the projectile leaves the muzzle, can one aim directly at the target without making corrections for the free-fall motion of the projectile.

Shape of a Projectile’s Trajectory
Let’s now examine the trajectory of a projectile in two dimensions. To find \( y \) as a function of \( x \), we solve the equation \( x = x_0 + v_{x0}t \) for the time, \( t = (x - x_0)/v_{x0} \), and then substitute for \( t \) in the equation \( y = y_0 + v_{y0}t - \frac{1}{2}gt^2 \):

\[
y = y_0 + v_{y0}t - \frac{1}{2}gt^2 \Rightarrow
y = y_0 + v_{y0} \frac{x-x_0}{v_{x0}} - \frac{1}{2}g \left( \frac{x-x_0}{v_{x0}} \right)^2 \Rightarrow
y = \left( y_0 - \frac{v_{y0}x_0}{v_{x0}} \right) + \frac{v_{y0}}{v_{x0}} \frac{gx_0^2}{2v_{x0}^2} + \frac{v_{y0}}{v_{x0}} \frac{gx_0}{2v_{x0}^2} x - \frac{g}{2v_{x0}^2} x^2.
\]  
(3.18)

Thus, the trajectory follows an equation of the general form \( y = c + bx + ax^2 \), with constants \( a \), \( b \), and \( c \). This is the form of an equation for a parabola in the xy-plane. It is customary to set the x-component of the initial point of the parabola equal to zero: \( x_0 = 0 \). In this case, the equation for the parabola becomes

\[
y = y_0 + \frac{v_{y0}}{v_{x0}} x - \frac{g}{2v_{x0}^2} x^2.
\]  
(3.19)

The trajectory of the projectile is completely determined by three input constants. These constants are the initial height of the release of the projectile, \( y_0 \), and the x- and y-components of the initial velocity vector, \( v_{x0} \) and \( v_{y0} \), as shown in Figure 3.8.

We can also express the initial velocity vector \( \vec{v}_0 \) in terms of its magnitude, \( v_0 \), and direction, \( \theta_0 \). Expressing \( \vec{v}_0 \) in this manner involves the transformation

\[
v_0 = \sqrt{v_{x0}^2 + v_{y0}^2}
\]  
and

\[
\theta_0 = \tan^{-1} \frac{v_{y0}}{v_{x0}}.
\]  
(3.20)

In Chapter 1, we discussed this transformation from Cartesian coordinates to length and angle of the vector, as well as the inverse transformation:

\[
\begin{align*}
v_{x0} &= v_0 \cos \theta_0 \\
v_{y0} &= v_0 \sin \theta_0.
\end{align*}
\]  
(3.21)
Expressed in terms of the magnitude and direction of the initial velocity vector, the equation for the path of the projectile becomes

\[ y = y_0 + (\tan \theta_0)x - \frac{g}{2v_0^2 \cos^2 \theta_0} x^2. \]  

(3.22)

The fountain shown in Figure 3.9 is in the Detroit Metropolitan Wayne County (DTW) airport. You can clearly see that the water shot out of many pipes traces almost perfect parabolic trajectories.

Note that because a parabola is symmetric, a projectile takes the same amount of time and travels the same distance from its launch point to the top of its trajectory as from the top of its trajectory back to launch level. Also, the speed of a projectile at a given height on its way up to the top of its trajectory is the same as its speed at that same height going back down.

**Time Dependence of the Velocity Vector**

From equation 3.12, we know that the \( x \)-component of the velocity is constant in time: \( v_x = v_{x0} \). This result means that a projectile will cover the same horizontal distance in each time interval of the same duration. Thus, in a video of projectile motion, such as a basketball player shooting a free throw as in Figure 3.6, or the path of the dart in the shoot-the-monkey demonstration in Figure 3.7, the horizontal displacement of the projectile from one frame of the video to the next will be constant.

The \( y \)-component of the velocity vector changes according to equation 3.15, \( v_y = v_{y0} - gt \); that is, the projectile falls with constant acceleration. Typically, projectile motion starts with a positive value, \( v_{y0} \). The apex (highest point) of the trajectory is reached at the point where \( v_y = 0 \) and the projectile moves only in the horizontal direction. At the apex, the \( y \)-component of the velocity is zero, and it changes sign from positive to negative.

We can indicate the instantaneous values of the \( x \)- and \( y \)-components of the velocity vector on a plot of \( y \) versus \( x \) for the flight path of a projectile (Figure 3.10). The \( x \)-components, \( v_x \), of the velocity vector are shown by green arrows, and the \( y \)-components, \( v_y \), by red arrows. Note the identical lengths of the green arrows, demonstrating the fact that \( v_x \) remains constant. Each blue arrow is the vector sum of the \( x \)- and \( y \)-velocity components and depicts the instantaneous velocity vector along the path. Note that the direction of the velocity vector is always tangential to the trajectory. This is because the slope of the velocity vector is

\[ \frac{v_y}{v_x} = \frac{dy}{dt} = \frac{dy}{dx} = \frac{v_y}{v_x}, \]

which is also the local slope of the flight path. At the top of the trajectory, the green and blue arrows are identical because the velocity vector has only an \( x \)-component—that is, it points in the horizontal direction.

Although the vertical component of the velocity vector is equal to zero at the top of the trajectory, the gravitational acceleration has the same constant value as on any other part of the trajectory. Beware of the common misconception that the gravitational acceleration is equal to zero at the top of the trajectory. The gravitational acceleration has the same constant value everywhere along the trajectory.

Finally, let’s explore the functional dependence of the absolute value of the velocity vector on time and/or the \( y \)-coordinate. We start with the dependence of \( |\mathbf{v}| \) on \( y \). We use the fact that the absolute value of a vector is given as the square root of the sum of the squares of the components. Then we use kinematical equation 3.12 for the \( x \)-component and kinematical equation 3.17 for the \( y \)-component. We obtain

\[ |\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{v_{x0}^2 + v_{y0}^2 - 2g(y - y_0)} = \sqrt{v_{y0}^2 - 2g(y - y_0)}. \]  

(3.23)

Note that the initial launch angle does not appear in this equation. The absolute value of the velocity—the speed—depends only on the initial value of the speed and the difference between the \( y \)-coordinate and the initial launch height. Thus, if we release a projectile from the top of the trajectory of any projectile, which of the following statement(s), if any, is (are) true?

a) The acceleration is zero.

b) The \( x \)-component of the acceleration is zero.

c) The \( y \)-component of the acceleration is zero.

d) The speed is zero.

e) The \( x \)-component of the velocity is zero.

f) The \( y \)-component of the velocity is zero.

---

**FIGURE 3.9** A fountain with water following parabolic trajectories.

**FIGURE 3.10** Graph of a parabolic trajectory with the velocity vector and its Cartesian components shown at constant time intervals.
3.4 Maximum Height and Range of a Projectile

When launching a projectile, for example, throwing a ball, we are often interested in the range \( R \), or how far the projectile will travel horizontally before returning to its original vertical position, and the maximum height \( H \) it will reach. These quantities \( R \) and \( H \) are illustrated in Figure 3.11. We find that the maximum height reached by the projectile is

\[
H = y_0 + \frac{\nu_0^2}{2g}.
\]

(3.24)

We'll derive this equation below. We'll also derive this equation for the range:

\[
R = \frac{\nu_0^2}{g} \sin 2\theta_0,
\]

(3.25)

where \( \nu_0 \) is the absolute value of the initial velocity vector and \( \theta_0 \) is the launch angle. The maximum range, for a given fixed value of \( \nu_0 \), is reached when \( \theta_0 = 45^\circ \).

**DERIVATION 3.1**

Let’s investigate the maximum height first. To determine its value, we obtain an expression for the height, differentiate it, set the result equal to zero, and solve for the maximum height. Suppose \( \nu_0 \) is the initial speed and \( \theta_0 \) is the launch angle. We take the derivative of the path function \( y(x) \), equation 3.22, with respect to \( x \):

\[
\frac{dy}{dx} = \frac{d}{dx} \left( y_0 + (\tan \theta_0) x - \frac{g}{2
\nu_0^2 \cos^2 \theta_0} x^2 \right) = \tan \theta_0 - \frac{g}{\nu_0^2 \cos^2 \theta_0} x.
\]

Now we look for the point \( x_H \) where the derivative is zero:

\[
0 = \tan \theta_0 - \frac{g}{\nu_0^2 \cos^2 \theta_0} x_H
\]

\[
\Rightarrow x_H = \frac{\nu_0^2 \cos^2 \theta_0 \tan \theta_0}{g} = \frac{\nu_0^2}{g} \sin \theta_0 \cos \theta_0 = \frac{\nu_0^2}{2g} \sin 2\theta_0.
\]

In the second line above, we used the trigonometric identities \( \tan \theta = \sin \theta \cos \theta \) and \( 2\sin \theta \cos \theta = \sin 2\theta \). Now we insert this value for \( x \) into equation 3.22 and obtain the maximum height, \( H \):

\[
H = y(x_H) = y_0 + x_H \tan \theta_0 - \frac{g}{2\nu_0^2 \cos^2 \theta_0} x_H^2
\]

\[
= y_0 + \frac{\nu_0^2}{2g} \sin 2\theta_0 \tan \theta_0 - \frac{g}{2\nu_0^2 \cos^2 \theta_0} \left( \frac{\nu_0^2}{2g} \sin 2\theta_0 \right)^2
\]

\[
= y_0 + \frac{\nu_0^2}{g} \sin^2 \theta_0 - \frac{\nu_0^2}{2g} \sin^2 \theta_0
\]

\[
= y_0 + \frac{\nu_0^2}{2g} \sin^2 \theta_0.
\]

Because \( \nu_y = \nu_0 \sin \theta_0 \), we can also write

\[
H = y_0 + \frac{\nu_0^2}{2g},
\]

which is equation 3.24.
The range, \( R \), of a projectile is defined as the horizontal distance between the launching point and the point where the projectile reaches the same height from which it started, \( y(R) = y_0 \). Inserting \( x = R \) into equation 3.22:

\[
y_0 = y_0 + R \tan \theta_0 - \frac{g}{2v_0^2 \cos^2 \theta_0} R^2
\]

\[
\Rightarrow \tan \theta_0 = \frac{g}{2v_0^2 \cos^2 \theta_0} R
\]

\[
\Rightarrow R = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0 = \frac{v_0^2}{g} \sin 2\theta_0,
\]

which is equation 3.25.

Note that the range, \( R \), is twice the value of the \( x \)-coordinate, \( x_{hf} \), at which the trajectory reached its maximum height: \( R = 2x_{hf} \).

Finally, we consider how to maximize the range of the projectile. One way to maximize the range is to maximize the initial velocity, because the range increases with the absolute value of the initial velocity, \( v_0 \). The question then is, given a specific initial speed, what is the dependence of the range on the launch angle \( \theta_0 \)? To answer this question, we take the derivative of the range (equation 3.25) with respect to the launch angle:

\[
\frac{dR}{d\theta_0} = \frac{d}{d\theta_0} \left( \frac{v_0^2}{g} \sin 2\theta_0 \right) = 2 \frac{v_0^2}{g} \cos 2\theta_0.
\]

Then we set this derivative equal to zero and find the angle for which the maximum value is achieved. The angle between 0° and 90° for which \( \cos 2\theta_0 = 0 \) is 45°. So the maximum range of an ideal projectile is given by

\[
R_{\text{max}} = \frac{v_0^2}{g}.
\] (3.26)

We could have obtained this result directly from the formula for the range because, according to that formula (equation 3.25), the range is at a maximum when \( \sin 2\theta_0 \) has its maximum value of 1, and it has this maximum when \( 2\theta_0 = 90^\circ \), or \( \theta_0 = 45^\circ \).

Most sports involving balls provide numerous examples of projectile motion. We next consider a few examples where the effects of air resistance and spin do not dominate the motion, and so the findings are reasonably close to what happens in reality. In the next section, we’ll look at what effects air resistance and spin can have on a projectile.

**SOLVED PROBLEM 3.1 ** Throwing a Baseball

When listening to a radio broadcast of a baseball game, you often hear the phrase “line drive” or “frozen rope” for a ball hit really hard and at a low angle with respect to the ground. Some announcers even use “frozen rope” to describe a particularly strong throw from second or third base to first base. This figure of speech implies movement on a straight line—but we know that the ball’s actual trajectory is a parabola.

**PROBLEM**

What is the maximum height that a baseball reaches if it is thrown from second base to first base and from third base to first base, released from a height of 6.0 ft, with a speed of 90 mph, and caught at the same height?

**SOLUTION**

**THINK**

The dimensions of a baseball infield are shown in Figure 3.12. (In this problem, we’ll need to perform lots of unit conversions. Generally, this book uses SI units, but baseball is...
full of British units.) The baseball infield is a square with sides 90 ft long. This is the distance between second and first base, and we get \(d_{12} = 90 \text{ ft} = 90 \cdot 0.3048 \text{ m} = 27.4 \text{ m}\).

The distance from third to first base is the length of the diagonal of the infield square: 
\[
d_{13} = d_{12} \sqrt{2} = 38.8 \text{ m}.
\]

A speed of 90 mph (the speed of a good Major League fastball) translates into 
\[
v_0 = 90 \text{ mph} = 90 \cdot 0.4469 \text{ m/s} = 40.2 \text{ m/s}.
\]

As with most trajectory problems, there are many ways to solve this problem. The most straightforward way follows from our considerations of range and maximum height. We can equate the base-to-base distance with the range of the projectile because the ball is released and caught at the same height, \(y_0 = 6 \text{ ft} = 6 \cdot 0.3048 \text{ m} = 1.83 \text{ m}\).

**Sketch**

![Image of a baseball infield](image)

**Figure 3.12** Dimensions of a baseball infield.

**Research**

In order to obtain the initial launch angle of the ball, we use equation 3.25, setting the range equal to the distance between first and second base:

\[
d_{12} = \frac{v_0^2}{g} \sin 2\theta_0 \implies \theta_0 = \frac{1}{2} \sin^{-1} \left( \frac{d_{12} g}{v_0^2} \right).
\]

However, we already have an equation for the maximum height:

\[
H = y_0 + \frac{v_0^2 \sin^2 \theta_0}{2g}.
\]

**Simplify**

Substituting our expression for the launch angle into the equation for the maximum height results in

\[
H = y_0 + \frac{v_0^2 \sin^2 \left( \frac{1}{2} \sin^{-1} \left( \frac{d_{12} g}{v_0^2} \right) \right)}{2g}.
\]

**Calculate**

We are ready to insert numbers:

\[
H = 1.83 \text{ m} + \frac{(40.2 \text{ m/s})^2 \sin^2 \left( \frac{1}{2} \sin^{-1} \left( \frac{27.4 \text{ m}(9.81 \text{ m/s}^2)}{40.2 \text{ m/s}^2} \right) \right)}{2(9.81 \text{ m/s}^2)} = 2.40367 \text{ m}.
\]

**Round**

The initial precision specified was two significant digits. So we round our final result to 
\[H = 2.4 \text{ m}.
\]
Thus, a 90-mph throw from second to first base is $2.39 \text{ m} - 1.83 \text{ m} = 0.56 \text{ m}$—that is, almost 2 ft—above a straight line at the middle of its trajectory. This number is even bigger for the throw from third to first base, for which we find an initial angle of $6.8^\circ$ and a maximum height of $3.0 \text{ m}$, or $1.2 \text{ m}$ (almost 4 ft) above the straight line connecting the points of release and catch.

**DOUBLE-CHECK**

Common sense says that the longer throw from third to first needs to have a greater maximum height than the throw from second to first, and our answers agree with that. If you watch a baseball game from the stands or on television, these calculated heights may seem too large. However, if you watch a game from ground level, you’ll see that the other infielders really do have to get some height on the ball to make a good throw to first base.

Let’s consider one more example from baseball and calculate the trajectory of a batted ball (see Figure 3.13).

**EXAMPLE 3.2  Batting a Baseball**

During the flight of a batted baseball, in particular, a home run, air resistance has a quite noticeable impact. For now though, we want to neglect it. Section 3.5 will discuss the effect of air resistance.

**PROBLEM**

If the ball comes off the bat with a launch angle of $35^\circ$ and an initial speed of 110 mph, how far will the ball fly? How long will it be in the air? What will its speed be at the top of its trajectory? What will its speed be when it lands?

**SOLUTION**

Again, we need to convert to SI units first: $v_0 = 110 \text{ mph} = 49.2 \text{ m/s}$. We first find the range:

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{(49.2 \text{ m/s})^2}{9.81 \text{ m/s}^2 \sin 70^\circ} = 231.5 \text{ m}.$$  

This distance is about 760 feet, which would be a home run in even the biggest ballpark. However, this calculation does not take air resistance into account. If we took friction due to air resistance into account, the distance would be reduced to approximately 400 feet. (See Section 3.5 on realistic projectile motion.)

In order to find the baseball’s time in the air, we can divide the range by the horizontal component of the velocity assuming the ball is hit at about ground level.

$$t = \frac{R}{v_0 \cos \theta_0} = \frac{231.5 \text{ m}}{(49.2 \text{ m/s})(\cos 35^\circ)} = 5.74 \text{ s}.$$  

Now we will calculate the speeds at the top of the trajectory and at landing. At the top of the trajectory, the velocity has only a horizontal component, which is $v_0 \cos \theta_0 = 40.3 \text{ m/s}$. When the ball lands, we can calculate its speed using equation 3.23: $|v| = \sqrt{v_0^2 - 2g(y - y_0)}$. Because we assume that the altitude at which it lands is the same as the one from which it was launched, we see that the speed is the same at the landing point as at the launching point, 49.2 m/s.

A real baseball would not quite follow the trajectory calculated here. If instead we launched a small steel ball bearing with the same angle and speed, neglecting air resistance would have led to a very good approximation, and the trajectory parameters just found would be verified in such an experiment. The reason we can comfortably neglect air resistance for the steel ball bearing is that it has a much higher mass density and smaller surface area than a baseball, so drag effects (which depend on cross-sectional area) are small compared to gravitational effects.
Baseball is not the only sport that provides examples of projectile motion. Let's consider an example from football.

**SOLVED PROBLEM 3.2 Hang Time**

When a football team is forced to punt the ball away to the opponent, it is very important to kick the ball as far as possible but also to attain a sufficiently long hang time—that is, the ball should remain in the air long enough that the punt-coverage team has time to run down field and tackle the receiver right after the catch.

**PROBLEM**

What are the initial angle and speed with which a football has to be punted so that its hang time is 4.41 s and it travels a distance of 49.8 m (= 54.5 yd)?

**SOLUTION**

**THINK**

A punt is a special case of projectile motion for which the initial and final values of the vertical coordinate are both zero. If we know the range of the projectile, we can figure out the hang time from the fact that the horizontal component of the velocity vector remains at a constant value; thus, the hang time must simply be the range divided by this horizontal component of the velocity vector. The equations for hang time and range give us two equations in the two unknown quantities, \( v_0 \) and \( \theta_0 \), that we are looking for.

**SKETCH**

This is one of the few cases in which a sketch does not seem to provide additional information.

**RESEARCH**

We have already seen (equation 3.25) that the range of a projectile is given by

\[
R = \frac{v_0^2 \sin 2\theta_0}{g}.
\]

As already mentioned, the hang time can be most easily computed by dividing the range by the horizontal component of the velocity:

\[
t = \frac{R}{v_0 \cos \theta_0}.
\]

Thus, we have two equations in the two unknowns, \( v_0 \) and \( \theta_0 \). (Remember, \( R \) and \( t \) were given in the problem statement.)

**SIMPLIFY**

We solve both equations for \( v_0^2 \) and set them equal:

\[
R = \frac{v_0^2 \sin 2\theta_0}{g} \Rightarrow v_0^2 = \frac{gR}{\sin 2\theta_0} \Rightarrow v_0^2 = \frac{gR}{2\sin 2\theta_0} = \frac{R^2}{2\sin 2\theta_0}.
\]

\[
t = \frac{R}{v_0 \cos \theta_0} \Rightarrow v_0^2 = \frac{R^2}{t^2 \cos^2 \theta_0} = \frac{R^2}{t^2 \sin 2\theta_0}.
\]

Now, we can solve for \( \theta_0 \). Using \( \sin 2\theta_0 = 2 \sin \theta_0 \cos \theta_0 \), we find

\[
\frac{g}{2 \sin \theta_0 \cos \theta_0} = \frac{R}{t^2 \sin 2\theta_0} \Rightarrow \theta_0 = \tan^{-1} \left( \frac{g^2}{2R} \right).
\]
Next, we substitute this expression in either of the two equations we started with. We select the equation for hang time and solve it for $v_0$:

$$t = \frac{R}{v_0 \cos \theta_0} \Rightarrow v_0 = \frac{R}{t \cos \theta_0}.$$ 

**CALCULATE**

All that remains is to insert numbers into the equations we have obtained:

$$\theta_0 = \tan^{-1}\left(\frac{\left(9.81 \text{ m/s}^2\right)\left(4.41 \text{ s}\right)^2}{2(49.8 \text{ m})}\right) = 62.4331^\circ$$

$$v_0 = \frac{49.8 \text{ m}}{(4.41 \text{ s})(\cos 1.08966)} = 24.4013 \text{ m/s}.$$ 

**ROUND**

The range and hang time were specified to three significant figures, so we state our final results to this precision:

$$\theta_0 = 62.4^\circ$$

and

$$v_0 = 24.4 \text{ m/s}.$$ 

**DOUBLE-CHECK**

We know that the maximum range is reached with a launch angle of $45^\circ$. The punted ball here is launched at an initial angle that is significantly steeper, at $62.4^\circ$. Thus, the ball does not travel as far as it could go with the value of the initial speed that we computed. Instead, it travels higher and thus maximizes the hang time. If you watch good college or pro punters practice their skills during football games, you’ll see that they try to kick the ball with an initial angle larger than $45^\circ$, in agreement with what we found in our calculations.

### 3.5 Realistic Projectile Motion

If you are familiar with tennis or golf or baseball, you know that the parabolic model for the motion of a projectile is only a fairly crude approximaion to the actual trajectory of any real ball. However, by ignoring some factors that affect real projectiles, we were able to focus on the physical principles that are most important in projectile motion. This is a common technique in science: Ignore some factors involved in a real situation in order to work with fewer variables and come to an understanding of the basic concept. Then go back and consider how the ignored factors affect the model. Let’s briefly consider the most important factors that affect real projectile motion: air resistance, spin, and surface properties of the projectile.

The first modifying effect that we need to take into account is air resistance. Typically, we can parameterize air resistance as a velocity-dependent acceleration. The general analysis exceeds the scope of this book; however, the resulting trajectories are called *ballistic curves*.

Figure 3.14 shows the trajectories of baseballs launched at an initial angle of $35^\circ$ with respect to the horizontal at initial speeds of 90 and 110 mph. Compare the trajectory shown for the launch speed of 110 mph with the result we calculated in Example 3.2: The real range of this ball is only slightly more than 400 ft, whereas we found 760 ft when we neglected air resistance. Obviously, for a long fly ball, neglecting air resistance is not valid.

Another important effect that the parabolic model neglects is the spin of the projectile as it moves through the air. When a quarterback throws a “spiral” in football, for example, the spin is important for the stability of the flight motion and prevents the ball from rotating end-over-end. In tennis, a ball with topspin drops much faster than a ball without noticeable spin, given the same initial values of speed and launch angle. Conversely, a tennis ball with underspin, or backspin, “floats” deeper into the court. In golf, backspin is sometimes

### 3.2 In-Class Exercise

The same range as in Solved Problem 3.2 could be achieved with the same initial speed of 24.4 m/s but a launch angle different from $62.4^\circ$. What is the value of this angle?

- a) 12.4°
- b) 27.6°
- c) 45.0°
- d) 55.2°

### 3.3 In-Class Exercise

What is the hang time for that other launch angle found in In-Class Exercise 3.2?

- a) 2.30 s
- b) 3.14 s
- c) 4.41 s
- d) 5.14 s
desired, because it causes a steeper landing angle and thus helps the ball come to rest closer
to its landing point than a ball hit without backspin. Depending on the magnitude and
direction of rotation, sidespin of a golf ball can cause a deviation from a straight-line path
along the ground (draws and fades for good players or hooks and slices for the rest of us).

In baseball, sidespin is what enables a pitcher to throw a curveball. By the way, there is
no such thing as a “rising fastball” in baseball. However, balls thrown with severe backspin
do not drop as fast as the batter expects and are thus sometimes perceived as rising—an
optical illusion. In the graph of ballistic baseball trajectories in Figure 3.14, an initial back-
spin of 2000 rpm was assumed.

Curving and practically all other effects of spin on the trajectory of a moving ball are a
result of the air molecules bouncing with higher speeds off the side of the ball (and the bound-
ary layer of air molecules) that is rotating in the direction of the flight motion (and thus has
a higher velocity relative to the incoming air molecules) than off the side of the ball rotating
against the flight direction. We will return to this topic in Chapter 13 on fluid motion.

The surface properties of projectiles also have significant effects on their trajectories.
Golf balls have dimples to make them fly farther. Balls that are otherwise identical to typical
golf balls but have a smooth surface can be driven only about half as far. This surface effect
is also the reason why sandpaper found in a pitcher’s glove leads to ejection of that player
from the game, because a baseball that is roughened on parts of its surface moves differently
from one that is not.

Figure 3.14 Trajectories of baseballs initially launched at an angle of 35° above the horizontal at
speeds of 90 mph (green) and 110 mph (red). Solid curves neglect air resistance and backspin; dotted
curves reflect air resistance and backspin.

3.6 Relative Motion

To study motion, we have allowed ourselves to shift the origin of the coordinate system by
properly choosing values for \(x_0\) and \(y_0\). In general, \(x_0\) and \(y_0\) are constants that can be chosen
freely. If this choice is made intelligently, it can help make a problem more manageable. For
example, when we calculated the path of the projectile, \(y(x)\), we set \(x_0 = 0\) to simplify our
calculations. The freedom to select values for \(x_0\) and \(y_0\) arises from the fact that our ability to
describe any kind of motion does not depend on the location of the origin of the coordinate
system.

So far, we have examined physical situations where we have kept the origin of the coordi-
nate system at a fixed location during the motion of the object we wanted to consider.
However, in some physical situations, it is impractical to choose a reference system with a
fixed origin. Consider, for example, a jet plane landing on an aircraft carrier that is going
forward at full throttle at the same time. You want to describe the plane’s motion in a coordi-
nate system fixed to the carrier, even though the carrier is moving. The reason why this is
important is that the plane needs to come to rest relative to the carrier at some fixed location
on the deck. The reference frame from which we view motion makes a big difference in how
we describe the motion, producing an effect known as relative velocity.

Another example of a situation for which we cannot neglect relative motion is a trans-
atlantic flight flying from Detroit, Michigan, to Frankfurt, Germany, which takes 8 h and 10
min. Using the same aircraft and going in the reverse direction, from Frankfurt to Detroit,
takes 9 h and 10 min, a full hour longer. The primary reason for this difference is that the
prevailing wind at high altitudes, the jet stream, tends to blow from west to east at speeds
as high as 67 m/s (150 mph). Even though the airplane’s speed relative to the air around it is the same in both directions, that air is moving with its own speed. Thus, the relationship of the coordinate system of the air inside the jet stream to the coordinate system in which the locations of Detroit and Frankfurt remain fixed is important in understanding the difference in flight times.

For a more easily analyzed example of a moving coordinate system, let’s consider motion on a moving walkway, as is typically found in airport terminals. This system is an example of one-dimensional relative motion. Suppose that the walkway surface moves with a certain velocity, \( v_{\text{w}} \), relative to the terminal. We use the subscripts \( w \) for walkway and \( t \) for terminal. Then a coordinate system that is fixed to the walkway surface has exactly velocity \( v_{\text{w}} \) relative to a coordinate system attached to the terminal. The man shown in Figure 3.15 is walking with a velocity \( v_{\text{m}} \) as measured in a coordinate system on the walkway, and he has a velocity \( v_{\text{mt}} = v_{\text{mw}} + v_{\text{w}} \) with respect to the terminal. The two velocities \( v_{\text{mw}} \) and \( v_{\text{w}} \) add as vectors since the corresponding displacements add as vectors. (We will show this explicitly when we generalize to three dimensions.) For example, if the walkway moves with \( v_{\text{w}} = 1.5 \text{ m/s} \) and the man moves with \( v_{\text{mw}} = 2.0 \text{ m/s} \), then he will progress through the terminal with a velocity of \( v_{\text{mt}} = v_{\text{mw}} + v_{\text{w}} = 2.0 \text{ m/s} + 1.5 \text{ m/s} = 3.5 \text{ m/s} \).

One can achieve a state of no motion relative to the terminal by walking in the direction opposite of the motion of the walkway with a velocity that is exactly the negative of the walkway velocity. Children often try to do this. If a child were to walk with \( v_{\text{mw}} = -1.5 \text{ m/s} \) on this walkway, her velocity would be zero relative to the terminal.

It is essential for this discussion of relative motion that the two coordinate systems have a velocity relative to each other that is constant in time. In this case, we can show that the accelerations measured in both coordinate systems are identical: \( v_{\text{mt}} = \text{const.} \Rightarrow \frac{dv_{\text{mt}}}{dt} = 0 \). From \( v_{\text{mt}} = v_{\text{mw}} + v_{\text{w}} \) we then obtain:

\[
\frac{dv_{\text{mt}}}{dt} = \frac{d(v_{\text{mw}} + v_{\text{w}})}{dt} = \frac{dv_{\text{mw}}}{dt} + \frac{dv_{\text{w}}}{dt} + 0
\]

\[\Rightarrow a_{\text{m}} = a_{\text{w}}, \quad (3.27)\]

Therefore, the accelerations measured in both coordinate systems are indeed the same. This type of velocity addition is also known as a Galilean transformation. Before we go on to the two- and three-dimensional cases, note that this type of transformation is valid only for speeds that are small compared to the speed of light. Once the speed approaches the speed of light, we must use a different transformation, which we discuss in detail in Chapter 35 on the theory of relativity.

Now let’s generalize this result to more than one spatial dimension. We assume that we have two coordinate systems: \( x_{\text{m}}, y_{\text{m}}, z_{\text{m}} \) and \( x_{\text{w}}, y_{\text{w}}, z_{\text{w}} \). (Here we use the subscripts \( l \) for the coordinate system that is at rest in the laboratory and \( m \) for the one that is moving.) At time \( t = 0 \), suppose the origins of both coordinate systems are located at the same point, with their axes exactly parallel to one another. As indicated in Figure 3.16, the origin of the moving \( x_{\text{m}}, y_{\text{m}}, z_{\text{m}} \) coordinate system moves with a constant translational velocity \( \vec{v}_{\text{m}} \) (blue arrow) relative to the origin of the laboratory \( x_{\text{l}}, y_{\text{l}}, z_{\text{l}} \) coordinate system. After a time \( t \), the origin of the moving \( x_{\text{m}}, y_{\text{m}}, z_{\text{m}} \) coordinate system is thus located at the point \( \vec{r}_{\text{m}} = \vec{v}_{\text{m}}t \).

We can now describe the motion of any object in either coordinate system. If the object is located at coordinate \( \vec{r}_{\text{l}} \) in the \( x_{\text{l}}, y_{\text{l}}, z_{\text{l}} \) coordinate system and at coordinate \( \vec{r}_{\text{m}} \) in the \( x_{\text{m}}, y_{\text{m}}, z_{\text{m}} \) coordinate system, then the position vectors are related to each other via simple vector addition:

\[
\vec{r}_{\text{l}} = \vec{r}_{\text{m}} + \vec{v}_{\text{m}}t. \quad (3.28)
\]

A similar relationship holds for the object’s velocities, as measured in the two coordinate systems. If the object has velocity \( \vec{v}_{\text{m}} \) in the \( x_{\text{m}}, y_{\text{m}}, z_{\text{m}} \) coordinate system and velocity \( \vec{v}_{\text{ml}} \) in the \( x_{\text{l}}, y_{\text{l}}, z_{\text{l}} \) coordinate system, these two velocities are related via:

\[
\vec{v}_{\text{ml}} = \vec{v}_{\text{om}} + \vec{v}_{\text{m}}. \quad (3.29)
\]

This equation can be obtained by taking the time derivative of equation 3.28, because \( \vec{v}_{\text{m}} \) is constant. Note that the two inner subscripts on the right-hand side of this equation are the same (and will be in any application of this equation). This makes the equation...
understandable on an intuitive level, because it says that the velocity of the object in the lab frame (subscript \( ol \)) is equal to the sum of the velocity with which the object moves relative to the moving frame (subscript \( om \)) and the velocity with which the moving frame moves relative to the lab frame (subscript \( ml \)).

Taking another time derivative produces the accelerations. Again, because \( \vec{v}_{ml} \) is constant and thus has a derivative equal to zero, we obtain, just as in the one-dimensional case,

\[
\vec{a}_{l} = \vec{a}_{m}.
\] (3.30)

The magnitude and direction of the acceleration for an object is the same in both coordinate systems.

**Example 3.3 Airplane in a Crosswind**

Airplanes move relative to the air that surrounds them. Suppose a pilot points his plane in the northeast direction. The airplane moves with a speed of 160 m/s relative to the wind, and the wind is blowing at 32.0 m/s in a direction from east to west (measured by an instrument at a fixed point on the ground).

**Problem**

What is the velocity vector—speed and direction—of the airplane relative to the ground? How far off course does the wind blow this plane in 2.0 h?

**Solution**

Figure 3.17 shows a vector diagram of the velocities. The airplane heads in the northeast direction, and the yellow arrow represents its velocity vector relative to the wind. The velocity vector of the wind is represented in orange and points due west. Graphical vector addition results in the green arrow that represents the velocity of the plane relative to the ground. To solve this problem, we apply the basic transformation of equation 3.29 embodied in the equation

\[
\vec{v}_{pg} = \vec{v}_{pw} + \vec{v}_{wg}.
\]

Here \( \vec{v}_{pg} \) is the velocity of the plane with respect to the wind and has these components:

\[
\begin{align*}
\vec{v}_{pg,x} &= v_{pw,x} \cos \theta = 160 \text{ m/s} \cdot \cos 45^\circ = 113 \text{ m/s} \\
\vec{v}_{pg,y} &= v_{pw,y} \sin \theta = 160 \text{ m/s} \cdot \sin 45^\circ = 113 \text{ m/s}.
\end{align*}
\]

The velocity of the wind with respect to the ground, \( \vec{v}_{wg} \), has these components:

\[
\begin{align*}
v_{wg,x} &= -32 \text{ m/s} \\
v_{wg,y} &= 0.
\end{align*}
\]

We next obtain the components of the airplane’s velocity relative to a coordinate system fixed to the ground, \( \vec{v}_{pg} \):

\[
\begin{align*}
\vec{v}_{pg,x} &= v_{pw,x} + v_{wg,x} = 113 \text{ m/s} - 32 \text{ m/s} = 81 \text{ m/s} \\
\vec{v}_{pg,y} &= v_{pw,y} + v_{wg,y} = 113 \text{ m/s}.
\end{align*}
\]

The absolute value of the velocity vector and its direction in the ground-based coordinate system are therefore

\[
\begin{align*}
v_{pg} &= \sqrt{v_{pg,x}^2 + v_{pg,y}^2} = 139 \text{ m/s} \\
\theta &= \tan^{-1}\left(\frac{v_{pg,y}}{v_{pg,x}}\right) = 54.4^\circ.
\end{align*}
\]

Now we need to find the course deviation due to the wind. To find this quantity, we can multiply the plane’s velocity vectors in each coordinate system by the elapsed time of 2 h = 7200 s, then take the vector difference, and finally obtain the magnitude of the
vector difference. The answer can be obtained more easily if we use equation 3.29 multiplied by the elapsed time to reflect that the course deviation, $\vec{r}_T$, due to the wind is the wind velocity, $\vec{v}_{wg}$, times 7200 s:

$$|\vec{r}_T| = |\vec{v}_{wg}| t = 32.0 \text{ m/s} \cdot 7200 \text{ s} = 230.4 \text{ km}.$$  

**DISCUSSION**

The Earth itself moves a considerable amount in 2 h, as a result of its own rotation and its motion around the Sun, and you might think we have to take these motions into account. That the Earth moves is true, but is irrelevant for the present example: The airplane, the air, and the ground all participate in this rotation and orbital motion, which is superimposed on the relative motion of the objects described in the problem. Thus, we simply perform our calculations in a coordinate system in which the Earth is at rest and not rotating.

Another interesting consequence of relative motion can be seen when observing rain while in a moving car. You may have wondered why the rain seems to come almost straight at you as you are driving. The following example answers this question.

**EXAMPLE 3.4 Driving through Rain**

Let’s suppose rain is falling straight down on a car, as indicated by the white lines in Figure 3.18. A stationary observer outside the car would be able to measure the velocities of the rain (blue arrow) and of the moving car (red arrow).

However, if you are sitting inside the moving car, the outside world of the stationary observer (including the street, as well as the rain) moves with a relative velocity of $\vec{v} = -\vec{v}_{car}$. The velocity of this relative motion has to be added to all outside events as observed from inside the moving car. This motion results in a velocity vector $\vec{v}_{rain}'$ for the rain as observed from inside the moving car (Figure 3.19); mathematically, this vector is a sum, $\vec{v}_{rain}' = \vec{v}_{rain} - \vec{v}_{car}$, where $\vec{v}_{rain}$ and $\vec{v}_{car}$ are the velocity vectors of the rain and the car as observed by the stationary observer.

**WHAT WE HAVE LEARNED | EXAM STUDY GUIDE**

- In two or three dimensions, any change in the magnitude or direction of an object’s velocity corresponds to acceleration.
- Projectile motion of an object can be separated into motion in the $x$-direction, described by the equations
  
  $$(1) \quad x = x_0 + v_{x0}t$$  
  
  $$(2) \quad v_x = v_{x0}$$

  and motion in the $y$-direction, described by
  
  $$(3) \quad y = y_0 + v_{y0}t - \frac{1}{2}gt^2$$  
  
  $$(4) \quad y = y_0 + \vec{v}_y t$$  
  
  $$(5) \quad v_y = v_{y0} - gt$$  
  
  $$(6) \quad \vec{v}_y = \frac{1}{2}(v_y + v_{y0})$$  
  
  $$(7) \quad v_y^2 = v_{y0}^2 - 2g(y - y_0)$$
The relationship between the $x$- and $y$-coordinates for ideal projectile motion can be described by a parabola given by the formula
$$y = y_0 + (\tan \theta_0) x - \frac{g}{2 v_0^2 \cos^2 \theta_0} x^2,$$
where $y_0$ is the initial vertical position, $v_0$ is the initial speed of the projectile, and $\theta_0$ is the initial angle with respect to the horizontal at which the projectile is launched.

- The range $R$ of a projectile is given by
  $$R = \frac{v_0^2 \sin 2\theta_0}{g}.$$
- The maximum height $H$ reached by an ideal projectile is given by
  $$H = y_0 + \frac{v_{\text{yo}}}{2g},$$
  where $v_{\text{yo}}$ is the vertical component of the initial velocity.

- Projectile trajectories are not parabolas when air resistance is taken into account. In general, the trajectories of realistic projectiles do not reach the maximum predicted height, and have a significantly shorter range.

- The velocity $\vec{v}_{\text{lab}}$ of an object with respect to a stationary laboratory reference frame can be calculated using a Galilean transformation of the velocity, $\vec{v}_{\text{lab}} = \vec{v}_{\text{om}} + \vec{v}_{\text{mf}}$, where $\vec{v}_{\text{om}}$ is the velocity of the object with respect to a moving reference frame and $\vec{v}_{\text{mf}}$ is the constant velocity of the moving reference frame with respect to the laboratory frame.

### Key Terms
- right-handed coordinate system, p. 72
- ideal projectile, p. 74
- ideal projectile motion, p. 74
- trajectory, p. 76
- range, p. 78
- maximum height, p. 78
- relative velocity, p. 84
- Galilean transformation, p. 85

### New Symbols
- $\vec{v}_{\text{lab}}$, relative velocity of an object with respect to a laboratory frame of reference

### Answers to Self-Test Opportunities
3.1 Use equation 3.23 and $t = (x - x_0)/v_{\text{xo}} = (x - x_0)/(v_0 \cos \theta_0)$ to find
$$\sqrt{\vec{v}} = \sqrt{v_0^2 - 2g(x - x_0)(\tan \theta_0) + g^2(x - x_0)^2(v_0 \cos \theta_0)^2}.$$

3.2 The time to reach the top is $v_{\text{yo}} = v_{\text{yo}} - gt_{\text{top}} = 0 \Rightarrow t_{\text{top}} = v_{\text{yo}}/g = v_0 \sin \theta / g$. The total flight time is $t_{\text{total}} = 2t_{\text{top}}$ because of the symmetry of the parabolic projectile trajectory. The range is the product of the total flight time and the horizontal velocity component: $R = t_{\text{total}}v_{\text{xo}} = 2t_{\text{top}}v_0 \cos \theta = 2(v_0 \sin \theta / g) v_0 \cos \theta = v_0^2 \sin(2\theta)/g$.

### Problem-Solving Practice

**Problem-Solving Guidelines**

1. In all problems involving moving reference frames, it is important to clearly distinguish which object has what motion in which frame and relative to what. It is convenient to use subscripts consisting of two letters, where the first letter stands for a particular object and the second letter for the object it is moving relative to. The moving walkway situation discussed at the opening of Section 3.6 is a good example of this use of subscripts.

2. In all problems concerning ideal projectile motion, the motion in the $x$-direction is independent of that in the $y$-direction. To solve these, you can almost always use the seven kinematical equations (3.11 through 3.17), which describe motion with constant velocity in the horizontal direction and free-fall motion with constant acceleration in the vertical direction. In general, you should avoid cookie-cutter-style application of formulas, but in exam situations, these seven kinematic equations can be your first line of defense. Keep in mind, however, that these equations work only in situations in which the horizontal acceleration component is zero and the vertical acceleration component is constant.
SOLVED PROBLEM 3.3 Time of Flight

You may have participated in Science Olympiad during middle school or high school. In one of the events in Science Olympiad, the goal is to hit a horizontal target at a fixed distance with a golf ball launched by a trebuchet. Competing teams build their own trebuchets. Your team has constructed a trebuchet that is able to launch the golf ball with an initial speed of 17.2 m/s, according to extensive tests performed before the competition.

PROBLEM

If the target is located at the same height as the elevation from which the golf ball is released and at a horizontal distance of 22.42 m away, how long will the golf ball be in the air before it hits the target?

SOLUTION

THINK

Let’s first eliminate what does not work. We cannot simply divide the distance between trebuchet and target by the initial speed, because this would imply that the initial velocity vector is in the horizontal direction. Since the projectile is in free fall in the vertical direction during its flight, it would certainly miss the target. So we have to aim the golf ball with an angle larger than zero relative to the horizontal. But at what angle do we need to aim?

If the golf ball, as stated, is released from the same height as the height of the target, then the horizontal distance between the trebuchet and the target is equal to the range. Because we also know the initial speed, we can calculate the release angle. Knowing the release angle and the initial speed lets us determine the horizontal component of the velocity vector. Since this horizontal component does not change in time, the flight time is simply given by the range divided by the horizontal component of the velocity.

SKETCH

We don’t need a sketch at this point because it would simply show a parabola, as for all projectile motion. However, we do not know the initial angle yet, so we will need a sketch later.

RESEARCH

The range of a projectile is given by equation 3.25:

\[ R = \frac{v_0^2 \sin 2\theta_0}{g} \]

If we know the value of this range and the initial speed, we can find the angle:

\[ \sin 2\theta_0 = \frac{gR}{v_0^2} \]

Once we have the value for the angle, we can use it to calculate the horizontal component of the initial velocity:

\[ v_{x0} = v_0 \cos \theta_0. \]

Finally, as noted previously, we obtain the flight time as the ratio of the range and the horizontal component of the velocity:

\[ t = \frac{R}{v_{x0}}. \]

SIMPLIFY

If we solve the equation for the angle, \( \sin 2\theta_0 = \frac{gR}{v_0^2} \), we see that it has two solutions: one for an angle of less than 45° and one for an angle of more than 45°. Figure 3.20 plots the function \( \sin 2\theta_0 \) (in red) for all possible values of the initial angle \( \theta_0 \) and shows where that curve crosses the plot of \( gR/v_0^2 \) (blue horizontal line). We call the two solutions \( \theta_a \) and \( \theta_b \).

\[ \text{FIGURE 3.20 Two solutions for the initial angle.} \]

Continued—
Algebraically, these solutions are given as

$$\theta_{a,b} = \frac{1}{2} \sin^{-1} \left( \frac{Rg}{v_0} \right).$$

Substituting this result into the formula for the horizontal component of the velocity results in

$$t = \frac{R}{v_x} = \frac{R}{v_0 \cos \theta} = \frac{R}{v_0 \cos \left( \frac{1}{2} \sin^{-1} \left( \frac{Rg}{v_0} \right) \right)}.$$

**CALCULATE**

Inserting numbers, we find:

$$\theta_{a,b} = \frac{1}{2} \sin^{-1} \left( \frac{(22.42 \text{ m})(9.81 \text{ m/s}^2)}{(17.2 \text{ m/s})^2} \right) = 24.0128^\circ \text{ or } 65.9872^\circ$$

$$t_a = \frac{R}{v_0 \cos \theta_a} = \frac{22.42 \text{ m}}{(17.2 \text{ m/s})(\cos 24.0128^\circ)} = 1.42699 \text{ s}$$

$$t_b = \frac{R}{v_0 \cos \theta_b} = \frac{22.42 \text{ m}}{(17.2 \text{ m/s})(\cos 65.9872^\circ)} = 3.20314 \text{ s}.$$

**ROUND**

The range was specified to four significant figures, and the initial speed to three. Therefore, we also state our final results to three significant figures:

$$t_a = 1.43 \text{ s}, \quad t_b = 3.20 \text{ s}.$$

Note that both solutions are valid in this case, and the team can select either one.

**DOUBLE-CHECK**

Back to the approach that does not work: simply taking the distance from the trebuchet to the target and dividing it by the speed. This incorrect procedure leads to $$t_{\text{min}} = \frac{d}{v_0} = 1.30 \text{ s}.$$ We write $$t_{\text{min}}$$ to symbolize this value to indicate that it is some lower boundary representing the case in which the initial velocity vector points horizontally and in which we neglect the free-fall motion of the projectile. Thus, $$t_{\text{min}}$$ serves as an absolute lower boundary, and it is reassuring to note that the shorter time we obtained above is a little larger than this lowest possible, but physically unrealistic, value.

**SOLVED PROBLEM 3.4 Moving Deer**

The zookeeper who captured the monkey in Example 3.1 now has to capture a deer. We found that she needed to aim directly at the monkey for that earlier capture. She decides to fire directly at her target again, indicated by the bull’s-eye in Figure 3.21.

**PROBLEM**

Where will the tranquilizer dart hit if the deer is $$d = 25 \text{ m}$$ away from the zookeeper and running from her right to her left with a speed of $$v_d = 3.0 \text{ m/s}$$? The tranquilizer dart leaves her rifle horizontally with a speed of $$v_0 = 90. \text{ m/s}$$.

**SOLUTION**

**THINK**

The deer is moving at the same time as the dart is falling, which introduces two complications. It is easiest to think about this problem in the moving reference frame of the deer.
In that frame, the sideways horizontal component of the dart’s motion has a constant velocity of \(-v_d\). The vertical component of the motion is again a free-fall motion. The total displacement of the dart is then the vector sum of the displacements caused by both of these motions.

**SKETCH**

We draw the two displacements in the reference frame of the deer (Figure 3.22). The blue arrow is the displacement due to the free-fall motion, and the red arrow is the sideways horizontal motion of the dart in the reference frame of the deer. The advantage of drawing the displacements in this moving reference frame is that the bull’s-eye is attached to the deer and is moving with it.

**RESEARCH**

First, we need to calculate the time it takes the tranquilizer dart to move 25 m in the direct line of sight from the gun to the deer. Because the dart leaves the rifle in the horizontal direction, the initial forward horizontal component of the dart’s velocity vector is 90 m/s. For projectile motion, the horizontal velocity component is constant. Therefore, for the time the dart takes to cross the 25-m distance, we have

\[
t_d = \frac{d}{v_0}.
\]

During this time, the dart falls under the influence of gravity, and this vertical displacement is

\[
\Delta y = -\frac{1}{2}gt^2.
\]

Also, during this time, the deer has a sideways horizontal displacement in the reference frame of the zookeeper of \(x = -v_d t\) (the deer moves to the left, hence the negative value of the horizontal velocity component). Therefore, the displacement of the dart in the reference frame of the deer is (see Figure 3.22)

\[
\Delta x = v_d t.
\]

**SIMPLIFY**

Substituting the expression for the time into the equations for the two displacements results in

\[
\Delta x = v_d \frac{d}{v_0} = \frac{v_d}{v_0} d
\]

\[
\Delta y = -\frac{1}{2}gt^2 = -\frac{d^2g}{2v_0^2}.
\]

**CALCULATE**

We are now ready to put in the numbers:

\[
\Delta x = \frac{(3.0 \text{ m/s})(25 \text{ m})}{(90. \text{ m/s})} = 0.833333 \text{ m}
\]

\[
\Delta y = -\frac{(25 \text{ m})^2(9.81 \text{ m/s}^2)}{2(90. \text{ m/s})^2} = -0.378472 \text{ m}.
\]

**ROUND**

Rounding our results to two significant figures gives:

\[
\Delta x = 0.83 \text{ m}
\]

\[
\Delta y = -0.38 \text{ m}.
\]

The net effect is the vector sum of the sideways horizontal and vertical displacements, as indicated by the green diagonal arrow in Figure 3.22: The dart will miss the deer and hit the ground behind the deer.

Continued—
DOUBLE-CHECK

Where should the zookeeper aim? If she wants to hit the running deer, she has to aim approximately 0.38 m above and 0.83 m to the left of her intended target. A dart fired in this direction will hit the deer, but not in the center of the bull’s-eye. Why? With this aim, the initial velocity vector does not point in the horizontal direction. This lengthens the flight time, as we just saw in Solved Problem 3.3. A longer flight time translates into a larger displacement in both x- and y-directions. This correction is small, but calculating it is a bit too involved to show here.

MULTIPLE-CHOICE QUESTIONS

3.1 An arrow is shot horizontally with a speed of 20 m/s from the top of a tower 60 m high. The time to reach the ground will be
a) 8.9 s  
 b) 7.1 s  
 c) 3.5 s  
 d) 2.6 s  
 e) 1.0 s

3.2 A projectile is launched from the top of a building with an initial velocity of 30 m/s at an angle of 60° above the horizontal. The magnitude of its velocity at t = 5 s after the launch is
a) –23.0 m/s  
 b) 7.3 m/s  
 c) 15.0 m/s  
 d) 27.5 m/s  
 e) 50.4 m/s

3.3 A ball is thrown at an angle between 0° and 90° with respect to the horizontal. Its velocity and acceleration vectors are parallel to each other at
a) 0°  
 b) 45°  
 c) 60°  
 d) 90°  
 e) none of the above

3.4 An outfielder throws the baseball to first base, located 80 m away from the fielder, with a velocity of 45 m/s. At what launch angle above the horizontal should he throw the ball for the first baseman to catch the ball in 2 s at the same height?

a) 50.74°  
 b) 25.4°  
 c) 22.7°  
 d) 18.5°  
 e) 12.6°

3.5 A 50-g ball rolls off a countertop and lands 2 m from the base of the counter. A 100-g ball rolls off the same counter top with the same speed. It lands ______ from the base of the counter.

a) less than 1 m  
 b) 1 m  
 c) 2 m  
 d) 4 m  
 e) more than 4 m

3.6 For a given initial speed of an ideal projectile, there is (are) ______ launch angle(s) for which the range of the projectile is the same.

a) only one  
 b) two different  
 c) more than two but a finite number of  
 d) only one if the angle is 45° but otherwise two different  
 e) an infinite number of

3.7 A cruise ship moves southward in still water at a speed of 20.0 km/h, while a passenger on the deck of the ship walks toward the east at a speed of 5.0 km/h. The passenger’s velocity with respect to Earth is

a) 20.6 km/h, at an angle of 14.04° east of south.  
 b) 20.6 km/h, at an angle of 14.04° south of east.  
 c) 25.0 km/h, south.  
 d) 25.0 km/h, east.  
 e) 20.6 km/h, south.

3.8 Two cannonballs are shot from different cannons at angles $\theta_01 = 20^\circ$ and $\theta_02 = 30^\circ$, respectively. Assuming ideal projectile motion, the ratio of the launching speeds, $v_{01}/v_{02}$, for which the two cannonballs achieve the same range is

a) 0.742 m  
 b) 0.862 m  
 c) 1.212 m  
 d) 1.093 m  
 e) 2.222 m

3.9 The acceleration due to gravity on the Moon is 1.62 m/s², approximately a sixth of the value on Earth. For a given initial velocity $v_0$ and a given launch angle $\theta_0$, the ratio of the range of an ideal projectile on the Moon to the range of the same projectile on Earth, $R_{\text{Moon}}/R_{\text{Earth}}$, will be

a) 6 m  
 b) 3 m  
 c) 12 m  
 d) 5 m  
 e) 1 m

3.10 A baseball is launched from the bat at an angle $\theta_0 = 30^\circ$ with respect to the positive x-axis and with an initial speed of 40 m/s, and it is caught at the same height from which it was hit. Assuming ideal projectile motion (positive y-axis upward), the velocity of the ball when it is caught is

a) $(20.00 \hat{x} + 34.64 \hat{y})$ m/s.  
 b) $(-20.00 \hat{x} + 34.64 \hat{y})$ m/s.  
 c) $(34.64 \hat{x} - 20.00 \hat{y})$ m/s.  
 d) $(34.64 \hat{x} + 20.00 \hat{y})$ m/s.

3.11 In ideal projectile motion, the velocity and acceleration of the projectile at its maximum height are, respectively,

a) horizontal, vertical downward.  
 b) horizontal, zero.  
 c) zero, zero.  
 d) zero, vertical downward.  
 e) zero, horizontal.
3.12 In ideal projectile motion, when the positive y-axis is chosen to be vertically upward, the y-component of the velocity of the object during the ascending part of the motion and the y-component of the acceleration during the descending part of the motion are, respectively,

- a) positive, negative.
- b) negative, positive.
- c) positive, positive.
- d) negative, negative.

3.13 In ideal projectile motion, when the positive y-axis is chosen to be vertically upward, the y-component of the velocity of the object during the ascending part of the motion and the y-component of the velocity during the descending part of the motion are, respectively,

- a) positive, negative.
- b) negative, positive.
- c) positive, positive.
- d) negative, negative.

3.14 A ball is thrown from ground at an angle between 0° and 90°. Which of the following remain constant: x, y, vx, vy, ax, ay?

3.15 A ball is thrown straight up by a passenger in a train that is moving with a constant velocity. Where would the ball land—back in his hands, in front of him, or behind him? Does your answer change if the train is accelerating in the forward direction? If yes, how?

3.16 A rock is thrown at an angle 45° below the horizontal from the top of a building. Immediately after release will its acceleration be greater than, equal to, or less than the acceleration due to gravity?

3.17 Three balls of different masses are thrown horizontally from the same height with different initial speeds, as shown in the figure. Rank in order, from the shortest to the longest, the times the balls take to hit the ground.

3.18 To attain maximum height for the trajectory of a projectile, what angle would you choose between 0° and 90°, assuming that you can launch the projectile with the same initial speed independent of the launch angle. Explain your reasoning.

3.19 An airplane is traveling at a constant horizontal speed v, at an altitude h above a lake when a trapdoor at the bottom of the airplane opens and a package is released (falls) from the plane. The airplane continues horizontally at the same altitude and velocity. Neglect air resistance.

- a) What is the distance between the package and the plane when the package hits the surface of the lake?
- b) What is the horizontal component of the velocity vector of the package when it hits the lake?
- c) What is the speed of the package when it hits the lake?

3.20 Two cannonballs are shot in sequence from a cannon, into the air, with the same muzzle velocity, at the same launch angle. Based on their trajectory and range, how can you tell which one is made of lead and which one is made of wood. If the same cannonballs where launched in vacuum, what would the answer be?

3.21 One should never jump off a moving vehicle (train, car, bus, etc.). Assuming, however, that one does perform such a jump, from a physics standpoint, what would be the best direction to jump in order to minimize the impact of the landing? Explain.

3.22 A boat travels at a speed of v\textsubscript{BW} relative to the water in a river of width D. The speed at which the water is flowing is v\textsubscript{W}.

- a) Prove that the time required to cross the river to a point exactly opposite the starting point and then to return is 
  \[ T = \frac{2D}{v\textsubscript{BW}^2 - v\textsubscript{W}^2}. \]
- b) Also prove that the time for the boat to travel a distance D downstream and then return is 
  \[ T_1 = \frac{2Dv\textsubscript{BW}}{v\textsubscript{BW}^2 - v\textsubscript{W}^2}. \]

3.23 A rocket-powered hockey puck is moving on a (frictionless) horizontal air-hockey table. The x- and y-components of its velocity as a function of time are presented in the graphs below. Assuming that at \( t = 0 \) the puck is at \( (x_0, y_0) = (1, 2) \), draw a detailed graph of the trajectory \( y(x) \).
In a three-dimensional motion, the x-, y-, and z-coordinates of the object as a function of time are given by
\[ x(t) = \frac{\sqrt{2}}{2}t, \quad y(t) = \frac{\sqrt{2}}{2}t, \quad \text{and} \quad z(t) = -4.9t^2 + \sqrt{3}t. \]
Describe the motion and the trajectory of the object in an xyz coordinate system.

An object moves in the xy-plane. The x- and y-coordinates of the object as a function of time are given by the following equations:
\[ x(t) = 4.9t^2 + 2t + 1 \quad \text{and} \quad y(t) = 3t + 2. \]
What is the velocity vector of the object as a function of time? What is its acceleration vector at a time \( t = 2 \) s?

A particle's motion is described by the following two parametric equations:
\[ x(t) = 5\cos(2\pi t) \quad \text{and} \quad y(t) = 5\sin(2\pi t) \]
where the displacements are in meters and \( t \) is the time, in seconds.

a) Draw a graph of the particle's trajectory (that is, a graph of \( y \) versus \( x \)).

b) Determine the equations that describe the \( x \)- and \( y \)-components of the velocity, \( v_x \) and \( v_y \), as functions of time.

c) Draw a graph of the particle's speed as a function of time.

In a proof-of-concept experiment for an antiballistic missile defense system, a missile is fired from the ground of a shooting range toward a stationary target on the ground. The system detects the missile by radar, analyzes in real time its parabolic motion, and determines that it was fired from a distance \( x_0 = 5000 \) m, with an initial speed of 600 m/s at a launch angle \( \theta_0 = 20^\circ \). The defense system then calculates the required time delay measured from the launch of the missile and fires a small rocket situated at \( y_0 = 500 \) m with an initial velocity of \( v_0 \), m/s at a launch angle \( \alpha_0 = 60^\circ \) in the \( yz \)-plane, to intercept the missile. Determine the initial speed \( v_0 \) of the intercept rocket and the required time delay.

A projectile is launched at an angle of \( 45^\circ \) above the horizontal. What is the ratio of its horizontal range to its maximum height? How does the answer change if the initial speed of the projectile is doubled?

In a projectile motion, the horizontal range and the maximum height attained by the projectile are equal.

a) What is the launch angle?

b) If everything else stays the same, how should the launch angle, \( \theta_0 \), of a projectile be changed for the range of the projectile to be halved?

An air-hockey puck has a model rocket rigidly attached to it. The puck is pushed from one corner along the long side of the 2-m long air-hockey table, with the rocket pointing along the short side of the table, and at the same time the rocket is fired. If the rocket thrust imparts an acceleration of 2 m/s\(^2\) to the puck, and the table is 1 m wide, with what minimum initial velocity should the puck be pushed to make it to the opposite short side of the table without bouncing off either long side of the table?

Draw the trajectory of the puck for three initial velocities: \( v < v_{\text{min}} \), \( v = v_{\text{min}} \), and \( v > v_{\text{min}} \). Neglect friction and air resistance.

On a battlefield, a cannon fires a cannonball up a slope, from ground level, with an initial velocity \( v_0 \) at an angle \( \theta_0 \) above the horizontal. The ground itself makes an angle \( \alpha \) above the horizontal (\( \alpha < \theta_0 \)). What is the range \( R \) of the cannonball, measured along the inclined ground? Compare your result with the equation for the range on horizontal ground (equation 3.25).

Two swimmers with a soft spot for physics engage in a peculiar race that models a famous optics experiment: the Michelson-Morley experiment. The race takes place in a river 50 m wide that is flowing at a steady rate of 3 m/s. Both swimmers start at the same point on one bank and swim at the same speed of 5 m/s with respect to the stream. One of the swimmers swims directly across the river to the closest point on the opposite bank and then turns around and swims back to the starting point. The other swimmer swims along the river bank, first upstream a distance exactly equal to the width of the river and then downstream back to the starting point. Who gets back to the starting point first?
3.36 A truck travels 3.02 km north and then makes a 90° left turn and drives another 4.30 km. The whole trip takes 5.00 min.

a) With respect to a two-dimensional coordinate system on the surface of Earth such that the y-axis points north, what is the net displacement vector of the truck for this trip?

b) What is the magnitude of the average velocity for this trip?

c) What is the speed at \( t = 10 \) s?

\[ \vec{r}(t) = 2.0 \text{ m/s}^2 (2.43 \text{ m/s}^2), \]

\[ (74.4 \text{ m}) + \vec{r} (1.80 \text{ m/s}^2) - \vec{r} (0.130 \text{ m/s}^2) \]

\[ \text{a) Calculate the rabbit’s position (magnitude and direction) at } t = 10 \text{ s.} \]

\[ \text{b) Calculate the rabbit’s velocity at } t = 10 \text{ s.} \]

\[ \text{c) Determine the acceleration vector at } t = 10 \text{ s.} \]

\[ 3.37 \text{ A rabbit runs in a garden such that the } x \text{- and } y \text{-components of its displacement as function of times are given by } x(t) = -0.45t^2 - 6.5t + 25 \text{ and } y(t) = 0.35t^2 + 8.3t + 34. \]

(Both \( x \) and \( y \) are in meters and \( t \) is in seconds.)

\[ \text{a) Calculate the rabbit’s position (magnitude and direction) at } t = 10 \text{ s.} \]

\[ \text{b) Calculate the rabbit’s velocity at } t = 10 \text{ s.} \]

\[ \text{c) Determine the acceleration vector at } t = 10 \text{ s.} \]

\[ 3.38 \text{ Some rental cars have a GPS unit installed, which allows the rental car company to check where you are at all times and thus also know your speed at any time. One of these rental cars is driven by an employee in the company’s lot, and, during the time interval from 0 to 10 s, is found to have a position vector as a function of time of} \]

\[ \vec{r}(t) = \begin{pmatrix} 24.4 \text{ m} \\ (12.3 \text{ m/s}) \text{ + } t^2 \text{ (2.43 m/s}^2) \end{pmatrix} \text{,} \]

\[ \begin{pmatrix} 74.4 \text{ m} \\ t^2 \text{ (1.80 m/s}^2) \text{ – } t^3 \text{ (0.130 m/s}^3) \end{pmatrix} \]

\[ \text{a) What is the distance of this car from the origin of the coordinate system at } t = 5.00 \text{ s?} \]

\[ \text{b) What is the velocity vector as a function of time?} \]

\[ \text{c) What is the speed at } t = 5.00 \text{ s?} \]

\[ \text{Extra credit: Can you produce a plot of the trajectory of the car in the } xy \text{-plane?} \]

\[ \text{Section 3.3} \]

3.39 A skier launches off a ski jump with a horizontal velocity of 30.0 m/s (and no vertical velocity component). What are the magnitudes of the horizontal and vertical components of her velocity the instant before she lands 2 s later?

3.40 An archer shoots an arrow from a height of 1.14 m above ground with an initial velocity of 47.5 m/s and an initial angle of 35.2° above the horizontal. At what time after the release of the arrow from the bow will the arrow be flying exactly horizontally?

3.41 A football is punted with an initial velocity of 27.5 m/s and an initial angle of 56.7°. What is its hang time (the time until it hits the ground again)?

3.42 You serve a tennis ball from a height of 1.8 m above the ground. The ball leaves your racket with a speed of 18.0 m/s at an angle of 7.00° above the horizontal. The horizontal distance from the court’s baseline to the net is 11.83 m, and the net is 1.07 m high. Neglect spin imparted on the ball as well as air resistance effects. Does the ball clear the net? If yes, by how much? If not, by how much did it miss?

3.43 Stones are thrown horizontally with the same velocity from two buildings. One stone lands twice as far away from its building as the other stone. Determine the ratio of the heights of the two buildings.

3.44 You are practicing throwing darts in your dorm. You stand 3.0 m from the wall on which the board hangs. The dart leaves your hand with a horizontal velocity at a point 2.0 m above the ground. The dart strikes the board at a point 1.65 m from the ground. Calculate:

a) the time of flight of the dart;

b) the initial speed of the dart;

c) the velocity of the dart when it hits the board.

3.45 A football player kicks a ball with a speed of 22.4 m/s at an angle of 49° above the horizontal from a distance of 39 m from the goal line.

a) By how much does the ball clear or fall short of clearing the crossbar of the goalpost if that bar is 3.05 m high?

b) What is the vertical velocity of the ball at the time it reaches the goalpost?

3.46 An object fired at an angle of 35° above the horizontal takes 1.5 s to travel the last 15 m of its vertical distance and the last 10 m of its horizontal distance. With what velocity was the object launched?

3.47 A conveyor belt is used to move sand from one place to another in a factory. The conveyor is tilted at an angle of 14° from the horizontal and the sand is moved without slipping at the rate of 7 m/s. The sand is collected in a big drum 3 m below the end of the conveyor belt. Determine the horizontal distance between the end of the conveyor belt and the middle of the collecting drum.

3.48 Your friend’s car is parked on a cliff overlooking the ocean on an incline that makes an angle of 17.0° below the horizontal. The brakes fail, and the car rolls from rest down the incline for a distance of 29.0 m to the edge of the cliff, which is 55.0 m above the ocean, and, unfortunately, continues over the edge and lands in the ocean.

a) Find the car’s position relative to the base of the cliff when the car lands in the ocean.

b) Find the length of time the car is in the air.

3.49 An object is launched at a speed of 20.0 m/s from the top of a tall tower. The height \( y \) of the object as a function of the time \( t \) elapsed from launch is \( y(t) = -4.9t^2 + 19.32t + 60 \), where \( h \) is in meters and \( t \) is in seconds. Determine:

a) the height \( H \) of the tower;

b) the launch angle;

c) the horizontal distance traveled by the object before it hits the ground.

3.50 A projectile is launched at a 60° angle above the horizontal on level ground. The change in its velocity between launch and just before landing is found to be \( \Delta \vec{v} = \vec{v}_{\text{landing}} - \vec{v}_{\text{launch}} = 20 \text{ m/s} \). What is the initial velocity of the projectile? What is its final velocity just before landing?
3.51 The figure shows the paths of a tennis ball your friend drops from the window of her apartment and of the rock you throw from the ground at the same instant. The rock and the ball collide at $x = 50\,\text{m}$, $y = 10\,\text{m}$ and $t = 3\,\text{s}$. If the ball was dropped from a height of 54 m, determine the velocity of the rock initially and at the time of its collision with the ball.

3.52 For a science fair competition, a group of high school students build a kicker-machine that can launch a golf ball from the origin with a velocity of 11.2 m/s and initial angle of 31.5° with respect to the horizontal.
   a) Where will the golf ball fall back to the ground?
   b) How high will it be at the highest point of its trajectory?
   c) What is the ball’s velocity vector (in Cartesian components) at the highest point of its trajectory?
   d) What is the ball’s acceleration vector (in Cartesian components) at the highest point of its trajectory?

3.53 If you want to use a catapult to throw rocks and the maximum range you need these projectiles to have is 0.67 km, what initial speed do your projectiles have to have as they leave the catapult?

3.54 What is the maximum height above ground a projectile of mass 0.79 kg, launched from ground level, can achieve if you are able to give it an initial speed of 80.3 m/s?

3.55 During one of the games, you were asked to punt for your football team. You kicked the ball at an angle of 35° with a velocity of 25 m/s. If your punt goes straight down the field, determine the average velocity at which the running back of the opposing team standing at 70 m from you must run to catch the ball at the same height as you released it? Assume that the running back starts running as the ball leaves your foot and that the air resistance is negligible.

3.56 By trial and error, a frog learns that it can leap a maximum horizontal distance of 1.3 m. If, in the course of an hour, the frog spends 20% of the time resting and 80% of the time performing identical jumps of that maximum length, in a straight line, what is the distance traveled by the frog?

3.57 A circus juggler performs an act with balls that he tosses with his right hand and catches with his left hand. Each ball is launched at an angle of 75° and reaches a maximum height of 90 cm above the launching height. If it takes the juggler 0.2 s to catch a ball with his left hand, pass it to his right hand and toss it back into the air, what is the maximum number of balls he can juggle?

3.58 In an arcade game, a ball is launched from the corner of a smooth inclined plane. The inclined plane makes a 30° angle with the horizontal and has a width of $w = 50\,\text{cm}$. The spring-loaded launcher makes an angle of 45° with the lower edge of the inclined plane. The goal is to get the ball in a small hole at the opposite corner of the inclined plane. With what initial velocity should you launch the ball to achieve this goal?

3.59 A copy-cat daredevil tries to reenact Evel Knievel’s 1974 attempt to jump the Snake River Canyon in a rocket-powered motorcycle. The canyon is $L = 400\,\text{m}$ wide, with the opposite rims at the same height. The height of the launch ramp at one rim of the canyon is $h = 8\,\text{m}$ above the rim, and the angle of the end of the ramp is 45° with the horizontal.

   a) What is the minimum launch speed required for the daredevil to make it across the canyon? Neglect the air resistance and wind.
   b) Famous after his successful first jump, but still recovering from the injuries sustained in the crash caused by a strong bounce upon landing, the daredevil decides to jump again but to add a landing ramp with a slope that will match the angle of his velocity at landing. If the height of the landing ramp at the opposite rim is 3 m, what is the new required launch speed, and how far from the launch rim and at what height should the edge of the landing ramp be?

3.60 A golf ball is hit with an initial angle of 35.5° with respect to the horizontal and an initial velocity of 83.3 mph. It lands a distance of 86.8 m away from where it was hit. By how much did the effects of wind resistance, spin, and so forth reduce the range of the golf ball from the ideal value?

3.61 You are walking on a moving walkway in an airport. The length of the walkway is 59.1 m. If your velocity relative to the walkway is 2.35 m/s and the walkway moves with a velocity of 1.77 m/s, how long will it take you to reach the other end of the walkway?
3.62 The captain of a boat wants to travel directly across a river that flows due east with a speed of 1.00 m/s. He starts from the south bank of the river and heads toward the north bank. The boat has a speed of 6.10 m/s with respect to the water. What direction (in degrees) should the captain steer the boat? Note that 90° is east, 180° is south, 270° is west, and 360° is north.

3.63 The captain of a boat wants to travel directly across a river that flows due east. He starts from the south bank of the river and heads toward the north bank. The boat has a speed of 5.57 m/s with respect to the water. The captain steers the boat in the direction 315°. How fast is the water flowing? Note that 90° is east, 180° is south, 270° is west, and 360° is north.

• 3.64 The air speed indicator of a plane that took off from Detroit reads 350 km/h and the compass indicates that it is heading due east to Boston. A steady wind is blowing due north at 40 km/h. Calculate the velocity of the plane with reference to the ground. If the pilot wishes to fly directly to Boston (due east) what must the compass read?

• 3.65 You want to cross a straight section of a river that has a uniform current of 5.33 m/s and is 127. m wide. Your motorboat has an engine that can generate a speed of 17.5 m/s for your boat. Assume that you reach top speed right away (that is, neglect the time it takes to accelerate the boat to top speed).

a) If you want to go directly across the river with a 90° angle relative to the riverbank, at what angle relative to the riverbank should you point your boat?

b) How long will it take to cross the river in this way?

c) In which direction should you aim your boat to achieve minimum crossing time?

d) What is the minimum time to cross the river?

e) What is the minimum speed of your boat that will still enable you to cross the river with a 90° angle relative to the riverbank?

• 3.66 During a long airport layover, a physicist father and his 8-year-old daughter try a game that involves a moving walkway. They have measured the walkway to be 42.5 m long. The father has a stopwatch and times his daughter. First, the daughter walks with a constant speed in the same direction as the conveyor. It takes 15.2 s to reach the end of the walkway. Then, she turns around and walks with the same speed relative to the conveyor as before in the opposite direction. The return leg takes 70.8 s. What is the speed of the walkway conveyor relative to the terminal, and with what speed was the girl walking?

• 3.67 An airplane has an air speed of 126.2 m/s and is flying due north, but the wind blows from the northeast to the southwest at 55.5 m/s. What is the plane's actual ground speed?

Additional Problems

3.68 A cannon is fired from a hill 116.7 m high at an angle of 22.7° with respect to the horizontal. If the muzzle velocity is 36.1 m/s, what is the speed of a 4.35-kg cannonball when it hits the ground 116.7 m below?

3.69 A baseball is thrown with a velocity of 31.1 m/s at an angle of $\theta = 33.4°$ above horizontal. What is the horizontal component of the ball’s velocity at the highest point of the ball’s trajectory?

3.70 A rock is thrown horizontally from the top of a building with an initial speed of $v = 10.1$ m/s. If it lands $d = 57.1$ m from the base of the building, how high is the building?

3.71 A car is moving at a constant 19.3 m/s, and rain is falling at 8.9 m/s straight down. What angle $\theta$ (in degrees) does the rain make with respect to the horizontal as observed by the driver?

3.72 You passed the salt and pepper shakers to your friend at the other end of a table of height 0.85 m by sliding them across the table. They both missed your friend and slid off the table, with velocities of 5 m/s and 2.5 m/s, respectively.

a) Compare the times it takes the shakers to hit the floor.

b) Compare the distance that each shaker travels from the edge of the table to the point it hits the floor.

3.73 A box containing food supplies for a refugee camp was dropped from a helicopter flying horizontally at a constant elevation of 500 m. If the box hit the ground at a distance of 150 m horizontally from the point of its release, what was the speed of the helicopter? With what speed did the box hit the ground?

3.74 A car drives straight off the edge of a cliff that is 60 m high. The police at the scene of the accident note that the point of impact is 150 m from the base of the cliff. How fast was the car traveling when it went over the cliff?

3.75 At the end of the spring term, a high school physics class celebrates by shooting a bundle of exam papers into the town landfill with a homemade catapult. They aim for a point that is 30 m away and at the same height from which the catapult releases the bundle. The initial horizontal velocity component is 3.9 m/s. What is the initial velocity component in the vertical direction? What is the launch angle?

3.76 Salmon often jump upstream through waterfalls to reach their breeding grounds. One salmon came across a waterfall 1.05 m in height, which she jumped in 2.1 s at an angle of 35° to continue upstream. What was the initial speed of her jump?

3.77 A firefighter, 60 m away from a burning building, directs a stream of water from a ground-level fire hose at an angle of 37° above the horizontal. If the water leaves the hose at 40.3 m/s, which floor of the building will the stream of water strike? Each floor is 4 m high.
3.78 A projectile leaves ground level at an angle of 68° above the horizontal. As it reaches its maximum height, \( H \), it has traveled a horizontal distance, \( d \), in the same amount of time. What is the ratio \( H/d? \)

3.79 The McNamara Northwest terminal at the Metro Detroit Airport has moving walkways for the convenience of the passengers. Robert walks beside one walkway and takes 30.0 s to cover its length. John simply stands on the walkway and covers the same distance in 13.0 s. Kathy walks on the walkway with the same speed as Robert’s. How long does Kathy take to complete her stroll?

3.80 Rain is falling vertically at a constant speed of 7.0 m/s. At what angle from the vertical do the raindrops appear to be falling to the driver of a car traveling on a straight road with a speed of 60 km/h?

3.81 To determine the gravitational acceleration at the surface of a newly discovered planet, scientists perform a projectile motion experiment. They launch a small model rocket at an initial speed of 50 m/s and an angle of 30° above the horizontal and measure the (horizontal) range on flat ground to be 2165 m. Determine the value of \( g \) for the planet.

3.82 A diver jumps from a 40 m high cliff into the sea. Rocks stick out of the water for a horizontal distance of 7 m from the foot of the cliff. With what minimum horizontal speed must the diver jump off the cliff in order to clear the rocks and land safely in the sea?

3.83 An outfielder throws a baseball with an initial speed of 32 m/s at an angle of 23° to the horizontal. The ball leaves his hand from a height of 1.83 m. How long is the ball in the air before it hits the ground?

3.84 A rock is tossed off the top of a cliff of height 34.9 m. Its initial speed is 29.3 m/s, and the launch angle is 29.9° with respect to the horizontal. What is the speed with which the rock hits the ground at the bottom of the cliff?

3.85 During the 2004 Olympic Games, a shot putter threw a shot put with a speed of 13.0 m/s at an angle of 43° above the horizontal. She released the shot put from a height of 2 m above the ground.

a) How far did the shot put travel in the horizontal direction?

b) How long was it until the shot put hit the ground?

3.86 A salesman is standing on the Golden Gate Bridge in a traffic jam. He is at a height of 71.8 m above the water below. He receives a call on his cell phone that makes him so mad that he throws his phone horizontally off the bridge with a speed of 23.7 m/s.

a) How far does the cell phone travel horizontally before hitting the water?

b) What is the speed with which the phone hits the water?

3.87 A security guard is chasing a burglar across a rooftop, both running at 4.2 m/s. Before the burglar reaches the edge of the roof, he has to decide whether or not to try jumping to the roof of the next building, which is 5.5 m away and 4.0 m lower. If he decides to jump horizontally to get away from the guard, can he make it? Explain your answer.

3.88 A blimp is ascending at the rate of 7.5 m/s at a height of 80 m above the ground when a package is thrown from its cockpit horizontally with a speed of 4.7 m/s.

a) How long does it take for the package to reach the ground?

b) With what velocity (magnitude and direction) does it hit the ground?

3.89 Wild geese are known for their lack of manners. One goose is flying northward at a level altitude of \( h_g = 30.0 \) m above a north-south highway, when it sees a car ahead in the distance moving in the southbound lane and decides to deliver (drop) an “egg.” The goose is flying at a speed of \( v_g = 15.0 \) m/s , and the car is moving at a speed of \( v_c = 100.0 \) km/h.

a) Given the details in the figure, where the separation between the goose and the front bumper of the car, \( d = 104.1 \) m, is specified at the instant when the goose takes action, will the driver have to wash the windshield after this encounter? (The center of the windshield is \( h_c = 1.00 \) m off the ground.)

b) If the delivery is completed, what is the relative velocity of the “egg” with respect to the car at the moment of the impact?

3.90 You are at the mall on the top step of a down escalator when you lean over laterally to see your 1.8 m tall physics professor on the bottom step of the adjacent up escalator. Unfortunately, the ice cream you hold in your hand falls out of its cone as you lean. The two escalators have identical angles of 40° with the horizontal, a vertical height of 10 m, and move at the same speed of 0.4 m/s. Will the ice cream land on your professor’s head? Explain. If it does land on his head, at what time and at what vertical height does that happen? What is the relative speed of the ice cream with respect to the head at the time of impact?
• 3.91 A basketball player practices shooting three-pointers from a distance of 7.50 m from the hoop, releasing the ball at a height of 2.00 m above ground. A standard basketball hoop's rim top is 3.05 m above the floor. The player shoots the ball at an angle of 48° with the horizontal. At what initial speed must he shoot to make the basket?

• 3.92 Wanting to invite Juliet to his party, Romeo is throwing pebbles at her window with a launch angle of 37° from the horizontal. He is standing at the edge of the rose garden 7.0 m below her window and 10.0 m from the base of the wall. What is the initial speed of the pebbles?

• 3.93 An airplane flies horizontally above the flat surface of a desert at an altitude of 5 km and a speed of 1000 km/h. If the airplane is to drop a care package that is supposed to hit a target on the ground, where should the plane be with respect to the target when the package is released? If the target covers a circular area with a diameter of 50 m, what is the “window of opportunity” (or margin of error allowed) for the release time?

• 3.94 A plane diving with constant speed at an angle of 49° degrees with the vertical, releases a package at an altitude of 600 m. The package hits the ground 3.5 s after release. How far horizontally does the package travel?

• 3.95 Ten seconds after being fired, a cannonball strikes a point 500 m horizontally from and 100 m vertically above the point of launch.
   a) With what initial velocity was the cannonball launched?
   b) What maximum height was attained by the ball?
   c) What is the magnitude and direction of the ball’s velocity just before it strikes the given point?

• 3.96 Neglect air resistance for the following. A soccer ball is kicked from the ground into the air. When the ball is at a height of 12.5 m, its velocity is \((5.6\hat{x} + 4.1\hat{y})\) m/s.
   a) To what maximum height will the ball rise?
   b) What horizontal distance will be traveled by the ball?
   c) With what velocity (magnitude and direction) will it hit the ground?