Niagara Falls is one of the most spectacular sights in the world, with about 2000 cubic meters of water dropping 49 m (161 ft) every second! Horse-shoe Falls on the Canadian border, shown in Figure 6.1, has a length of 790 m (2592 ft); American Falls on the United States side extends another 305 m (1001 ft). Together, they are one of the great tourist attractions of North America. However, Niagara Falls is more than a scenic wonder. It is also one of the largest sources of electric power in the world, producing over 2500 megawatts (see Solved Problem 6.1). Humans have exploited the energy of falling water since ancient times, using it to turn large paddlewheels for mills and factories. Today, the conversion of energy in falling water to electrical energy by hydroelectric dams is a major source of energy throughout the world.

As we saw in Chapter 5, energy is a fundamental concept in physics that governs many of the interactions involving forces and motions of objects. In this chapter, we continue our study of energy, introducing several new forms of energy and new laws that govern its use. We will return to the laws of energy in the chapters on thermodynamics, building on much of the material presented here. Until then, we will continue our study of mechanics, relying heavily on the ideas discussed here.
Chapter 5 examined in detail the relationship between kinetic energy and work, and one of the main points was that work and kinetic energy can be converted into one another. Now, this section introduces another kind of energy, called potential energy.

Potential energy, \( U \), is the energy stored in the configuration of a system of objects that exert forces on one another. For example, we have seen that work is done by an external force in lifting a load against the force of gravity, and this work is given by \( W = \text{mg}h \), where \( m \) is the mass of the load and \( h = y - y_0 \) is the height to which the load is lifted above its initial position. (In this chapter, we will assume the y-axis points upward unless specified differently.) This lifting can be accomplished without changing the kinetic energy, as in the case of a weightlifter who lifts a mass above his head and holds it there. There is energy stored in holding the mass above the head. If the weightlifter lets go of the mass, this energy can be converted back into kinetic energy as the mass accelerates and falls to the ground.

We can express the gravitational potential energy as

\[
U_g = mgy. \tag{6.1}
\]

The change in the gravitational potential energy of the mass is then

\[
\Delta U_g = U_g(y) - U_g(y_0) = mg(y - y_0) = mgh. \tag{6.2}
\]

(Equation 6.1 is valid only near the surface of the Earth, where \( F_g = \text{mg} \), and in the limit that Earth is infinitely massive relative to the object. We will encounter a more general expression for \( U_g \) in Chapter 12.) In Chapter 5, we found that the work done by the gravitational force on an object that is lifted through a height \( h \) is \( W_g = -mgh \). From this, we see that the work done by the gravitational force and the gravitational potential energy for an object lifted from rest to a height \( h \) are related by

\[
\Delta U_g = -W_g. \tag{6.3}
\]

**Example 6.1: Weightlifting**

**Problem**

Let's consider the gravitational potential energy in a specific situation: a weightlifter lifting a barbell of mass \( m \). What is the gravitational potential energy and the work done during the different phases of lifting the barbell?

– Continued
Equation 6.3 is true even for complicated paths involving horizontal as well as vertical motion of the object, because the gravitational force does no work during horizontal segments of the motion. In horizontal motion, the displacement is perpendicular to the force of gravity (which always points vertically down), and thus the scalar product between the force and displacement vectors is zero; hence, no work is done.

Lifting any mass to a higher elevation involves doing work against the force of gravity and generates an increase in gravitational potential energy of the mass. This energy can be stored for later use. This principle is employed, for example, at many hydroelectric dams. The excess electricity generated by the turbines is used to pump water to a reservoir at a higher elevation. There it constitutes a reserve that can be tapped in times of high energy demand and/or low water supply. Stated in general terms, if \( U_g \) is positive, there exists the potential (hence the name potential energy) to allow \( U_g \) to be negative in the future, thereby extracting positive work, since \( W_g = -U_g \).

**Solved Problem 6.1 Power Produced by Niagara Falls**

**Problem**
The Niagara River delivers an average of 5520 m³ of water per second to the top of Niagara Falls, where it drops 49.0 m. If all the potential energy of that water could be converted to electrical energy, how much electrical power could Niagara Falls generate?

**Solution**

The mass of 1 m³ of water is 1000 kg. The work done by the falling water is equal to the change in its gravitational potential energy. The average power is the work per unit time.
Before we can calculate the potential energy from a given force, we have to ask: Can all kinds of forces be used to store potential energy for later retrieval? If not, what kinds of forces can we use? To answer this question, we need to consider what happens to the work done by a force when the direction of the path taken by an object is reversed. In the case of the gravitational force, we have already seen what happens. As shown in Figure 6.4, the work done by $F_g$ when an object of mass $m$ is lifted from elevation $y_A$ to $y_B$ has the same magnitude as, but the opposite sign to, the work done by $F_g$ when lowering the same object from elevation $y_B$ to $y_A$. This means that the total work done by $F_g$ in lifting the object from some elevation to a different one and then returning it to the same elevation is zero. This fact is the basis for the definition of a conservative force (refer to Figure 6.5a).

**Definition**

A **conservative force** is any force for which the work done over any closed path is zero. A force that does not fulfill this requirement is called a **nonconservative force**.
For conservative forces, we can immediately state two consequences of this definition:

1. If we know the work, \( W_{A \rightarrow B} \), done by a conservative force on an object as the object moves along a path from point A to point B, then we also know the work, \( W_{B \rightarrow A} \), that the same force does on the object as it moves along the path in the reverse direction, from point B to point A (see Figure 6.5b):
   \[
   W_{B \rightarrow A} = -W_{A \rightarrow B} \quad \text{(for conservative forces).} \tag{6.4}
   \]
   The proof of this statement is obtained from the condition of zero work over a closed loop. Because the path from A to B to A forms a closed loop, the sum of the work contributions from the loop has to equal zero. In other words,
   \[
   W_{A \rightarrow B} + W_{B \rightarrow A} = 0,
   \]
   from which equation 6.4 follows immediately.

2. If we know the work, \( W_{A \rightarrow B, \text{path } 1} \), done by a conservative force on an object moving along path 1 from point A to point B, then we also know the work, \( W_{B \rightarrow A, \text{path } 2} \), done by the same force on the object when it uses any other path (path 2) to go from point A to point B (see Figure 6.5c). The work is the same; the work done by a conservative force is independent of the path taken by the object:
   \[
   W_{A \rightarrow B, \text{path } 2} = W_{A \rightarrow B, \text{path } 1} \quad \text{(for arbitrary paths 1 and 2, for conservative forces).} \tag{6.5}
   \]
   This statement is also easy to prove from the definition of a conservative force as a force for which the work done over any closed path is zero. The path from point A to point B on path 1 and then back from B to A on path 2 is a closed loop; therefore, \( W_{A \rightarrow B, \text{path } 2} + W_{B \rightarrow A, \text{path } 1} = 0 \). Now we use equation 6.4 for reversal of the path direction, \( W_{B \rightarrow A, \text{path } 1} = -W_{A \rightarrow B, \text{path } 1} \). Combining these two results gives us
   \[
   W_{A \rightarrow B, \text{path } 2} - W_{A \rightarrow B, \text{path } 1} = 0, \quad \text{from which equation 6.5 follows.}
   \]

One physical application of the mathematical results just given involves riding a bicycle from one point, such as your home, to another point, such as the swimming pool. Assuming that your home is located at the foot of a hill and the pool at the top, we can use the Figure 6.5c to illustrate this example, with point A representing your home and point B the pool. What the preceding statements regarding conservative forces mean is that you do the same amount of work riding your bike from home to the pool, independent of the route you select. You can take a shorter and steeper route or a flatter and longer route; you can even take a route that goes up and down between points A and B. The total work will be the same. As with almost all real-world examples, however, there
are some complications here: It matters whether you use the handbrakes; there is air resistance and tire friction to consider; and your body also performs other metabolic functions during the ride, in addition to moving your mass and that of the bicycle from point \( A \) to \( B \). But this example can help you develop a mental picture of the concepts of path-independence of work and conservative forces.

The gravitational force, as we have seen, is an example of a conservative force. Another example of a conservative force is the spring force. Not all forces are conservative, however. Which forces are nonconservative?

### Friction Forces

Let’s consider what happens in sliding a box across a horizontal surface, from point \( A \) to point \( B \) and then back to point \( A \), if the coefficient of kinetic friction between the box and the surface is \( \mu_k \) (Figure 6.6). As we have learned, the friction force is given by \( f = \mu_k N = \mu_k mg \) and always points in the direction opposite to that of the motion. Let’s use results from Chapter 5 to find the work done by this friction force. Since the friction force is constant, the amount of work it does is found by simply taking the scalar product between the friction force and displacement vectors.

For the motion from \( A \) to \( B \), we use the general scalar product formula for work done by a constant force:

\[
W_{f1} = \vec{f} \cdot \Delta \vec{r} = -f \cdot (x_B - x_A) = -\mu_k mg \cdot (x_B - x_A).
\]

We have assumed that the positive \( x \)-axis is pointing to the right, as is conventional, so the friction force points in the negative \( x \)-direction. For the motion from \( B \) back to \( A \), then, the friction force points in the positive \( x \)-direction. Therefore, the work done for this part of the path is

\[
W_{f2} = \vec{f} \cdot \Delta \vec{r} = f \cdot (x_A - x_B) = \mu_k mg \cdot (x_A - x_B).
\]

This result leads us to conclude that the total work done by the friction force while the box slides across the surface on the closed path from point \( A \) to point \( B \) and back to point \( A \) is not zero, but instead

\[
W_f = W_{f1} + W_{f2} = -2\mu_k mg(x_B - x_A) < 0. \tag{6.6}
\]

There appears to be a contradiction between this result and the work–kinetic energy theorem. The box starts with zero kinetic energy and at a certain position, and it ends up with zero kinetic energy and at the same position. According to the work–kinetic energy theorem, the total work done should be zero. This leads us to conclude that the friction force does not do work in the way that a conservative force does. Instead, the friction force converts kinetic and/or potential energy into internal excitation energy of the two objects that exert friction on each other (the box and the support surface, in this case). This internal excitation energy can take the form of vibrations or thermal energy or even chemical or electrical energy. The main point is that the conversion of kinetic and/or potential energy to internal excitation energy is not reversible; that is, the internal excitation energy cannot be fully converted back into kinetic and/or potential energy.

Thus, we see that the friction force is an example of a nonconservative force. Because the friction force always acts in a direction opposite to the displacement, the dissipation of energy due to the friction force always reduces the total mechanical energy, whether or not the path is closed. Work done by a conservative force, \( W_c \), can be positive or negative, but the dissipation from the friction force, \( W_f \), is always negative, withdrawing kinetic and/or potential energy and converting it into internal excitation energy. Using the symbol \( W_f \) for this dissipated energy is a reminder that we use the same procedures to calculate it as to calculate work for conservative forces.

The decisive fact is that the friction force switches direction as a function of the direction of motion and causes dissipation. The friction force vector is always antiparallel to the velocity vector; any force with this property cannot be conservative. Dissipation...
converts kinetic energy into internal energy of the object, which is another important characteristic of a nonconservative force. In Section 6.7, we will examine this point in more detail.

Another example of a nonconservative force is the force of air resistance. It is also velocity dependent and always points in the direction opposite to the velocity vector, just like the force of kinetic friction. Yet another example of a nonconservative force is the damping force (discussed in Chapter 14). It, too, is velocity dependent and opposite in direction to the velocity.

### 6.3 Work and Potential Energy

In considering the work done by the gravitational force and its relationship to gravitational potential energy in Section 6.1, we found that the change in potential energy is equal to the negative of the work done by the force, \( \Delta U_g = -W_g \). This relationship is true for all conservative forces. In fact, we can use it to define the concept of potential energy.

For any conservative force, the change in potential energy due to some spatial rearrangement of a system is equal to the negative of the work done by the conservative force during this spatial rearrangement:

\[
\Delta U = -W.
\]

We have already seen that work is given by

\[
W = \int_{x_0}^{x} F(x')dx'.
\]

Combining equations 6.7 and 6.8 gives us the relationship between the conservative force and the potential energy:

\[
\Delta U = U(x) - U(x_0) = -\int_{x_0}^{x} F(x')dx'.
\]

We could use equation 6.9 to calculate the potential energy change due to the action of any given conservative force. Why should we bother with the concept of potential energy when we can deal directly with the conservative force itself? The answer is that the change in potential energy depends only on the beginning and final states of the system and is independent of the path taken to get to the final state. Often, we have a simple expression for the potential energy (and thus its change) prior to working a problem! In contrast, evaluating the integral on the right-hand side of equation 6.9 could be quite complicated. And, in fact, the computational savings is not the only rationale, as the use of energy considerations is based on an underlying physical law (the law of energy conservation, to be introduced in Section 6.5).

In Chapter 5, we evaluated the integral in equation 6.9 for the force of gravity and for the spring force. The result for the gravitational force is

\[
\Delta U_g = U_g(y) - U_g(y_0) = -\int_{y_0}^{y} (-mg)dy' = mg\int_{y_0}^{y} dy' = mg(y - y_0).
\]

This is in accord with the result we found in Section 6.1. Consequently, the gravitational potential energy is

\[
U_g(y) = mgy + \text{constant}.
\]

Note that we are able to determine the potential energy at coordinate \( y \) only to within an additive constant. The only physically observable quantity, the work done, is related to the difference in the potential energy. If we add an arbitrary constant to the value of the potential energy everywhere, the difference in the potential energies remains unchanged.
In the same way, we find for the spring force that

\[
\Delta U_s = U_s(x) - U_s(x_0)
\]

\[
= -\int_{x_0}^{x} F_s(x')dx'
\]

\[
= -\int_{x_0}^{x} (-kx')dx'
\]

\[
= k\int_{x_0}^{x} x'dx'
\]

\[
\Delta U_s = \frac{1}{2}kx^2 - \frac{1}{2}kx_0^2.
\] (6.12)

Thus, the potential energy associated with elongating a spring from its equilibrium position, at \(x = 0\), is

\[
U_s(x) = \frac{1}{2}kx^2 + \text{constant}.
\] (6.13)

Again, the potential energy is determined only to within an additive constant. However, keep in mind that the physical situation will often force a choice of this additive constant.

### 6.4 Potential Energy and Force

How can we find the conservative force when we have information on the corresponding potential energy? In calculus, taking the derivative is the inverse operation of integrating, and integration is used in equation 6.9 for the change in potential energy. Therefore, we take the derivative of that expression to obtain the force from the potential energy:

\[
F_s(x) = -\frac{dU(x)}{dx}.
\] (6.14)

Equation 6.14 is an expression for the force from the potential energy for the case of motion in one dimension. As you can see from this expression, any constant that you add to the potential energy will not have any influence on the result you obtain for the force, because taking the derivative of a constant term results in zero. This is further evidence that the potential energy can be determined only within an additive constant.

We will not consider motion in three-dimensional situations until later in this book. However, for completeness, we can state the expression for the force from the potential energy for the case of three-dimensional motion:

\[
\vec{F}(\vec{r}) = -\left(\frac{\partial U(\vec{r})}{\partial x}\hat{x} + \frac{\partial U(\vec{r})}{\partial y}\hat{y} + \frac{\partial U(\vec{r})}{\partial z}\hat{z}\right).
\] (6.15)

Here, the force components are given as partial derivatives with respect to the corresponding coordinates. If you major in engineering or science, you will encounter partial derivatives in many situations, certainly in higher-level calculus classes during your undergraduate studies. Basically, to take a partial derivative with respect to \(x\), you follow the usual calculus process for finding a derivative and treat the variables \(y\) and \(z\) as constants. To take a partial derivative with respect to \(y\), you treat \(x\) and \(z\) as constants, and to take a partial derivative with respect to \(z\), you treat \(x\) and \(y\) as constants. We will revisit multidimensional potential energy surfaces in Chapter 11, when we talk about stability.
Chapter 6  Potential Energy and Energy Conservation

**Potential Energy and Energy Conservation**

Empirically, the potential energy associated with the interaction of two atoms in a molecule as a function of the separation of the atoms has a form that is called the Lennard-Jones potential. This potential energy as a function of the separation, \( x \), is given by

\[
U(x) = 4U_0 \left( \frac{x_0}{x} \right)^{12} - \left( \frac{x_0}{x} \right)^6.
\]

Here \( U_0 \) is a constant energy and \( x_0 \) is a constant distance. The Lennard-Jones potential is one of the most important concepts in atomic physics and is used for most numerical simulations of molecular systems.

**Lennard-Jones Potential**

Concept Check 6.2

The potential energy, \( U(x) \), is shown as a function of position, \( x \), in the figure. In which region is the magnitude of the force the highest?

![Image of potential energy as a function of position](b.png)

(a) (b) (c) (d)

**Problem**

What is the force resulting from the Lennard-Jones potential?

**Solution**

We simply take the negative of the derivative of the potential energy with respect to \( x \):

\[
F_x(x) = -\frac{dU(x)}{dx} = -4U_0 \left( \frac{x_0}{x} \right)^{12} \left( \frac{x_0}{x} \right)^6 + 4U_0 \left( \frac{x_0}{x} \right)^{12} \left( \frac{x_0}{x} \right)^6 = 24U_0 \left( \frac{x_0}{x} \right)^{12} \left( \frac{x_0}{x} \right)^6.
\]

**Problem**

At what value of \( x \) does the Lennard-Jones potential have its minimum?

**Solution**

Since we just found that the force is the derivative of the potential energy function, all we have to do is find the point(s) where \( F(x) = 0 \). This leads to

\[
F_x(x) = 24U_0 \left( \frac{x_0}{x} \right)^{12} \left( \frac{x_0}{x} \right)^6 = 0.
\]

This condition can be fulfilled only if the expression in the larger parentheses is zero; thus

\[
2 \left( \frac{x_0}{x_{\text{min}}} \right)^{12} = 0.
\]
Potential Energy and Force

Figure 6.7a shows the shape of the Lennard-Jones potential, plotted from equation 6.16, with $x_0 = 0.34 \text{ nm}$ and $U_0 = 1.70 \cdot 10^{-21} \text{ J}$, for the interaction of two argon atoms as a function of the separation of the centers of the two atoms. Figure 6.7b plots the corresponding molecular force, using the expression we found in Example 6.2. The vertical gray dashed line marks the coordinate where the potential has a minimum and where consequently the force is zero. Also note that close to the minimum point of the potential (within ±0.1 nm), the force can be closely approximated by a linear function, $F_x(x) \approx -k(x - x_{\text{min}})$. This means that close to the minimum, the molecular force due to the Lennard-Jones potential behaves like a spring force.

Chapter 5 mentioned that forces similar to the spring force appear in many physical systems, and the connection between potential energy and force just described tells us why. Look, for example, at the skateboarder in the half-pipe in Figure 6.8. The curved surface of the half-pipe approximates the shape of the Lennard-Jones potential close to the minimum. If the skateboarder is at $x = x_{\text{min}}$, he can remain there at rest. If he is to the left of the minimum, where $x < x_{\text{min}}$, then the half-pipe exerts a force on him, which points to the right, $F_x > 0$; the further to the left he moves, the bigger the force becomes. On the right side of the half-pipe, for $x > x_{\text{min}}$, the force points to the left, that is, $F_x < 0$. Again, these observations can be summarized with a force expression that approximately follows Hooke’s Law: $F_x(x) = -k(x - x_{\text{min}})$.

In addition, we can reach this same conclusion mathematically, by writing a Taylor expansion for $F_x(x)$ around $x_{\text{min}}$:

$$F_x(x) = F_x(x_{\text{min}}) + \left(\frac{dF_x}{dx}\right)_{x=x_{\text{min}}} \cdot (x - x_{\text{min}}) + \frac{1}{2} \left(\frac{d^2F_x}{dx^2}\right)_{x=x_{\text{min}}} \cdot (x - x_{\text{min}})^2 + \cdots.$$  

Since we are expanding around the potential energy minimum and since we have just shown that the force is zero there, we have $F_x(x_{\text{min}}) = 0$. If there is a potential minimum at $x = x_{\text{min}}$, then the second derivative of the potential must be positive. Since, according to equation 6.14, the force is $F_x(x) = -dU(x)/dx$, this means that the derivative of the force is $dF_x(x)/dx = -d^2U(x)/dx^2$. At the minimum of the potential, we thus have $(dF_x/dx)_{x=x_{\text{min}}} < 0$.

Figure 6.7a shows the shape of the Lennard-Jones potential, plotted from equation 6.16, with $x_0 = 0.34 \text{ nm}$ and $U_0 = 1.70 \cdot 10^{-21} \text{ J}$, for the interaction of two argon atoms as a function of the separation of the centers of the two atoms. Figure 6.7b plots the corresponding molecular force, using the expression we found in Example 6.2. The vertical gray dashed line marks the coordinate where the potential has a minimum and where consequently the force is zero. Also note that close to the minimum point of the potential (within ±0.1 nm), the force can be closely approximated by a linear function, $F_x(x) = -k(x - x_{\text{min}})$. This means that close to the minimum, the molecular force due to the Lennard-Jones potential behaves like a spring force.

Chapter 5 mentioned that forces similar to the spring force appear in many physical systems, and the connection between potential energy and force just described tells us why. Look, for example, at the skateboarder in the half-pipe in Figure 6.8. The curved surface of the half-pipe approximates the shape of the Lennard-Jones potential close to the minimum. If the skateboarder is at $x = x_{\text{min}}$, he can remain there at rest. If he is to the left of the minimum, where $x < x_{\text{min}}$, then the half-pipe exerts a force on him, which points to the right, $F_x > 0$; the further to the left he moves, the bigger the force becomes. On the right side of the half-pipe, for $x > x_{\text{min}}$, the force points to the left, that is, $F_x < 0$. Again, these observations can be summarized with a force expression that approximately follows Hooke’s Law: $F_x(x) = -k(x - x_{\text{min}})$.

In addition, we can reach this same conclusion mathematically, by writing a Taylor expansion for $F_x(x)$ around $x_{\text{min}}$:

$$F_x(x) = F_x(x_{\text{min}}) + \left(\frac{dF_x}{dx}\right)_{x=x_{\text{min}}} \cdot (x - x_{\text{min}}) + \frac{1}{2} \left(\frac{d^2F_x}{dx^2}\right)_{x=x_{\text{min}}} \cdot (x - x_{\text{min}})^2 + \cdots.$$  

Since we are expanding around the potential energy minimum and since we have just shown that the force is zero there, we have $F_x(x_{\text{min}}) = 0$. If there is a potential minimum at $x = x_{\text{min}}$, then the second derivative of the potential must be positive. Since, according to equation 6.14, the force is $F_x(x) = -dU(x)/dx$, this means that the derivative of the force is $dF_x(x)/dx = -d^2U(x)/dx^2$. At the minimum of the potential, we thus have $(dF_x/dx)_{x=x_{\text{min}}} < 0$.

Figure 6.8 Skateboarder in a half-pipe.
Expressing the value of the first derivative of the force at coordinate $x_{\text{min}}$ as some constant,

$$\left(\frac{dF}{dx}\right)_{x = x_{\text{min}}} = -k \quad (\text{with } k > 0),$$

we find

$$F_x(x) = -k(x - x_{\text{min}}),$$

if we are sufficiently close to $x_{\text{min}}$ that we can neglect terms proportional to $(x - x_{\text{min}})^2$ and higher powers.

These physical and mathematical arguments establish why it is important to study Hooke's Law and the resulting equations of motion in detail. In this chapter, we study the work done by the spring force. In Chapter 14 on oscillations, we will analyze the motion of an object under the influence of the spring force.

### 6.5 Conservation of Mechanical Energy

We have defined potential energy in reference to a system of objects. We will examine different kinds of general systems in later chapters, but here we focus on one particular kind of system: an isolated system, which by definition is a system of objects that exert forces on one another but for which no force external to the system causes energy changes within the system. This means that no energy is transferred into or out of the system. This very common situation is highly important in science and engineering and has been extensively studied. One of the fundamental concepts of physics involves energy within an isolated system.

To investigate this concept, we begin with a definition of mechanical energy, $E$, as the sum of kinetic energy and potential energy:

$$E = K + U.$$  \hspace{1cm} (6.17)

(Later, when we move beyond mechanics, we will add other kinds of energy to this sum and call it the total energy.)

For any mechanical process that occurs inside an isolated system and involves only conservative forces, the total mechanical energy is conserved. This means that the total mechanical energy remains constant in time:

$$\Delta E = \Delta K + \Delta U = 0.$$  \hspace{1cm} (6.18)

An alternative way of writing this result (which we'll derive below) is

$$K + U = K_0 + U_0,$$  \hspace{1cm} (6.19)

where $K_0$ and $U_0$ are the initial kinetic energy and potential energy, respectively. This relationship, which is called the law of conservation of mechanical energy, does not imply that the kinetic energy of the system cannot change, or that the potential energy alone remains constant. Rather, it states that their changes are exactly compensating and thus offset each other. It is worth repeating that conservation of mechanical energy is valid only for conservative forces and for an isolated system, for which the influence of external forces can be neglected.

### DERIVATION 6.1

As we have already seen in equation 6.7, if a conservative force does work, then the work causes a change in potential energy:

$$\Delta U = -W.$$  \hspace{1cm} (If the force under consideration is not conservative, this relationship does not hold in general, and conservation of mechanical energy is not valid.)

In Chapter 5, we learned that the relationship between the change in kinetic energy and the work done by a force is (equation 5.7):

$$\Delta K = W.$$  \hspace{1cm} (6.7)

Combining these two results, we obtain

$$\Delta U = -\Delta K \Rightarrow \Delta U + \Delta K = 0.$$  \hspace{1cm} (6.18)

Using $\Delta U = U - U_0$ and $\Delta K = K - K_0$, we find

$$0 = \Delta U + \Delta K = U - U_0 + K - K_0 = U + K - (U_0 + K_0) \Rightarrow$$

$$U + K = U_0 + K_0.$$
Note that Derivation 6.1 did not make any reference to the particular path along which the force did the work that caused the rearrangement. In fact, you do not need to know any detail about the work or the force, other than that the force is conservative. Nor do you need to know how many conservative forces are acting. If more than one conservative force is present, you interpret $\Delta U$ as the sum of all the potential energy changes and $W$ as the total work done by all of the conservative forces, and the derivation is still valid.

The law of energy conservation enables us to easily solve a huge number of problems that involve only conservative forces, problems that would have been very hard to solve without this law. Later in this chapter, the more general work-energy theorem for mechanics, which includes nonconservative forces, will be presented. This law will enable us to solve an even wider range of problems, including those involving friction.

Equation 6.19 introduces our first conservation law, the law of conservation of mechanical energy. Chapters 18 and 20 will extend this law to include thermal energy (heat) as well. Chapter 7 will present a conservation law for linear momentum. When we discuss rotation in Chapter 10, we will encounter a conservation law for angular momentum. In studying electricity and magnetism, we will find a conservation law for net charge (Chapter 21), and in looking at elementary particle physics (Chapter 39), we will find conservation laws for several other quantities. This list is intended to give you a flavor of a central theme of physics—the discovery of conservation laws and their use in determining the dynamics of various systems.

Before we solve a sample problem, one more remark on the concept of an isolated system is in order. In situations that involve the motion of objects under the influence of the Earth's gravitational force, the isolated system to which we apply the law of conservation of energy actually consists of the moving object plus the entire Earth. However, in using the approximation that the gravitational force is a constant, we assume that the Earth is infinitely massive and thus immobile (and that the moving object is close to the surface of Earth). Therefore, no change in the kinetic energy of the Earth can result from the rearrangement of the system. Thus, we calculate all changes in kinetic energy and potential energy only for the “junior partner”—the object moving under the influence of the gravitational force. This force is conservative and internal to the system consisting of Earth plus moving object, so all the conditions for the utilization of the law of energy conservation are fulfilled.

Specific examples of situations that involve objects moving under the influence of the gravitational force are projectile motion and pendulum motion occurring near the Earth's surface.

**SOLVED PROBLEM 6.2 The Catapult Defense**

Your task is to defend Neuschwanstein Castle from attackers (Figure 6.9). You have a catapult with which you can lob a rock with a launch speed of 14.2 m/s from the courtyard over the castle walls onto the attackers' camp in front of the castle at an elevation 7.20 m below that of the courtyard.

**PROBLEM**

What is the speed with which a rock will hit the ground at the attackers' camp? (Neglect air resistance.)

**SOLUTION**

**THINK** We can solve this problem by applying the conservation of mechanical energy. Once the catapult launches a rock, only the conservative force of gravity is acting on the rock. Thus, the total mechanical energy is conserved, which means the sum of the kinetic and potential energies of the rock always equals the total mechanical energy.

**SKETCH** The trajectory of the rock is shown in Figure 6.10, where the initial speed of the rock is $v_0$, the initial kinetic energy $K_0$, the initial potential energy $U_0$, and the initial height $y_0$. The final speed is $v$, the final kinetic energy $K$, the final potential energy $U$, and the final height $y$.

--- Continued
In Solved Problem 6.2, we neglected air resistance. Discuss qualitatively how our final answer would have changed if we had included the effects of air resistance.

**FIGURE 6.10** Trajectory of the rock launched by the catapult.

**RESEARCH** We can use conservation of mechanical energy to write

\[ E = K + U = K_0 + U_0, \]

where \( E \) is the total mechanical energy. The kinetic energy of the projectile can be expressed as

\[ K = \frac{1}{2}mv^2, \]

where \( m \) is the mass of the projectile and \( v \) is its speed when it hits the ground. The potential energy of the projectile can be expressed as

\[ U = mgy, \]

where \( y \) is the vertical component of the position vector of the projectile when it hits the ground.

**SIMPLIFY** We substitute for \( K \) and \( U \) in \( E = K + U \) to get

\[ E = \frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 + mg(y_0 - y). \]

The mass of the rock, \( m \), cancels out, and we are left with

\[ \frac{1}{2}v^2 + gy = \frac{1}{2}v_0^2 + gy_0. \]

We solve this for the speed:

\[ v = \sqrt{v_0^2 + 2g(y_0 - y)}. \]

**CALCULATE** According to the problem statement, \( y_0 - y = 7.20 \text{ m} \) and \( v_0 = 14.2 \text{ m/s} \). Thus, for the final speed, we find

\[ v = \sqrt{(14.2 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(7.20 \text{ m})} = 18.51766724 \text{ m/s}. \]

**ROUND** The relative height was given to three significant figures, so we report our final answer as

\[ v = 18.5 \text{ m/s}. \]

**DOUBLE-CHECK** Our answer for the speed of the rock when it hits the ground in front of the castle is 18.5 m/s, compared with the initial launch speed of 14.2 m/s, which seems reasonable. This speed has to be bigger due to the gain from the difference in gravitational potential energy, and it is comforting that our answer passes this simple test.

Because we were only interested in the speed at impact, we did not even need to know the initial launch angle \( \theta_0 \) to solve the problem. All launch angles will give the same result (for a given launch speed), which is a somewhat surprising finding. (Of course, if you were in this situation, you would obviously want to aim high enough to clear the castle wall and accurately enough to strike the attackers’ camp.)

We can also solve this problem using the concepts of projectile motion, which is a way to double-check our answer and to show the power of applying the concept of energy conservation. We start by writing the components of the initial velocity vector \( \mathbf{v}_0 \):

\[ v_{x0} = v_0 \cos \theta_0 \]

and

\[ v_{y0} = v_0 \sin \theta_0. \]

The final \( x \)-component of the velocity, \( v_x \), is equal to the initial \( x \)-component of the initial velocity \( v_{x0} \):

\[ v_x = v_{x0} = v_0 \cos \theta_0. \]

The final component of the velocity in the \( y \)-direction can be obtained from a result of the analysis of projectile motion in Chapter 3:

\[ v_y^2 = v_{y0}^2 - 2g(y - y_0). \]

Therefore, the final speed of the rock as it hits the ground is

\[
\begin{align*}
v &= \sqrt{v_x^2 + v_y^2} \\
&= \sqrt{(v_0 \cos \theta_0)^2 + (v_{y0}^2 - 2g(y - y_0))} \\
&= \sqrt{v_0^2 \cos^2 \theta_0 + v_{y0}^2 \sin^2 \theta_0 - 2g(y - y_0)}. \end{align*}
\]

Remembering that \( \sin^2 \theta + \cos^2 \theta = 1 \), we can further simplify and get:

\[ v = \sqrt{v_0^2 (\cos^2 \theta_0 + \sin^2 \theta_0)} - 2g(y - y_0) = \sqrt{v_0^2 - 2g(y - y_0)} = \sqrt{v_0^2 + 2g(y_0 - y)}. \]

This is the same as equation 6.20, which we obtained using energy conservation. Even though the final result is the same, the solution process based on energy conservation was by far easier than that based on kinematics.

**Self-Test Opportunity 6.2**

In Solved Problem 6.2, we neglected air resistance. Discuss qualitatively how our final answer would have changed if we had included the effects of air resistance.
As you can see from Solved Problem 6.2, applying the conservation of mechanical energy provides us with a powerful technique for solving problems that seem rather complicated at first sight.

In general, we can determine final speed as a function of the elevation in situations where the gravitational force is at work. For instance, consider the image sequence in Figure 6.11. Two balls are released at the same time from the same height at the top of two ramps with different shapes. At the bottom end of the ramps, both balls reach the same lower elevation. Therefore, in both cases, the height difference between the initial and final points is the same. Both balls also experience normal forces in addition to the gravitational force; however, the normal forces do no work because they are perpendicular to the contact surface, by definition, and the motion is parallel to the surface. Thus, the scalar product of the normal force and displacement vectors is zero. (There is a small friction force, but it is negligible in this case.) Energy conservation considerations (see equation 6.20 in Solved Problem 6.2) tell us that the speed of both balls at the bottom end of the ramps has to be the same:

\[ v = \sqrt{2gy} \]

This equation is a special case of equation 6.20 with \( v_0 = 0 \). Note that, depending on the curve of the bottom ramp, this result could be rather difficult to obtain using Newton’s Second Law. However, even though the velocities at the top and the bottom of the ramps are the same for both balls, you cannot conclude from this result that both balls arrive at the bottom at the same time. The image sequence clearly shows that this is not the case.

**Self-Test Opportunity 6.3**

Why does the lighter-colored ball arrive at the bottom in Figure 6.11 before the other ball?

---

**SOLVED PROBLEM 6.3 Trapeze Artist**

**PROBLEM**

A circus trapeze artist starts her motion with the trapeze at rest at an angle of 45.0° relative to the vertical. The trapeze ropes have a length of 5.00 m. What is her speed at the lowest point in her trajectory?

**SOLUTION**

**THINK** Initially, the trapeze artist has only gravitational potential energy. We can choose a coordinate system such that \( y = 0 \) is at her trajectory’s lowest point, so the potential energy is zero at that lowest point. When the trapeze artist is at the lowest point, her kinetic energy will be a maximum. We can then equate the initial gravitational potential energy to the final kinetic energy of the trapeze artist.

**SKETCH** We represent the trapeze artist in Figure 6.12 as an object of mass \( m \) suspended by a rope of length \( \ell \). We indicate the position of the trapeze artist at a given value of the angle \( \theta \) by the blue disk. The lowest point of the trajectory is reached at \( \theta = 0 \), and we indicate this in Figure 6.12 by a gray disk.

The figure shows that the trapeze artist is at a distance \( \ell \) (the length of the rope) below the ceiling at the lowest point and at a distance \( \ell \cos \theta \) below the ceiling for all other values of \( \theta \). This means that she is at a height \( h = \ell - \ell \cos \theta = \ell (1 - \cos \theta) \) above the lowest point in the trajectory when the trapeze forms an angle \( \theta \) with the vertical.

**RESEARCH** The trapeze is pulled back to an initial angle \( \theta_0 \) relative to the vertical, and thus the trapeze artist is at a height \( h = \ell (1 - \cos \theta_0) \) above the lowest point in the trajectory, according to our analysis of Figure 6.12. The potential energy at this maximum deflection, \( E_{\text{pot}} \), is therefore

\[ E = K + U = 0 + U = mg\ell (1 - \cos \theta_0). \]

---
This is also the value for the total mechanical energy, because the trapeze artist has zero kinetic energy at the point of maximum deflection. For any other deflection, the energy is the sum of kinetic and potential energies:

\[ E = mg((1 - \cos \theta)) + \frac{1}{2}mv^2. \]

**Simplify** Solving the preceding equation for the speed, we obtain

\[ mg((1 - \cos \theta)) = mg((1 - \cos \theta)) + \frac{1}{2}mv^2 \Rightarrow \]

\[ mg(\cos \theta - \cos \theta) = \frac{1}{2}mv^2 \Rightarrow \]

\[ v = \sqrt{2gL(\cos \theta - \cos \theta_0)}. \]

Here, we are interested in the speed for \( \theta = 0 \), which is

\[ v(\theta = 0) = \sqrt{2gL(\cos 0 - \cos \theta_0)} = \sqrt{2gL(1 - \cos \theta_0)}. \]

**Calculate** The initial condition is \( \theta_0 = 45.0^\circ \). Inserting the numbers, we find

\[ v(0^\circ) = \sqrt{2(9.81\text{ m/s}^2)(5.00\text{ m})(1 - \cos 45.0^\circ)} = 5.360300809 \text{ m/s}. \]

**Round** All of the numerical values were specified to three significant figures, so we report our answer as

\[ v(0^\circ) = 5.36 \text{ m/s}. \]

**Double-check** First, the obvious check of units: m/s is the SI unit for velocity and speed. The speed of the trapeze artist at the lowest point is 5.36 m/s (12 mph), which seems in line with what we see in the circus.

We can perform another check on the formula \( v(\theta = 0) = \sqrt{2gL(1 - \cos \theta_0)} \) by considering the limiting cases for the initial angle \( \theta_0 \) to see if they yield reasonable results. In this situation, the limiting values for \( \theta_0 \) are 90°, where the trapeze starts out horizontal, and 0°, where it starts out vertical. If we use \( \theta_0 = 0^\circ \), the trapeze is just hanging at rest, and we expect zero speed, an expectation borne out by our formula. On the other hand, if we use \( \theta_0 = 90^\circ \), or \( \cos \theta_0 = \cos 90^\circ = 0 \), we obtain the limiting result \( \sqrt{2gL} \), which is the same result as a free fall from the ceiling to the bottom of the trapeze swing. Again, this limit is as expected, which gives us additional confidence in our solution.

---

**6.6 Work and Energy for the Spring Force**

In Section 6.3, we found that the potential energy stored in a spring is \( U_s = \frac{1}{2}kx^2 + \text{constant} \), where \( k \) is the spring constant and \( x \) is the displacement from the equilibrium position.

Here we choose the additive constant to be zero, corresponding to having \( U_s = 0 \) at \( x = 0 \).

Using the principle of energy conservation, we can find the velocity \( v \) as a function of the position. First, we can write, in general, for the total mechanical energy:

\[ E = K + U_s = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \]  

(6.21)

Once we know the total mechanical energy, we can solve this equation for the velocity. What is the total mechanical energy? The point of maximum elongation of a spring from the equilibrium position is called the **amplitude**, \( A \). When the displacement reaches the amplitude, the velocity is briefly zero. At this point, the total mechanical energy of an object oscillating on a spring is

\[ E = \frac{1}{2}kA^2. \]

However, conservation of mechanical energy means that this is the value of the energy for any point in the spring’s oscillation. Inserting the above expression for \( E \) into equation 6.21 yields

\[ \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \]  

(6.22)

From equation 6.22, we can get an expression for the speed as a function of the position:

\[ v = \frac{\sqrt{(A^2 - x^2)}k}{m}. \]  

(6.23)
Note that we did not rely on kinematics to get this result, as that approach is rather challenging—another piece of evidence that using principles of conservation (in this case, conservation of mechanical energy) can yield powerful results. We will return to the equation of motion for a mass on a spring in Chapter 14.

**SOLVED PROBLEM 6.4 Human Cannonball**

In a favorite circus act, called the “human cannonball,” a person is shot from a long barrel, usually with a lot of smoke and a loud bang added for theatrical effect. Before the Italian Zacchini brothers invented the compressed air cannon for shooting human cannonballs in the 1920s, the Englishman George Farini used a spring-loaded cannon for this purpose in the 1870s.

Suppose someone wants to recreate Farini’s spring-loaded human cannonball act with a spring inside a barrel. Assume the barrel is 4.00 m long, with a spring that extends the entire length of the barrel. Further, the barrel is upright, so it points vertically toward the ceiling of the circus tent. The human cannonball is lowered into the barrel and compresses the spring to some degree. An external force is added to compress the spring even further, to a length of only 0.70 m. At a height of 7.50 m above the top of the barrel is a spot on the tent that the human cannonball, of height 1.75 m and mass 68.4 kg, is supposed to touch at the top of his trajectory. Removing the external force releases the spring and fires the human cannonball vertically upward.

**PROBLEM 1**

What is the value of the spring constant needed to accomplish this stunt?

**SOLUTION 1**

**THINK** Let’s apply energy conservation considerations to solve this problem. Potential energy is stored in the spring initially and then converted to gravitational potential energy at the top of the human cannonball’s flight. As a reference point for our calculations, we select the top of the barrel and place the origin of our coordinate system there. To accomplish the stunt, enough energy has to be provided, through compressing the spring, that the top of the head of the human cannonball is elevated to a height of 7.50 m above the zero point we have chosen. Since the person has a height of 1.75 m, his feet need to be elevated by only 
\[ h = 7.50\,\text{m} - 1.75\,\text{m} = 5.75\,\text{m}. \]

We can specify all position values for the human cannonball on the \( y \)-coordinate as the position of the bottom of his feet.

**SKETCH** To clarify this problem, let’s apply energy conservation at different instants of time. Figure 6.13a shows the initial equilibrium position of the spring. In Figure 6.13b, the external force \( F \) and the weight of the human cannonball compress the spring by 3.30 m to a length...
Concept Check 6.3
A spring with spring constant \( k \) is oriented vertically and compressed downward a distance \( x \) from its equilibrium position. An object of mass \( m \) is placed on the upper end of the spring, and the spring is released. The object rises a distance \( h \) (with \( h \geq x \)) above the equilibrium position of the spring. If the spring is then compressed downward by the same distance \( x \) and an object of mass \( 3m \) is placed on it, how high will the object rise when the spring is released?

a) \( h \) 

b) \( 3h \) 

c) \( h/3 \) 

d) \( h^{1/2} \) 

e) \( h^{3} \)

Concept Check 6.4
A ball of mass \( m \) is thrown vertically into the air with an initial speed \( v \). Which of the following equations correctly describes the maximum height, \( h \), of the ball?

a) \( h = \frac{v^2}{2g} \) 

b) \( h = \frac{g}{v^2} \) 

c) \( h = \frac{2mv^2}{g} \) 

d) \( h = \frac{mv^2}{g} \) 

e) \( h = \frac{v^2}{2g} \)

Self-Test Opportunity 6.4
Graph the potential and kinetic energies of the human cannonball of Solved Problem 6.4 as a function of the \( y \)-coordinate. For what value of the displacement is the speed of the human cannonball at a maximum? (Hint: This occurs not exactly at \( y = 0 \) but at a value of \( y < 0 \).)
**Example 6.3** Bungee Jumper

A bungee jumper locates a suitable bridge that is 75.0 m above the river below, as shown in Figure 6.14. The jumper has a mass of \( m = 80.0 \text{ kg} \) and a height of \( L_{\text{jumper}} = 1.85 \text{ m} \). We can think of a bungee cord as a spring. The spring constant of the bungee cord is \( k = 50.0 \text{ N/m} \). Assume that the mass of the bungee cord is negligible compared with the jumper’s mass.

**Problem**
The jumper wants to know the maximum length of bungee cord he can safely use for this jump.

**Solution**
We are looking for the unstretched length of the bungee cord, \( L_0 \), as the jumper would measure it standing on the bridge. The distance from the bridge to the water is \( L_{\text{max}} = 75.0 \text{ m} \). Energy conservation tells us that the gravitational potential energy that the jumper has, as he dives off the bridge, will be converted to potential energy stored in the bungee cord. The jumper’s gravitational potential energy on the bridge is

\[
U_0 = mgL_{\text{max}}.
\]

assuming that gravitational potential energy is zero at the level of the water. Before he starts his jump, he has zero kinetic energy, and so his total energy when he is on top of the bridge is

\[
E_{\text{top}} = mgL_{\text{max}}.
\]

At the bottom of the jump, where the jumper’s head just touches the water, the potential energy stored in the bungee cord is

\[
U_i = \frac{1}{2}ky^2 = \frac{1}{2}k(L_{\text{max}} - L_{\text{jumper}} - L_0)^2,
\]

where \( L_{\text{max}} - L_{\text{jumper}} - L_0 \) is the length that the bungee cord stretches beyond its unstretched length. (Here we have to subtract the jumper’s height from the height of the bridge to obtain the maximum length, \( L_{\text{max}} - L_{\text{jumper}} \), to which the bungee cord is allowed to stretch, assuming that it is tied around his ankles.) Since the bungee jumper is momentarily at rest at this lowest point of his jump, the kinetic energy is zero at that point, and the total energy is then

\[
E_{\text{bottom}} = \frac{1}{2}k(L_{\text{max}} - L_{\text{jumper}} - L_0)^2.
\]

From the conservation of mechanical energy, we know that \( E_{\text{top}} = E_{\text{bottom}} \), and so we find

\[
mgL_{\text{max}} = \frac{1}{2}k(L_{\text{max}} - L_{\text{jumper}} - L_0)^2.
\]

Solving for the required unstretched length of bungee cord gives us

\[
L_0 = L_{\text{max}} - L_{\text{jumper}} - \sqrt{\frac{2mgL_{\text{max}}}{k}}.
\]

Putting in the given numbers, we get

\[
L_0 = (75.0 \text{ m}) - (1.85 \text{ m}) - \sqrt{\frac{2(80.0 \text{ kg})(9.81 \text{ m/s}^2)(75.0 \text{ m})}{50.0 \text{ N/m}}} = 24.6 \text{ m}.
\]

For safety, the jumper would be wise to use a bungee cord shorter than this and to test it with a dummy mass similar to his.

**Potential Energy of an Object Hanging from a Spring**

We saw in Solved Problem 6.4 that the initial potential energy of the human cannonball has contributions from the spring force and the gravitational force. In Example 5.3 and the discussion following it, we established that hanging an object of mass \( m \) from a spring with spring constant \( k \) shifts the equilibrium position of the spring from zero to \( y_0 \), given by the equilibrium condition,

\[
k y_0 = -mg \Rightarrow y_0 = -\frac{mg}{k}.
\]

Figure 6.15 shows the forces that act on an object suspended from a spring when it is in different positions. This figure shows two different choices for the origin of the vertical coordinate.
axis: In Figure 6.15a, the vertical coordinate is called $y$ and has zero at the equilibrium position of the end of the spring without the mass hanging from it; in Figure 6.15b the new equilibrium point, $y_0$, with the object suspended from the spring, is calculated according to equation 6.24. This new equilibrium point is the origin of the axis and the vertical coordinate is called $s$. The end of the spring is located at $s = 0$. The system is in equilibrium because the force exerted by the spring on the object balances the gravitational force acting on the object:

$$F_s (y_0) + F_g = 0.$$  

In Figure 6.15c, the object has been displaced downward away from the new equilibrium position, so $y = y_1$ and $s = s_1$. Now there is a net upward force tending to restore the object to the new equilibrium position:

$$F_{net} (s_1) = F_s (y_1) + F_g.$$  

If instead, the object is displaced upward, above the new equilibrium position, as shown in Figure 6.15d, there is a net downward force that tends to restore the object to the new equilibrium position:

$$F_{net} (s_2) = F_s (y_2) + F_g.$$  

We can calculate the potential energy of the object and the spring for these two choices of the coordinate system and show that they differ by only a constant. We start by defining the potential energy of the object connected to the spring, taking $y$ as the variable and assuming that the potential energy is zero at $y = 0$:

$$U(y) = \frac{1}{2} ky^2 + mgy.$$  

Using the relation $y = s + y_0$, we can express this potential energy in terms of the variable $s$:

$$U(s) = \frac{1}{2} k (s + y_0)^2 + mg(s + y_0).$$
Nonconservative Forces and the Work-Energy Theorem

Rearranging gives us
\[ U(s) = \frac{1}{2}ks^2 + ksy_0 + \frac{1}{2}ky_0^2 + mgs + mgy_0. \]
Substituting \( ky_0 = -mg \), from equation 6.24, into this equation, we get
\[ U(s) = \frac{1}{2}ks^2 - (mg)s + \frac{1}{2}(mg)y_0 + mgs - mgy_0. \]
Thus, we find that the potential energy in terms of \( s \) is
\[ U(s) = \frac{1}{2}ks^2 - \frac{1}{2}mgy_0. \]

Figure 6.16 shows the potential energy functions for these two coordinate axes. The blue curve in Figure 6.16 (a) shows the potential energy as a function of the vertical coordinate \( y \), with the choice of zero potential energy at \( y = 0 \) corresponding to the spring hanging vertically without the object connected to it. The new equilibrium position, \( y_0 \), is determined by the displacement that occurs when an object of mass \( m \) is attached to the spring, as calculated using equation 6.24. The red curve in Figure 6.16 (b) represents the potential energy as a function of the vertical coordinate \( s \), with the equilibrium position chosen to be \( s = 0 \). The potential energy curves \( U(y) \) and \( U(s) \) are both parabolas, which are offset from each other by a simple constant.

Thus, we can express the potential energy of an object of mass \( m \) hanging from a vertical spring in terms of the displacement \( s \) about an equilibrium point as
\[ U(s) = \frac{1}{2}ks^2 + C, \]
where \( C \) is a constant. For many problems, we can choose zero as the value of this constant, allowing us to write
\[ U(s) = \frac{1}{2}ks^2. \]
This result allows us to use the same spring force potential for different masses attached to the end of a spring by simply shifting the origin to the new equilibrium position. (Of course, this only works if we do not attach too much mass to the end of the spring and overstretch it beyond its elastic limit.)

6.7 Nonconservative Forces and the Work-Energy Theorem

With the introduction of the potential energy we can extend and augment the work–kinetic energy theorem of Chapter 5 for conservative forces. By including the potential energy as well, we find the work-energy theorem
\[ W = \Delta E = \Delta K + \Delta U \]
where \( W \) is the work done by an external force, \( \Delta K \) is the change of kinetic energy, and \( \Delta U \) is the change in potential energy. This relationship means that external work done to a system can change the total energy of the system.

Is energy conservation violated in the presence of nonconservative forces? The word nonconservative seems to imply that it is violated, and, indeed, the total mechanical energy is not conserved. Where, then, does the energy go? Section 6.2 showed that the friction force does not do work but instead dissipates mechanical energy into internal excitation energy, which can be vibration energy, deformation energy, chemical energy, or electrical energy, depending on the material of which the object is made and on the particular form of the friction force. In Section 6.2, we found that \( W_f \) is the total energy dissipated by nonconservative forces into internal energy and then into other energy forms besides mechanical energy. If we add this type of energy to the total mechanical energy, we obtain the total energy:
\[ E_{\text{total}} = E_{\text{mechanical}} + E_{\text{other}} = K + U + E_{\text{other}}. \]
Here \( E_{\text{other}} \) stands for all other forms of energy that are not kinetic or potential energies. The change in the other energy forms is exactly the negative of the energy dissipated by the friction force in going from the initial to the final state of the system:

\[ \Delta E_{\text{other}} = -W_f. \]

The total energy is conserved—that is, stays constant in time—even for nonconservative forces. This is the most important point in this chapter:

**The total energy—the sum of all forms of energy, mechanical or other—is always conserved in an isolated system.**

We can also write this law of energy conservation in the form that states that the change in the total energy of an isolated system is zero:

\[ \Delta E_{\text{total}} = 0. \]  
(6.28)

Since we do not yet know what exactly this internal energy is and how to calculate it, it may seem that we cannot use energy considerations when at least one of the forces acting is nonconservative. However, this is not the case. For the case in which only conservative forces are acting, we found that (see equation 6.18) the total mechanical energy is conserved, or \( \Delta E = \Delta K + \Delta U = 0 \), where \( E \) refers to the total mechanical energy. In the presence of nonconservative forces, combining equations 6.28, 6.27, and 6.26 gives

\[ W_f = \Delta K + \Delta U. \]  
(6.29)

This relationship is a generalization of the work-energy theorem. In the absence of nonconservative forces, \( W_f = 0 \), and equation 6.29 reduces to the law of the conservation of mechanical energy, equation 6.19. When applying either of these two equations, you must select two times—a beginning and an end. Usually this choice is obvious, but sometimes care must be taken, as demonstrated in the following solved problem.

### SOLVED PROBLEM 6.5 Block Pushed Off a Table

Consider a block on a table. This block is pushed by a spring attached to the wall, slides across the table, and then falls to the ground. The block has a mass \( m = 1.35 \text{ kg} \). The spring constant is \( k = 560 \text{ N/m} \), and the spring is initially compressed by 0.110 m. The block slides a distance \( d = 0.65 \text{ m} \) across the table of height \( h = 0.750 \text{ m} \). The coefficient of kinetic friction between the block and the table is \( \mu_k = 0.160 \).

**PROBLEM**

What speed will the block have when it lands on the floor?

**SOLUTION**

**THINK** At first sight, this problem does not seem to be one to which we can apply mechanical energy conservation, because the nonconservative force of friction is in play. However, we can utilize the work-energy theorem, equation 6.29. To be certain that the block actually leaves the table, though, we first calculate the total energy imparted to the block by the spring and make sure that the potential energy stored in the compressed spring is sufficient to overcome the friction force.

**SKETCH** Figure 6.17a shows the block of mass \( m \) pushed by the spring. The mass slides on the table a distance \( d \) and then falls to the floor, which is a distance \( h \) below the table.

We choose the origin of our coordinate system such that the block starts at \( x = y = 0 \), with the \( x \)-axis running along the bottom surface of the block and the \( y \)-axis running through its center (Figure 6.17b). The origin of the coordinate system can be placed at any point, but it is important to fix an origin, because all potential energies have to be expressed relative to some reference point.

**RESEARCH** Step 1: Let’s analyze the problem situation without the friction force. In this case, the block initially has potential energy from the spring and no kinetic energy, since it is...
at rest. When the block hits the floor, it has kinetic energy and negative gravitational potential energy. Conservation of mechanical energy results in
\[ K_0 + U_0 = K + U \Rightarrow 0 + \frac{1}{2}kx_0^2 = \frac{1}{2}mv^2 - mgh. \] (i)

Usually, we would solve this equation for the speed and put in the numbers later. However, because we will need them again, let’s evaluate the two expressions for potential energy:
\[ \frac{1}{2}kx_0^2 = 0.5(560 \text{ N/m})(0.11 \text{ m})^2 = 3.39 \text{ J} \]
\[ mgh = (1.35 \text{ kg})(9.81 \text{ m/s}^2)(0.75 \text{ m}) = 9.93 \text{ J}. \]

Now, solving equation (i) for the speed results in
\[ v = \sqrt{\frac{2}{m} \left( \frac{1}{2}kx_0^2 + mgh \right)} = \sqrt{\frac{2}{1.35 \text{ kg}} \left( 3.39 \text{ J} + 9.93 \text{ J} \right)} = 4.44 \text{ m/s}. \]

**Step 2:** Now we include friction. Our considerations remain almost unchanged, except that we have to include the energy dissipated by the nonconservative force of friction. We find the force of friction using the upper free-body diagram in Figure 6.17c. We can see that the normal force is equal to the weight of the block and write
\[ N = mg. \]

The friction force is given by
\[ F_f = \mu_k N = \mu_k mg. \]

We can then write the energy dissipated by the friction force as
\[ W_f = -\mu_k mgd. \]

In applying the generalization of the work-energy theorem, we choose the initial time to be when the block is about to start moving (see Figure 6.17a) and the final time to be when the block reaches the edge of the table and is about to start the free-fall portion of its path. Let \( K_{\text{top}} \) be the kinetic energy at the final time, chosen to make sure that the block makes it to the end of the table. Using equation 6.29 and the value we calculated above for the block’s initial potential energy, we find:
\[ W_f = \Delta K + \Delta U = K_{\text{top}} - \frac{1}{2}kx_0^2 = -\mu_k mgd \]
\[ = 3.39 \text{ J} - (0.16)(1.35 \text{ kg})(9.81 \text{ m/s}^2)(0.65 \text{ m}) \]
\[ = 3.39 \text{ J} - 1.38 \text{ J} = 2.01 \text{ J}. \]

Because the kinetic energy \( K_{\text{top}} > 0 \), the block can overcome friction and slide off the table. Now we can calculate the block’s speed when it hits the floor.

**Simplify** For this part of the problem, we choose the initial time to be when the block is at the table’s edge to exploit the calculations we have already done. The final time is when

-- Continued
As Solved Problem 6.5 shows, energy considerations are still a powerful tool for performing otherwise very difficult calculations, even in the presence of nonconservative forces. However, the principle of conservation of mechanical energy cannot be applied quite as straightforwardly when nonconservative forces are present, and you have to account for the energy dissipated by these forces. Let’s try one more solved problem to elaborate on these concepts.

**Concept Check 6.6**

A curling stone of mass $m$ is given an initial velocity $v$ on ice, where the coefficient of kinetic friction is $\mu_k$. The stone travels a distance $d$. If the initial velocity is doubled, how far will the stone slide?

a) $d$  

b) $2d$  

c) $d^2$  

d) $4d$  

e) $4d^2$

**CALCULATE** Putting in the numerical values gives us

$$v = \sqrt{\frac{2}{1.35 \text{ kg}}} (2.01 \text{ J} + 9.93 \text{ J}) = 4.20581608 \text{ m/s}.$$

**ROUND** All of the numerical values were given to three significant figures, so we have $v = 4.21 \text{ m/s}$.

**DOUBLE-CHECK** As you can see, the main contribution to the speed of the block at impact originates from the free-fall portion of its path. Why did we go through the intermediate step of figuring out the value of $K_{top}$, instead of simply using the formula $v = \sqrt{\frac{2}{m}}(K_{top} + mgd)$, which we got from a generalization of the work-energy theorem? We needed to calculate $K_{top}$ first to ensure that it is positive, meaning that the energy imparted to the block by the spring is sufficient to exceed the work to be done against the friction force. If $K_{top}$ had turned out to be negative, the block would have stopped on the table. For example, if we had attempted to solve the same problem with a coefficient of kinetic friction between the block and the table of $\mu_k = 0.50$ instead of $\mu_k = 0.16$, we would have found that

$$K_{top} = 3.39 \text{ J} - 4.30 \text{ J} = -0.91 \text{ J},$$

which is impossible.

As Solved Problem 6.5 shows, energy considerations are still a powerful tool for performing otherwise very difficult calculations, even in the presence of nonconservative forces. However, the principle of conservation of mechanical energy cannot be applied quite as straightforwardly when nonconservative forces are present, and you have to account for the energy dissipated by these forces. Let’s try one more solved problem to elaborate on these concepts.

**SOLVED PROBLEM 6.6** Sledding down a Hill

**PROBLEM** A boy on a sled starts from rest and slides down a snow-covered hill. Together the boy and sled have a mass of 23.0 kg. The hill’s slope makes an angle $\theta = 35.0^\circ$ with the horizontal. The surface of the hill is 25.0 m long. When the boy and the sled reach the bottom of the hill, they continue sliding on a horizontal snow-covered field. The coefficient of kinetic friction between the sled and the snow is 0.100. How far do the boy and sled move on the horizontal field before stopping?

**SOLUTION**

**THINK** The boy and sled start with zero kinetic energy and finish with zero kinetic energy and have gravitational potential energy at the top of the hill. As boy and sled go down the hill, they gain kinetic energy. At the bottom of the hill, their potential energy is zero, and they have kinetic energy. However, the boy and sled are continuously losing energy to friction. Thus, the change in potential energy will equal the energy lost to friction. We must take into account the fact that the friction force will be different when the sled is on the slope than when it is on the flat field.

**SKETCH** A sketch of the boy sledding is shown in Figure 6.18.

**RESEARCH** The boy and sled start with zero kinetic energy and finish with zero kinetic energy. We call the length of the slope of the hill $d_s$, and the distance that the boy and sled
travel on the flat field \(d_2\), as shown in Figure 6.18a. Assuming that the gravitational potential energy of the boy and sled is zero at the bottom of the hill, the change in the gravitational potential energy from the top of the hill to the flat field is

\[
\Delta U = -mgh,
\]

where \(m\) is the mass of the boy and sled together and \(h = d_1 \sin \theta\).

The force of friction is different on the slope and on the flat field because the normal force is different. From Figure 6.18b, the force of friction on the slope is

\[
f_{k1} = \mu_k N_1 = \mu_k mg \cos \theta.
\]

From Figure 6.18c, the force of friction on the flat field is

\[
f_{k2} = \mu_k N_2 = \mu_k mg.
\]

The energy dissipated by friction, \(W_f\), is equal to the energy dissipated by friction while sliding on the slope, \(W_{f1}\), plus the energy dissipated while sliding on the flat field, \(W_{f2}\):

\[
W_f = W_{f1} + W_{f2}.
\]

The energy dissipated by friction on the slope is

\[
W_{f1} = -f_{k1}d_1,
\]

and the energy dissipated by friction on the flat field is

\[
W_{f2} = -f_{k2}d_2.
\]

**Simplify** According to the preceding three equations, the total energy dissipated by friction is given by

\[
W_f = -f_{k1}d_1 - f_{k2}d_2.
\]

Substituting the two expressions for the friction forces into this equation gives us

\[
W_f = -(\mu_k mg \cos \theta)d_1 - (\mu_k mg)d_2.
\]

The change in potential energy is obtained by combining the equation \(\Delta U = -mgh\) with the expression obtained for the height, \(h = d_1 \sin \theta\):

\[
\Delta U = -mgd_1 \sin \theta.
\]

Since the sled is at rest at the top of the hill and at the end of the ride as well, we have \(\Delta K = 0\), and so according to equation 6.29, in this case, \(\Delta U = W_f\). Now we can equate the change in potential energy with the energy dissipated by friction:

\[
mgd_1 \sin \theta = (\mu_k mg \cos \theta)d_1 + (\mu_k mg)d_2.
\]

Canceling out \(mg\) on both sides and solving for the distance the boy and sled travel on the flat field, we get

\[
d_2 = \frac{d_1 (\sin \theta - \mu_k \cos \theta)}{\mu_k}.
\]

---

**FIGURE 6.18** (a) Sketch of the boy and sled on the slope and on the flat field, showing the angle of incline and distances. (b) Free-body diagram for the boy and sled on the slope. (c) Free-body diagram for the boy and sled on the flat field.
Let’s return to the relationship between force and potential energy. Perhaps it may help you gain physical insight into this relationship if you visualize the potential energy curve as the track of a roller coaster. This analogy is not a perfect one, because a roller coaster moves in a two-dimensional plane or even in three-dimensional space, not in one dimension, and there is some small amount of friction between the cars and the track. Still, it is a good approximation to assume that there is conservation of mechanical energy. The motion of the roller coaster car can then be described by a potential energy function.

Figure 6.19 shows plots of the potential energy (yellow line following the outline of the track), the total energy (horizontal orange line), and the kinetic energy (difference between these two, the red line) as a function of position for a segment of a roller coaster ride. You can see that the kinetic energy has a minimum at the highest point of the track, where the speed of the cars is smallest, and the speed increases as the cars roll down the incline. All of these effects are a consequence of the conservation of total mechanical energy. The motion of the roller coaster car can then be described by a potential energy function.

Figure 6.20 shows graphs of a potential energy function (part a) and the corresponding force (part b). Because the potential energy can be determined only within an additive
constant, the zero value of the potential energy in Figure 6.20a is set at the lowest value. However, for all physical considerations, this is irrelevant. On the other hand, the zero value for the force cannot be chosen arbitrarily.

**Equilibrium Points**

Three special points on the $x$-coordinate axis of Figure 6.20b are marked by vertical gray lines. These points indicate where the force has a value of zero. Because the force is the derivative of the potential energy with respect to the $x$-coordinate, the potential energy has an extremum—a maximum or minimum value—at such points. You can clearly see that the potential energies at $x_1$ and $x_3$ represent minima, and the potential energy at $x_2$ is a maximum. At all three points, an object would experience no acceleration, because it is located at an extremum where the force is zero. Because there is no force, Newton’s Second Law tells us that there is no acceleration. Thus, these points are equilibrium points.

The equilibrium points in Figure 6.20 represent two different kinds. Points $x_1$ and $x_3$ are stable equilibrium points, and $x_2$ is an unstable equilibrium point. What distinguishes stable and unstable equilibrium points is the response to perturbations (small changes in position around the equilibrium position).

**Definition**

At stable equilibrium points, small perturbations result in small oscillations around the equilibrium point. At unstable equilibrium points, small perturbations result in an accelerating movement away from the equilibrium point.

The roller coaster analogy may be helpful here: If you are sitting in a roller coaster car at point $x_1$ or $x_3$ and someone gives the car a push, it will just rock back and forth on the track, because you are sitting at a local point of lowest energy. However, if the car gets the same small push while sitting at $x_2$, it will roll down the slope.

What makes an equilibrium point stable or unstable from a mathematical standpoint is the value of the second derivative of the potential energy function, or the curvature. Negative curvature means a local maximum of the potential energy function, and therefore an unstable equilibrium point; a positive curvature indicates a stable equilibrium point. Of course, there is also the situation between a stable and unstable equilibrium, between a positive and negative curvature. This is a point of metastable equilibrium, with zero local curvature, that is, a value of zero for the second derivative of the potential energy function.

**Turning Points**

Figure 6.21a shows the same potential energy function as Figure 6.20, but with the addition of horizontal lines for four different values of the total mechanical energy ($E_1$ through $E_4$). For each value of this total energy and for each point on the potential energy curve, we can calculate the value of the kinetic energy by simple subtraction. Let’s first consider the largest value of the total mechanical energy shown in the figure, $E_1$ (blue horizontal line):

$$K(x) = E_1 - U(x).$$  \[6.30\]

The kinetic energy, $K_1(x)$, is shown in Figure 6.21b by the blue curve, which is clearly an upside-down version of the potential energy curve in Figure 6.21a. However, its absolute height is not arbitrary but results from equation 6.30. As previously mentioned, we can always add an arbitrary additive constant to the potential energy, but then we are forced to add the same additive constant to the total mechanical energy, so that their difference, the kinetic energy, remains unchanged.

For the other values of the total mechanical energy in Figure 6.21, an additional complication arises: the condition that the kinetic energy has to be larger than or equal to zero. This condition means that the kinetic energy is not defined in a region where $E_i - U(x)$ is negative. For the total mechanical energy of $E_2$, the kinetic energy is greater than zero only...
for \( x \geq a \), as indicated in Figure 6.21b by the green curve. Thus, an object moving with total energy \( E_4 \) from right to left in Figure 6.21 will reach the point \( x = a \) and have zero velocity there. Referring to Figure 6.20, you see that the force at that point is positive, pushing the object to the right, that is, making it turn around. This is why such a point is called a **turning point**. Moving to the right, this object will pick up kinetic energy and follow the same kinetic energy curve from left to right, making its path reversible. This behavior is a consequence of the conservation of total mechanical energy.

**Definition**

**Turning points** are points where the kinetic energy is zero and where a net force moves the object away from the point.

An object with total energy equal to \( E_i \) in Figure 6.21 has two turning points in its path: \( x = e \) and \( x = f \). The object can move only between these two points. It is trapped in this interval and cannot escape. Perhaps the roller coaster analogy is again helpful: A car released from point \( x = e \) will move through the dip in the potential energy curve to the right until it reaches the point \( x = f \), where it will reverse direction and move back to \( x = e \), never having enough total mechanical energy to escape the dip. The region in which the object is trapped is often referred to as a **potential well**.

Perhaps the most interesting situation is that for which the total energy is \( E_3 \). If an object moves in from the right in Figure 6.21 with energy \( E_3 \), it will be reflected at the turning point where \( x = d \), in complete analogy to the situation at \( x = a \) for the object with energy \( E_2 \). However, there is another allowed part of the path farther to the left, in the interval \( b \leq x \leq c \). If the object starts out in this interval, it remains trapped in a dip, just like the object with energy \( E_3 \). Between the allowed intervals \( b \leq x \leq c \) and \( x \geq d \) is a **forbidden region** that an object with total mechanical energy \( E_3 \) cannot cross.

**Preview: Atomic Physics**

In studying atomic physics, we will again encounter potential energy curves. Particles with energies such as \( E_i \) in Figure 6.21, which are trapped between two turning points, are said to be in **bound states**. One of the most interesting phenomena in atomic and nuclear physics, however, occurs in situations like the one shown in Figure 6.21 for a total mechanical energy of \( E_p \). From our considerations of classical mechanics in this chapter, we expect that an object sitting in a bound state between \( b \leq x \leq c \) cannot escape. However, in atomic and nuclear physics applications, a particle in such a bound state has a small probability of escaping out of this potential well and through the classically forbidden region into the region \( x \geq d \). This process is called **tunneling**. Depending on the height and width of the barrier, the tunneling probability can be quite large, leading to a fast escape, or quite small, leading to a very slow escape. For example, the isotope \(^{235}\text{U}\) of the element uranium, used in nuclear fission power plants and naturally occurring on Earth, has a half-life of over 700 million years, which is the average time that elapses until an alpha particle (a tightly bound cluster of two neutrons and two protons in the nucleus) tunnels through its potential barrier, causing the uranium nucleus to decay. In contrast, the isotope \(^{238}\text{U}\) has a half-life of 4500 million years. Thus, much of the original \(^{235}\text{U}\) present on Earth has decayed away. The fact that \(^{235}\text{U}\) comprises only 0.7% of all naturally occurring uranium means that the first sign that a nation is attempting to use nuclear power, for any purpose, is the acquisition of equipment that can separate \(^{235}\text{U}\) from the much more abundant (99.3%) \(^{238}\text{U}\), which is not suitable for nuclear fission power production.

The previous paragraph is included to whet your appetite for things to come. In order to understand the processes of atomic and nuclear physics, you’ll need to be familiar with quite a few more concepts. However, the basic considerations of energy introduced here will remain virtually unchanged.
**WHAT WE HAVE LEARNED**

- Potential energy, \( U \), is the energy stored in the configuration of a system of objects that exert forces on one another.
- Gravitational potential energy is defined as \( U_g = mgy \).
- The potential energy associated with elongating a spring from its equilibrium position at \( x = 0 \) is \( U(x) = \frac{1}{2} kx^2 \).
- A conservative force is a force for which the work done over any closed path is zero. A force that does not fulfill this requirement is a nonconservative force.
- For any conservative force, the change in potential energy due to some spatial rearrangement of a system is equal to the negative of the work done by the conservative force during this spatial rearrangement.
- The relationship between a potential energy and the corresponding conservative force is
  \[ \Delta U = U(x) - U(x_0) = -\int_{x_0}^{x} F(x')dx' \, . \]
- In one-dimensional situations, the force component can be obtained from the potential energy using
  \[ F(x) = -\frac{dU(x)}{dx} \, . \]
- The mechanical energy, \( E \), is the sum of the kinetic energy and the potential energy: \( E = K + U \).
- The total mechanical energy is conserved for any mechanical process inside an isolated system that involves only conservative forces: \( \Delta E = \Delta K + \Delta U = 0 \). An alternative way of expressing this law of conservation of mechanical energy is \( K + U = K_0 + U_0 \).
- The total energy—the sum of all forms of energy, mechanical or other—is always conserved in an isolated system. This holds for conservative as well as nonconservative forces:
  \[ E_{\text{total}} = E_{\text{mechanical}} + E_{\text{other}} = K + U + E_{\text{other}} = \text{constant} \, . \]
- Energy problems involving nonconservative forces can be solved using the work–energy theorem: \( W_f = \Delta K + \Delta U \).
- At stable equilibrium points, small perturbations result in small oscillations around the equilibrium point; at unstable equilibrium points, small perturbations result in an accelerating movement away from the equilibrium point.
- Turning points are points where the kinetic energy is zero and where a net force moves the object away from the point.

**ANSWERS TO SELF-TEST OPPORTUNITIES**

6.1 The potential energy is proportional to the inverse of the distance between the two objects. Examples of these forces are the force of gravity (see Chapter 12) and the electrostatic force (see Chapter 21).

6.2 To handle this problem with air resistance included, we would have introduced the work done by air resistance, which can be treated as a friction force. We would have modified our statement of energy conservation to reflect the fact that work, \( W_f \), is done by the friction force:

\[ W_f + K + U = K_0 + U_0 \, . \]

The solution would have to be done numerically because the work done by friction in this case would depend on the distance that the rock actually traveled through the air.

6.3 The lighter-colored ball descends to a lower elevation earlier in its motion and thus converts more of its potential energy to kinetic energy early on. Greater kinetic energy means higher speed. Thus, the lighter-colored ball reaches higher speeds earlier and is able to move to the bottom of the track faster, even though its path length is greater.

6.4 The speed is at a maximum where the kinetic energy is at a maximum:

\[
K(y) = U(-3.3 \text{ m}) - U(y) = (3856 \text{ J}) - (671 \text{ J/m})y - (557.5 \text{ J/m}^2)y^2
\]

\[
\frac{d}{dy} K(y) = -(671 \text{ J/m}) - (1115 \text{ J/m}^2)y = 0 \Rightarrow y = -0.602 \text{ m}
\]

\[
\sqrt{2K(-0.602 \text{ m})}/m = 10.89 \text{ m/s}
\]

Note that the value at which the speed is maximum is the equilibrium position of the spring once it is loaded with the human cannonball.

6.5 The net force at the maximum stretching is \( F = k(l_{\text{max}} - L_0) - mg \). Therefore, the acceleration at this point is

\[
a = k(l_{\text{max}} - L_0)/m - g
\]

Inserting the expression we found for \( L_0 \) gives

\[
a = \sqrt{2gL_{\text{max}}k/m} - g
\]

The maximum acceleration increases with the square root of the spring constant. If one wants to jump from a great height, \( L_{\text{max}} \), a very soft bungee cord is needed.
1. Many of the problem-solving guidelines given in Chapter 5 apply to problems involving conservation of energy as well. It is important to identify the system and determine the state of the objects in it at different key times, such as the beginning and end of each kind of motion. You should also identify which forces in the situation are conservative or nonconservative, because they affect the system in different ways.

2. Try to keep track of each kind of energy throughout the problem situation. When does the object have kinetic energy? Does gravitational potential energy increase or decrease? Where is the equilibrium point for a spring?

3. Remember that you can choose where potential energy is zero, so try to determine what choice will simplify the calculations.

4. A sketch is almost always helpful, and often a free-body diagram is useful as well. In some cases, drawing graphs of potential energy, kinetic energy, and total mechanical energy is a good idea.

### MULTIPLE-CHOICE QUESTIONS

**6.1** A block of mass 5.0 kg slides without friction at a speed of 8.0 m/s on a horizontal table surface until it strikes and sticks to a horizontal spring (with spring constant of \( k = 2000 \) N/m and very small mass), which in turn is attached to a wall. How far is the spring compressed before the mass comes to rest?

a) 0.40 m  
 b) 0.54 m  
 c) 0.30 m  
 d) 0.020 m

**6.2** A pendulum swings in a vertical plane. At the bottom of the swing, the kinetic energy is 8 J and the gravitational potential energy is 4 J. At the highest position of the swing, the kinetic and gravitational potential energies are

a) kinetic energy = 0 J and gravitational potential energy = 4 J.  
 b) kinetic energy = 12 J and gravitational potential energy = 0 J.  
 c) kinetic energy = 0 J and gravitational potential energy = 12 J.  
 d) kinetic energy = 4 J and gravitational potential energy = 8 J.  
 e) kinetic energy = 8 J and gravitational potential energy = 4 J.

**6.3** A ball of mass 0.50 kg is released from rest at point A, which is 5.0 m above the bottom of a tank of oil, as shown in the figure. At point B, which is 2.0 m above the bottom of the tank, the ball has a speed of 6.0 m/s. The work done on the ball by the force of fluid friction is

a) +15 J.  
 b) +9 J.  
 c) −9 J.  
 d) −5.7 J.  
 e) −5 J.

**6.4** A child throws three identical marbles from the same height above the ground so that they land on the flat roof of a building. The marbles are launched with the same initial speed. The first marble, marble A, is thrown at an angle of 75° above horizontal, while marbles B and C are thrown with launch angles of 60° and 45°, respectively. Neglecting air resistance, rank the marbles according to the speeds with which they hit the roof.

a) A < B < C  
 b) C < B < A  
 c) A and C have the same speed; B has a lower speed.  
 d) B has the highest speed; A and C have the same speed.  
 e) A, B, and C all hit the roof with the same speed.

**6.5** Which of the following is not a valid potential energy function for the spring force \( F = −kx \)?

a) \( \frac{1}{2}kx^2 \)  
 b) \( \frac{1}{2}kx^2 + 10 \)  
 c) None of the above  
 d) \( \frac{1}{2}kx^2 \) is valid.

**6.6** You use your hand to stretch a spring to a displacement \( x \) from its equilibrium position and then slowly bring it back to that position. Which is true?

a) The spring’s \( ΔU \) is positive.  
 b) The spring’s \( ΔU \) is negative.  
 c) The hand’s \( ΔU \) is positive.  
 d) The hand’s \( ΔU \) is negative.  
 e) None of the above statements is true.

**6.7** In Question 6, what is the work done by the hand?

a) \( −\frac{1}{2}kx^2 \)  
 b) \( \frac{1}{2}kx^2 \)  
 c) none of the above  
 d) zero  
 e) \( \frac{1}{2}mv^2 \), where \( v \) is the speed of the hand

**6.8** Which of the following is not a unit of energy?

a) newton-meter  
 b) joule  
 c) kilowatt-hour  
 d) kg m²/s²  
 e) All of the above are units of energy.

**6.9** A spring has a spring constant of 80 N/m. How much potential energy does it store when stretched by 1.0 cm?

a) 4.0 \( \times 10^{-3} \) J  
 b) 0.8 J  
 c) 80 J  
 d) 800 J

**6.10** What is the maximum acceleration that the human cannonball of Solved Problem 6.4 experiences?

a) 1.00g  
 b) 2.14g  
 c) 3.25g  
 d) 4.48g  
 e) 7.30g

**6.11** For an object sliding on the ground, the friction force

a) always acts in the same direction as the displacement.  
 b) always acts in a direction perpendicular to the displacement.  
 c) always acts in a direction opposite to the displacement.  
 d) acts either in the same direction as the displacement or in the direction opposite to the displacement depending on the value of the coefficient of kinetic friction.

**6.12** Some forces in nature vary with the inverse of the distance squared between two objects. For a force like this, how does the potential energy vary with the distance between the two objects?

a) The potential energy varies with the distance.  
 b) The potential energy varies with the distance squared.  
 c) The potential energy varies with the inverse of the distance.  
 d) The potential energy varies with the distance squared.  
 e) The potential energy does not depend on the distance.

**6.13** A baseball is dropped from the top of a building. Air resistance acts on the baseball as it drops. Which of the following statements is true?

a) The change in potential energy of the baseball as it falls is equal to the kinetic energy of the baseball just before it strikes the ground.  
 b) The change in potential energy of the baseball as it falls is greater than the kinetic energy of the baseball just before it strikes the ground.  
 c) The change in potential energy of the baseball as it falls is less than the kinetic energy of the baseball just before it strikes the ground.  
 d) The change in potential energy of the baseball is equal to the energy lost due to the friction from the air resistance while the ball is falling.

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**PROBLEM-SOLVING GUIDELINES: CONSERVATION OF ENERGY**

1. Many of the problem-solving guidelines given in Chapter 5 apply to problems involving conservation of energy as well. It is important to identify the system and determine the state of the objects in it at different key times, such as the beginning and end of each kind of motion. You should also identify which forces in the situation are conservative or nonconservative, because they affect the system in different ways.

2. Try to keep track of each kind of energy throughout the problem situation. When does the object have kinetic energy? Does gravitational potential energy increase or decrease? Where is the equilibrium point for a spring?

3. Remember that you can choose where potential energy is zero, so try to determine what choice will simplify the calculations.

4. A sketch is almost always helpful, and often a free-body diagram is useful as well. In some cases, drawing graphs of potential energy, kinetic energy, and total mechanical energy is a good idea.
CONCEPTUAL QUESTIONS

6.14 Can the kinetic energy of an object be negative? Can the potential energy of an object be negative?
6.15 a) If you jump off a table onto the floor, is your mechanical energy conserved? If not, where does it go? b) A car moving down the road smashes into a tree. Is the mechanical energy of the car conserved? If not, where does it go?
6.16 How much work do you do when you hold a bag of groceries while standing still? How much work do you do when carrying the same bag a distance d across the parking lot of the grocery store?
6.17 An arrow is placed on a bow, the bowstring is pulled back, and the arrow is shot straight upward. The arrow comes back down and sticks into the ground. Describe all of the changes in work and energy that occur.
6.18 Two identical billiard balls start at the same height and the same time and roll along different tracks, as shown in the figure:
   a) Which ball has the highest speed at the end?
   b) Which one will get to the end first?

6.19 A girl of mass 40.0 kg is on a swing, which has a mass of 1.0 kg. Suppose you pull her back until her center of mass is 2.0 m above the ground. Then you let her go, and she swings out and returns to the same point. Are all forces acting on the girl and swing conservative?
6.20 Can a potential energy function be defined for the force of friction?
6.21 Can the potential energy of a spring be negative?
6.22 One end of a rubber band is tied down, and you pull on the other end to trace a complicated closed trajectory. If you measured the elastic force F at every point, took its scalar product with the local displacements, F · dx, and then summed all of these, what would you get?
6.23 Can a unique potential energy function be identified with a particular conservative force?
6.24 In skydiving, the vertical velocity component of the skydiver is typically zero at the moment he or she leaves the plane; the vertical component of the velocity then increases until the skydiver reaches terminal speed (see Chapter 4). For a simplified model of this motion, we assume that the horizontal velocity component is zero and that the vertical velocity component increases linearly with acceleration a_v = -g until the skydiver reaches terminal velocity, after which it stays constant. Thus, our simplified model assumes free fall without air resistance followed by falling at constant speed. Sketch the kinetic energy, potential energy, and total energy as a function of time for this model.
6.25 A projectile of mass m is launched from the ground at t = 0 with a speed v_0 and at an angle θ_0 above the horizontal. Assuming that air resistance is negligible, write the kinetic, potential, and total energies of the projectile as explicit functions of time.
6.26 The energy height, H, of an aircraft of mass m at altitude h and with speed v is defined as its total energy (with the zero of the potential energy taken at ground level) divided by its weight. Thus, the energy height is a quantity with units of length.
   a) Derive an expression for the energy height, H, in terms of the quantities m, h, and v.
   b) A Boeing 747 jet with mass 3.5 × 10^7 kg is cruising in level flight at 250.0 m/s at an altitude of 10.0 km. Calculate the value of its energy height.
6.27 A body of mass m moves in one dimension under the influence of a force, F(x), which depends only on the body’s position.
   a) Prove that Newton’s Second Law and the law of conservation of energy for this body are exactly equivalent.
   b) Explain, then, why the law of conservation of energy is considered to be of greater significance than Newton’s Second Law.
6.28 The molecular bonding in a diatomic molecule such as the nitrogen (N₂) molecule can be modeled by the Lennard-Jones potential, which has the form
   \[ U(x) = 4U_0 \left( \frac{x_0}{x} \right)^{12} - 2 \frac{x_0}{x} \],
   where x is the separation distance between the two nuclei and x₀ and U₀ are constants. Determine, in terms of these constants, the following:
   a) the corresponding force function;
   b) the equilibrium separation xₑ, which is the value of x for which the two atoms experience zero force from each other; and
   c) the nature of the interaction (repulsive or attractive) for separations larger and smaller than xₑ.
6.29 A particle of mass m moving in the xy-plane is confined by a two-dimensional potential function, U(x, y) = \( \frac{1}{2} k(x^2 + y^2) \).
   a) Derive an expression for the net force, \( \vec{F} = F_x \hat{i} + F_y \hat{j} \).
   b) Find the equilibrium point on the xy-plane.
   c) Describe qualitatively the effect of the net force.
   d) What is the magnitude of the net force on the particle at the coordinate (x₀, 0) in centimeters if k = 10.0 N/cm²?
   e) What are the turning points if the particle has 10.0 J of total mechanical energy?
6.30 For a rock dropped from rest from a height h, to calculate the speed just before it hits the ground, we use the conservation of mechanical energy and write mgh = \( \frac{1}{2} mv^2 \). The mass cancels out, and we solve for v. A very common error made by some beginning physics students is to assume, based on the appearance of this equation, that they should set the kinetic energy equal to the potential energy at the same point in space. For example, to calculate the speed v₁ of the rock at some height h₁ < h, they often write mg(\( h - h_1 \)) = \( \frac{1}{2} mv^2 \) and solve for v. Explain why this approach is wrong.
6.31 What is the gravitational potential energy of a 2.00-kg book 1.50 m above the floor?
6.32 a) If the gravitational potential energy of a 40.0-kg rock is 500 J relative to a value of zero on the ground, how high is the rock above the ground?
   b) If the rock were lifted to twice its original height, how would the value of its gravitational potential energy change?

EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One * and two ** indicate increasing level of problem difficulty.

Section 6.1
6.31 What is the gravitational potential energy of a 2.00-kg book 1.50 m above the floor?
A uniform chain of total mass $m$ is laid out straight on a frictionless table and held stationary so that one-third of its length, $L = 1.00$ m, is hanging vertically over the edge of the table. The chain is then released. Determine the speed of the chain at the instant when only one-third of its length remains on the table.

6.46 A 20.0-kg child is on a swing attached to ropes that are $L = 1.50$ m long. Take the zero of the gravitational potential energy to be at the position of the child when the ropes are horizontal.

a) Determine the child's gravitational potential energy when the child is at the lowest point of the circular trajectory.

b) Determine the child's gravitational potential energy when the ropes make an angle of 45.0° relative to the vertical.

c) Based on these results, which position has the higher potential energy?

### Section 6.3

6.35 A 1.40-kg sphere travels 2.50 km up an incline at constant velocity. The incline has an angle of 3.00° with respect to the horizontal. What is the change in the car’s potential energy? What is the net work done on the car?

6.36 A constant force of 40.0 N is needed to keep a car traveling at constant speed as it moves 5.00 km along a road. How much work is done? Is the work done on or by the car?

6.37 A pitha of mass 3.27 kg is attached to a string tied to a hook in the ceiling. The length of the string is 0.810 m, and the pitha is released from rest from an initial position in which the string makes an angle of 56.5° with the vertical. What is the work done by gravity by the time the string is in a vertical position for the first time?

### Section 6.4

6.38 A particle is moving along the $x$-axis subject to the potential energy function $U(x) = ax^2 + bx^3 - d$, where $a = 7.00$ J/m, $b = 10.0$ J/m$^2$, $c = 6.00$ J/m, and $d = 28.0$ J.

a) Express the force felt by the particle as a function of $x$.

b) Plot this force and the potential energy function.

c) Determine the net force on the particle at the coordinate $x = 2.00$ m.

6.39 Calculate the force $F(y)$ associated with each of the following potential energies:

a) $U(y) = ay^2 - by^3$

b) $U(y) = U_0 \sin (cy)$

6.40 The potential energy of a certain particle is given by $U(x,z) = ax^2 + bx^2$, where $a$ and $b$ are constants. Find the force vector exerted on the particle.

### Section 6.5

6.41 A 0.624-kg basketball of mass 0.624 kg is shot from a vertical height of 1.20 m and at a speed of 20.0 m/s. After reaching its maximum height, the ball moves into the hoop on its downward path, at a distance of 0.331 m from the equilibrium position.

Determine the potential energy of the basketball when it passes the equilibrium point.

6.42 A flatbed of mass 100 kg is laid out straight on a frictionless incline that uses friction to stop a large air-filled 0.100-kg plastic ball is thrown up into the air with an initial speed of 10.0 m/s. After reaching its maximum height, the ball moves into the hoop on its downward path, at a distance of 0.331 m from the equilibrium position.

Determine the total mechanical energy of the system.

b) How fast is the stone moving as it passes the equilibrium point?

### Section 6.7

6.53 An 80.0-kg fireman slides down a 3.00-m pole by applying a frictional force of 400 N against the pole with his hands. If he slides from rest, how fast is he moving once he reaches the ground?

6.54 A large air-filled 0.100-kg plastic ball is thrown up into the air with an initial speed of 10.0 m/s. At a height of 3.00 m, the ball's speed is 3.00 m/s. What fraction of its original energy has been lost to air friction?

6.55 How much mechanical energy is lost to friction if a 55.0-kg skier slides down a ski slope at constant speed of 14.4 m/s? The slope is 123.5 m long and makes an angle of 14.7° with respect to the horizontal.

6.56 A block of mass 0.773 kg on a spring with spring constant 239.5 N/m oscillates vertically with amplitude 0.551 m. What is the speed of this block at a distance of 0.331 m from the equilibrium position?

6.57 A 50.0-kg car travels 2.50 km up an incline at constant velocity. The incline has an angle of 3.00° with respect to the horizontal. What is the change in the car’s potential energy? What is the net work done on the car?
a truck in such a situation; see the figure. In this case, the incline makes an angle of $\theta = 40.15^\circ$ with the horizontal, and the gravel has a coefficient of friction of 0.634 with the tires of the truck. How far along the incline ($\Delta x$) does the truck travel before it stops?

• 6.57 A snowboarder of mass 70.1 kg (including gear and clothing), starting with a speed of 5.10 m/s, slides down a slope at an angle $\theta = 37.1^\circ$ with the horizontal. The coefficient of kinetic friction is 0.116. What is the net work done on the snowboarder in the first 5.72 s of descent?

• 6.58 The greenskeepers of golf courses use a stimpmeter to determine how “fast” their greens are. A stimpmeter is a straight aluminum bar with a V-shaped groove on which a golf ball can roll. It is designed to release the golf ball once the angle of the bar with the ground reaches a value of $\theta = 20.0^\circ$. The golf ball (mass = 1.62 oz = 0.0459 kg) rolls 30.0 in down the bar and then continues to roll along the green for several feet. This distance is called the “reading.” The test is done on a level part of the green, and stimpmeter readings between 7 and 12 ft are considered acceptable. For a stimpmeter reading of 11.1 ft, what is the coefficient of friction between the ball and the green? (The ball is rolling and not sliding, as we usually assume when considering friction, but this does not change the result in this case.)

• 6.59 A 1.00-kg block is pushed up and down a rough plank of length $L = 2.00$ m, inclined at 30.0° above the horizontal. From the bottom, it is pushed a distance $L/2$ up the plank, then pushed back down a distance $L/4$, and finally pushed back up the plank until it reaches the top end. If the coefficient of kinetic friction between the block and plank is 0.300, determine the work done by the block against friction.

• 6.60 A 1.00-kg block initially at rest at the top of a 4.00-m incline with a slope of 45.0° begins to slide down the incline. The upper half of the incline is frictionless, while the lower half is rough, with a coefficient of kinetic friction $\mu_k = 0.300$.

  a) How fast is the block moving midway along the incline, before entering the rough section?
  b) How fast is the block moving at the bottom of the incline?

• 6.61 A spring with a spring constant of 500. N/m is used to propel a 0.500-kg mass up an inclined plane. The spring is compressed 30.0 cm from its equilibrium position and launches the mass from rest across a horizontal surface and onto the plane. The plane has a length of 4.00 m and is inclined at 30.0°. Both the plane and the horizontal surface have a coefficient of kinetic friction with the mass of 0.350. When the spring is compressed, the mass is 1.50 m from the bottom of the plane.

  a) What is the speed of the mass as it reaches the bottom of the plane?
  b) What is the speed of the mass as it reaches the top of the plane?
  c) What is the total work done by friction from the beginning to the end of the mass’s motion?

• 6.62 The sled shown in the figure leaves the starting point with a velocity of 20.0 m/s. Use the work-energy theorem to calculate the sled’s speed at the end of the track or the maximum height it reaches if it stops before reaching the end.

![Image](image_url)

Section 6.8

• 6.63 On the segment of roller coaster track shown in the figure, a cart of mass 237.5 kg starts at $x = 0$ with a speed of 16.5 m/s. Assuming that the dissipation of energy due to friction is small enough to be ignored, where is the turning point of this trajectory?

• 6.64 A 70.0-kg skier moving horizontally at 4.50 m/s encounters a 20.0° incline.

  a) How far up the incline will the skier move before she momentarily stops, ignoring friction?
  b) How far up the incline will the skier move if the coefficient of kinetic friction between the skis and snow is 0.100?

• 6.65 A 0.200-kg particle is moving along the $x$-axis, subject to the potential energy function shown in the figure, where $U_A = 50.0$ J, $U_B = 0$ J, $U_C = 25.0$ J, $U_D = 10.0$ J, and $U_E = 60.0$ J along the path. If the particle was initially at $x = 4.00$ m and had a total mechanical energy of 40.0 J, determine:

  a) the particle’s speed at $x = 3.00$ m,
  b) the particle’s speed at $x = 4.50$ m, and
  c) the particle’s turning points.

![Graph](graph_url)

Additional Exercises

6.66 A ball of mass 1.84 kg is dropped from a height $y_i = 1.49$ m and then bounces back up to a height of $y_f = 0.87$ m. How much mechanical energy is lost in the bounce? The effect of air resistance has been experimentally found to be negligible in this case, and you can ignore it.

6.67 A car of mass 987 kg is traveling on a horizontal segment of a freeway with a speed of 64.5 mph. Suddenly, the driver has to hit the brakes hard to try to avoid an accident up ahead. The car does not have an ABS (antilock
braking system), and the wheels lock, causing the car to slide some distance before it is brought to a stop by the friction force between its tires and the road surface. The coefficient of kinetic friction is 0.301. How much mechanical energy is lost to heat in this process?

6.68 Two masses are connected by a light string that goes over a light, frictionless pulley, as shown in the figure. The 10.0-kg mass is released and falls through a vertical distance of 1.00 m before hitting the ground. Use conservation of mechanical energy to determine:

a) how fast the 5.00-kg mass is moving just before the 10.0-kg mass hits the ground; and
b) the maximum height attained by the 5.00-kg mass.

6.69 In 1896 in Waco, Texas, William George Crush, owner of the K-T (or “Katy”) Railroad, parked two locomotives at opposite ends of a 6.4-km-long track, fired them up, tied their throttles open, and then allowed them to crash head-on at full speed in front of 30,000 spectators. Hundreds of people were hurt by flying debris; a few were killed. Assuming that each locomotive weighed 1.2 \times 10^6 N and its acceleration along the track was a constant 0.26 m/s², what was the total kinetic energy of the two locomotives just before the collision?

6.70 A baseball pitcher can throw a 5.00-oz baseball with a speed measured by a radar gun to be 90.0 mph. Assuming that the force exerted by the pitcher on the ball acts over a distance of two arm lengths, each 28.0 in, what is the average force exerted by the pitcher on the ball?

6.71 A 1.50-kg ball has a speed of 20.0 m/s when it is 15.0 m above the ground. What is the total energy of the ball?

6.72 If it takes an average force of 5.50 N to push a 4.50-g dart 6.00 cm into a dart gun, assuming that the barrel is frictionless, how fast will the dart exit the gun?

6.73 A high jumper approaches the bar at 9.00 m/s. What is the highest altitude the jumper can reach, if he does not use any additional push off the ground and is moving at 7.00 m/s as he goes over the bar and if he stays upright during the jump?

6.74 A roller coaster is moving at 2.00 m/s at the top of the first hill (h = 40.0 m). Ignoring friction and air resistance, how fast will the roller coaster be moving at the top of a subsequent hill, which is 15.0 m high?

6.75 You are on a swing with a chain 4.00 m long. If your maximum displacement from the vertical is 35.0°, how fast will you be moving at the bottom of the arc?

6.76 A truck is descending a winding mountain road. When the truck is 680. m above sea level and traveling at 15.0 m/s, its brakes fail. What is the maximum possible speed of the truck at the foot of the mountain, 550. m above sea level?

6.77 Tarzan swings on a taut vine from his tree house to a limb on a neighboring tree, which is located a horizontal distance of 10.0 m from and a vertical distance of 4.00 m below his starting point. Amazingly the vine neither stretches nor breaks; Tarzan’s trajectory is thus a portion of a circle. If Tarzan starts with zero speed, what is his speed when he reaches the limb?

6.78 The graph shows the component (F \cos θ) of the net force that acts on a 2.00-kg block as it moves along a flat horizontal surface. Find

a) the net work done on the block;
b) the final speed of the block if it starts from rest at s = 0.

6.79 A 3.00-kg model rocket is launched vertically upward with sufficient initial speed to reach a height of 1.00 \times 10^4 m, even though air resistance (a nonconservative force) performs –8.00 \times 10^2 J of work on the rocket. How high would the rocket have gone, if there were no air resistance?

6.80 A 0.500-kg mass is attached to a horizontal spring with k = 100. N/m. The mass slides across a frictionless surface. The spring is stretched 25.0 cm from equilibrium, and then the mass is released from rest.

a) Find the mechanical energy of the system.
b) Find the speed of the mass when it has moved 5.00 cm.
c) Find the maximum speed of the mass.

6.81 You have decided to move a refrigerator (mass = 81.3 kg, including all the contents) to the other side of a room. You slide it across the floor on a straight path of length 6.35 m, and the coefficient of kinetic friction between floor and fridge is 0.437. Happy about your accomplishment, you leave the apartment. Your roommate comes home, wonders why the fridge is on the other side of the room, picks it up (you have a strong roommate!), carries it back to where it was originally, and puts it down. How much net mechanical work have the two of you done together?

6.82 A 1.00-kg block compresses a spring for which k = 100. N/m by 20.0 cm; the spring is then released, and the block moves across a horizontal, frictionless table, where it hits and compresses another spring, for which k = 50.0 N/m. Determine

a) the total mechanical energy of the system,
b) the speed of the mass while moving freely between springs, and
c) the maximum compression of the second spring.

6.83 A 1.00-kg block is resting against a light, compressed spring at the bottom of a rough plane inclined at an angle of 30.0°; the coefficient of kinetic friction between block and plane is μ_k = 0.100. Suppose the spring is compressed 10.0 cm from its equilibrium length. The spring is then released, and the block separates from the spring and slides up the incline a distance of only 2.00 cm beyond the spring’s normal length before stopping. Determine

a) the change in total mechanical energy of the system, and
b) the spring constant k.

6.84 A 0.100-kg ball is dropped from a height of 1.00 m and lands on a light (approximately massless) cup mounted on top of a light, vertical spring initially at its equilibrium position. The maximum compression of the spring is to be 10.0 cm.

a) What is the required spring constant of the spring?
b) Suppose you ignore the change in the gravitational energy of the ball during the 10.0-cm compression. What is the percentage difference between the calculated spring constant for this case and the answer obtained in part (a)?
**6.85** A mass of 1.00 kg attached to a spring with a spring constant of 100. N/m oscillates horizontally on a smooth frictionless table with an amplitude of 0.500 m. When the mass is 0.250 m away from equilibrium, determine:

a) its total mechanical energy,
b) the system's potential energy and the mass's kinetic energy, and
c) the mass's kinetic energy when it is at the equilibrium point.
d) Suppose there was friction between the mass and the table so that the amplitude was cut in half after some time. By what factor has the mass's maximum kinetic energy changed?
e) By what factor has the maximum potential energy changed?

**6.86** Bolo, the human cannonball, is ejected from a 3.50-m long barrel. If Bolo (m = 80.0 kg) has a speed of 12.0 m/s at the top of his trajectory, 15.0 m above the ground, what was the average force exerted on him while in the barrel?

**6.87** A 1.00-kg mass is suspended vertically from a spring with \( k = 100. \) N/m and oscillates with an amplitude of 0.200 m. At the top of its oscillation, the mass is hit in such a way that it instantaneously moves down with a speed of 1.00 m/s. Determine

a) its total mechanical energy,
b) how fast it is moving as it crosses the equilibrium point, and
c) its new amplitude.

**6.88** A runner reaches the top of a hill with a speed of 6.50 m/s. He descends 50.0 m and then ascends 28.0 m to the top of the next hill. His speed is now 4.50 m/s. The runner has a mass of 83.0 kg. The total distance that the runner covers is 400. m, and there is a constant resistance to motion of 9.00 N. Use energy considerations to find the work done by the runner over the total distance.

**6.89** A package is dropped on a horizontal conveyor belt. The mass of the package is \( m \), the speed of the conveyor belt is \( v_c \), and the coefficient of kinetic friction between the package and the belt is \( \mu_k \).

a) How long does it take for the package to stop sliding on the belt?
b) What is the package's displacement during this time?
c) What is the energy dissipated by friction?
d) What is the total work done by the conveyor belt?

**6.90** A father exerts a 2.40·10^2 N force to pull a sled with his daughter on it (combined mass of 85.0 kg) across a horizontal surface. The rope with which he pulls the sled makes an angle of 70.0° with the horizontal. The coefficient of kinetic friction is 0.200, and the sled moves a distance of 8.00 m. Find

a) the work done by the father,
b) the work done by the friction force, and
c) the total work done by all the forces.

**6.91** A variable force acting on a 0.100-kg particle moving in the xy-plane is given by \( F(x, y) = (x^2 \hat{x} + y^2 \hat{y}) \) N, where \( x \) and \( y \) are in meters. Suppose that due to this force, the particle moves from the origin, \( O \), to point \( S \) with coordinates (10.0 m, 10.0 m). The coordinates of points \( P \) and \( Q \) are (0 m, 10.0 m) and (10.0 m, 0 m), respectively. Determine the work performed by the force as the particle moves along each of the following paths:

a) \( O P S \)
b) \( O Q S \)
c) \( O Q S P \)

d) \( O P S Q O \)

**6.92** In the situation in Problem 6.91, suppose there is friction between the 0.100-kg particle and the xy-plane, with \( \mu_k = 0.100 \). Determine the net work done by all forces on this particle when it takes each of the following paths:

a) \( O P S \)
b) \( O Q S \)
c) \( O Q S P \)

d) \( O P S Q O \)

**MULTI-VERSION EXERCISES**

**6.93** A snowboarder starts from rest, rides 38.09 m down a snow-covered slope that makes an angle of 30.15° with the horizontal, and reaches the flat snow near the lift. How far will she travel along the flat snow if the coefficient of kinetic friction between her board and the snow is 0.02501?

**6.94** A snowboarder starts from rest, rides 30.37 m down a snow-covered slope that makes an angle of 30.35° with the horizontal, and reaches the flat snow near the lift. She travels 506.4 m along the flat snow. What is the coefficient of kinetic friction between her board and the snow?

**6.95** A snowboarder starts from rest, rides down a snow-covered slope that makes an angle of 30.57° with the horizontal, and reaches the flat snow near the lift. She travels 478.0 m along the flat snow. The coefficient of kinetic friction between her board and the snow is 0.03281. How far did she travel down the slope?

**6.96** A batter hits a pop-up straight up in the air from a height of 1.397 m. The baseball rises to a height of 7.653 m above the ground. Ignoring air resistance, what is the speed of the baseball when the catcher gloves it 1.757 m above the ground?

**6.97** A batter hits a pop-up straight up in the air from a height of 1.581 m. The baseball rises to a height \( h \) above the ground. The speed of the baseball when the catcher gloves it 1.859 m above the ground is 10.74 m/s. To what height \( h \) did the ball rise?

**6.98** A batter hits a pop-up straight up in the air from a height of 1.273 m. The baseball rises to a height of 7.777 m above the ground. The speed of the baseball when the catcher gloves it 10.73 m/s. At what height above the ground did the catcher glove the ball?

**6.99** A ball is thrown horizontally from the top of a building that is 20.27 m high with a speed of 24.89 m/s. Neglecting air resistance, at what angle with respect to the horizontal will the ball strike the ground?

**6.100** A ball is thrown horizontally from the top of a building that is 26.01 m high. The ball strikes the ground at an angle of 41.86° with respect to the horizontal. Neglecting air resistance, with what speed was the ball thrown?

**6.101** A ball is thrown horizontally from the top of a building with a speed of 25.51 m/s. The ball strikes the ground at an angle of 44.37° with respect to the horizontal. What is the height of the building?