SECTION 7-5 Polar Coordinates and Graphs

- Polar Coordinate System
- Converting from Polar to Rectangular Form, and Vice Versa
- Graphing Polar Equations
- Some Standard Polar Curves
- Application

Up until now we have used only the rectangular coordinate system. Other coordinate systems have particular advantages in certain situations. Of the many that are possible, the polar coordinate system ranks second in importance to the rectangular coordinate system and forms the subject matter for this section.

To form a polar coordinate system in a plane (see Fig. 1), start with a fixed point $O$ and call it the pole, or origin. From this point draw a half line, or ray (usually horizontal and to the right), and call this line the polar axis.

If $P$ is an arbitrary point in a plane, then associate polar coordinates $(r, \theta)$ with it as follows: Starting with the polar axis as the initial side of an angle, rotate the terminal side until it, or the extension of it through the pole, passes through the point. The $\theta$ coordinate in $(r, \theta)$ is this angle, in degree or radian measure. The angle $\theta$ is positive if the rotation is counterclockwise and negative if the rotation is clockwise. The $r$ coordinate in $(r, \theta)$ is the directed distance from the pole to the point $P$. It is positive if measured from the pole along the terminal side of $\theta$ and negative if measured along the terminal side extended through the pole.

Figure 2 illustrates a point $P$ with three different sets of polar coordinates. Study this figure carefully. The pole has polar coordinates $(0, \theta)$ for arbitrary $\theta$. For example, $(0, 0^\circ)$, $(0, \pi/3)$, and $(0, -371^\circ)$ are all coordinates of the pole.
We now see a distinct difference between rectangular and polar coordinates for the given point. For a given point in a rectangular coordinate system, there exists exactly one set of rectangular coordinates. On the other hand, in a polar coordinate system, a point has infinitely many sets of polar coordinates.

Just as graph paper with a rectangular grid is readily available for plotting rectangular coordinates, polar graph paper is available for plotting polar coordinates.

**EXAMPLE 1** Plotting Points in a Polar Coordinate System

Plot the following points in a polar coordinate system:

- (A) \( A(3, 30^\circ), \ B(-8, 180^\circ), \ C(5, -135^\circ), \ D(-10, -45^\circ) \)
- (B) \( A(5, \pi/3), \ B(-6, 5\pi/6), \ C(7, -\pi/2), \ D(-4, -\pi/6) \)

**Solutions**

(A) [Diagram showing points plotted in polar coordinates]

(B) [Diagram showing points plotted in polar coordinates]

**Matched Problem 1** Plot the following points in a polar coordinate system:

- (A) \( A(8, 45^\circ), \ B(-5, 150^\circ), \ C(4, -210^\circ), \ D(-6, -90^\circ) \)
- (B) \( A(9, \pi/6), \ B(-3, -\pi), \ C(-7, 7\pi/4), \ D(5, -5\pi/6) \)

**EXPLORE-DISCUSS 1** A point in a polar coordinate system has coordinates \((5, 30^\circ)\). How many other polar coordinates does the point have for \(\theta\) restricted to \(-360^\circ \leq \theta \leq 360^\circ\)? Find the other coordinates of the point, and explain how they are found.

- **Converting from Polar to Rectangular Form, and Vice Versa**

Often, it is necessary to transform coordinates or equations in rectangular form to polar form, or vice versa. The following polar–rectangular relationships are useful in this regard:
Polar–Rectangular Relationships

We have the following relationships between rectangular coordinates \((x, y)\) and polar coordinates \((r, \theta)\):

\[
\begin{align*}
    r^2 & = x^2 + y^2 \\
    \sin \theta & = \frac{y}{r} \quad \text{or} \quad y = r \sin \theta \\
    \cos \theta & = \frac{x}{r} \quad \text{or} \quad x = r \cos \theta \\
    \tan \theta & = \frac{y}{x}
\end{align*}
\]

[Note: The signs of \(x\) and \(y\) determine the quadrant for \(\theta\). The angle \(\theta\) is chosen so that \(-\pi < \theta \leq \pi\) or \(-180^\circ < \theta \leq 180^\circ\), unless directed otherwise.]

Many calculators can automatically convert rectangular coordinates to polar form, and vice versa. (Read your manual for your particular calculator.) Example 2 illustrates calculator conversions in both directions.

**EXAMPLE 2** Converting from Polar to Rectangular Form, and Vice Versa

(A) Convert the polar coordinates \((-4, 1.077)\) to rectangular coordinates to three decimal places.

(B) Convert the rectangular coordinates \((-3.207, -5.719)\) to polar coordinates with \(\theta\) in degree measure, \(-180^\circ < \theta \leq 180^\circ\), and \(r \geq 0\).

**Solution**

(A) Use a calculator set in radian mode.

\((r, \theta) = (-4, 1.077)\)

\[
\begin{align*}
x & = r \cos \theta = (-4) \cos 1.077 = -1.896 \\
y & = r \sin \theta = (-4) \sin 1.077 = -3.522
\end{align*}
\]

Rectangular coordinates are \((-1.896, -3.522)\)

Figure 3 shows the same conversion done in a graphing utility with a built-in conversion routine.

(B) Use a calculator set in degree mode.

\((x, y) = (-3.207, -5.719)\)

\[
\begin{align*}
r & = \sqrt{x^2 + y^2} = \sqrt{(-3.207)^2 + (-5.719)^2} = 6.557 \\
\tan \theta & = \frac{y}{x} = \frac{-5.719}{-3.207}
\end{align*}
\]
Additional Topics in Trigonometry

θ is a third-quadrant angle and is to be chosen so that \(-180° < \theta \leq 180°\).

\[
\theta = -180° + \tan^{-1}\left(\frac{-5.719}{-3.207}\right) = -119.28°
\]

Polar coordinates are \((6.557, -119.28°)\).

Figure 4 shows the same conversion done in a graphing utility with a built-in conversion routine.

Matched Problem 2

(A) Convert the polar coordinates \((8.677, -1.385)\) to rectangular coordinates to three decimal places.

(B) Convert the rectangular coordinates \((-6.434, 4.023)\) to polar coordinates with \(\theta\) in degree measure, \(-180° < \theta \leq 180°\), and \(r \geq 0\).

Generally, a more important use of the polar–rectangular relationships is in the conversion of equations in rectangular form to polar form, and vice versa.

**EXAMPLE 3** Converting an Equation from Rectangular Form to Polar Form

Change \(x^2 + y^2 - 4y = 0\) to polar form.

**Solution** Use \(r^2 = x^2 + y^2\) and \(y = r \sin \theta\).

\[
x^2 + y^2 - 4y = 0
\]
\[
r^2 - 4r \sin \theta = 0
\]
\[
r(r - 4 \sin \theta) = 0
\]
\[
r = 0 \quad \text{or} \quad r - 4 \sin \theta = 0
\]

The graph of \(r = 0\) is the pole. Because the pole is included in the graph of \(r - 4 \sin \theta = 0\) (let \(\theta = 0\)), we can discard \(r = 0\) and keep only

\[
r - 4 \sin \theta = 0
\]

or

\[
r = 4 \sin \theta \quad \text{The polar form of } x^2 + y^2 - 4y = 0
\]

**Matched Problem 3** Change \(x^2 + y^2 - 6x = 0\) to polar form.
EXAMPLE 4 Converting an Equation from Polar Form to Rectangular Form

Change \( r = -3 \cos \theta \) to rectangular form.

Solution
The transformation of this equation as it stands into rectangular form is fairly difficult. With a little trick, however, it becomes easy. We multiply both sides by \( r \), which simply adds the pole to the graph. But the pole is already part of the graph of \( r = -3 \cos \theta \) (let \( \theta = \pi/2 \)), so we haven’t actually changed anything.

\[
\begin{align*}
  r &= -3 \cos \theta \\
  r^2 &= -3r \cos \theta \\
  x^2 + y^2 &= -3x \\
  r^2 &= x^2 + y^2 \text{ and } r \cos \theta = x \\
  x^2 + y^2 + 3x &= 0
\end{align*}
\]

Matched Problem 4 Change \( r + 2 \sin \theta = 0 \) to rectangular form.

• Graphing Polar Equations
We now turn to graphing polar equations. The graph of a polar equation, such as \( r = 3\theta \) or \( r = 6 \cos \theta \), in a polar coordinate system is the set of all points having coordinates that satisfy the polar equation. Certain curves have simpler representations in polar coordinates, and other curves have simpler representations in rectangular coordinates.

To establish fundamentals in graphing polar equations, we start with a point-by-point graph. We then consider a more rapid way of making rough sketches of certain polar curves. And, finally, we show how polar curves are graphed in a graphing utility.

To plot a polar equation using point-by-point plotting, just as in rectangular coordinates, make a table of values that satisfy the equation, plot these points, then join them with a smooth curve. Example 5 illustrates the process.

EXAMPLE 5 Point-by-Point Plotting

(A) Graph \( r = 8 \cos \theta \) with \( \theta \) in radians.
(B) Convert the polar equation in part A to rectangular form, and identify the graph.

Solution
(A) We construct a table using multiples of \( \pi/6 \), plot these points, then join the points with a smooth curve (Fig. 5):
Multiply both sides by \( r \).

Change to rectangular form.

Complete the square on the left side.

Standard equation of a circle

The graph in part A is a circle with center at \((4, 0)\) and radius 4 (see Section 2-1).

**Matched Problem 5**

(A) Graph \( r = 8 \sin \theta \) with \( \theta \) in degrees.

(B) Convert the polar equation in part A to rectangular form, and identify the graph.

If only a rough sketch of a polar equation involving \( \sin \theta \) or \( \cos \theta \) is desired, you can speed up the point-by-point graphing process by taking advantage of the uniform variation of \( \sin \theta \) and \( \cos \theta \) as \( \theta \) moves around a unit circle. This process is referred to as **rapid polar sketching**. It is convenient to visualize Figure 6 in the process. With a little practice most of the table work in rapid sketching can be done mentally, and a rough sketch can be made directly from the equation.
**EXAMPLE 6** Rapid Polar Sketching

Sketch $r = 4 + 4 \cos \theta$ using rapid sketching techniques with $\theta$ in radians.

**Solution** We set up a table that indicates how $r$ varies as we let $\theta$ vary through each set of quadrant values:

<table>
<thead>
<tr>
<th>$\theta$ varies from</th>
<th>$\cos \theta$ varies from</th>
<th>$4 \cos \theta$ varies from</th>
<th>$r = 4 + 4 \cos \theta$ varies from</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to $\pi/2$</td>
<td>1 to 0</td>
<td>4 to 0</td>
<td>8 to 4</td>
</tr>
<tr>
<td>$\pi/2$ to $\pi$</td>
<td>0 to $-1$</td>
<td>0 to $-4$</td>
<td>4 to 0</td>
</tr>
<tr>
<td>$\pi$ to $3\pi/2$</td>
<td>$-1$ to 0</td>
<td>$-4$ to 0</td>
<td>0 to 4</td>
</tr>
<tr>
<td>$3\pi/2$ to $2\pi$</td>
<td>0 to 1</td>
<td>0 to 4</td>
<td>4 to 8</td>
</tr>
</tbody>
</table>

Notice that as $\theta$ increases from 0 to $\pi/2$, $\cos \theta$ decreases from 1 to 0, $4 \cos \theta$ decreases from 4 to 0, and $r = 4 + 4 \cos \theta$ decreases from 8 to 4, and so on. Sketching these values, we obtain the graph in Figure 7, called a **cardioid**.

**FIGURE 7**

---

**Matched Problem 6** Sketch $r = 5 + 5 \sin \theta$ using rapid sketching techniques with $\theta$ in radians.

---

**EXAMPLE 7** Rapid Polar Sketching

Sketch $r = 8 \cos 2\theta$ with $\theta$ in radians.

**Solution** Start by letting $2\theta$ (instead of $\theta$) range through each set of quadrant values. That is, start with values for $2\theta$ in the second column of the table, fill in the table at the top of the next page, and then fill in the first column for $\theta$.
As \(2\theta\) increases from 0 to \(\pi/2\), \(\theta\) increases from 0 to \(\pi/4\), and \(r\) decreases from 8 to 0. As \(2\theta\) increases from \(\pi/2\) to \(\pi\), \(\theta\) increases from \(\pi/4\) to \(\pi/2\), and \(r\) decreases from 0 to \(-8\), and so on. Continue until the graph starts to repeat. Plotting the values, we obtain the graph in Figure 8, called a four-leafed rose:

**Matched Problem 7** Sketch \(r = 6 \sin 2\theta\) with \(\theta\) in radians.

We now turn to graphing polar equations in a graphing utility. Example 8 illustrates the process.
EXAMPLE 8  Graphing in a Graphing Utility

Graph each of the following polar equations in a graphing utility (parts B and C are from Examples 6 and 7).

(A) \( r = 3\theta, \quad 0 \leq \theta \leq 3\pi/2 \) (Archimedes’ spiral)
(B) \( r = 4 + 4 \cos \theta \) (cardioid)
(C) \( r = 8 \cos 2\theta \) (four-leafed rose)

Solution

Set the graphing utility in polar mode, and select polar coordinates and radian measure. Adjust window values to accommodate the whole graph. A squared graph is often desirable in showing the true shape of the curve and is used here. Many graphing utilities, including the one used here, do not show a polar grid. When using TRACE, many graphing utilities offer a choice between polar coordinates and rectangular coordinates for points on the polar curve. The graphs of the polar equations above are shown in Figure 9.

Matched Problem 8

Graph each of the following polar equations in a graphing utility.

(A) \( r = 2\theta, \quad 0 \leq \theta \leq 2\pi \)
(B) \( r = 5 + 5 \sin \theta \)  
(C) \( r = 6 \sin 2\theta \)

EXPLORE-DISCUSS 2

(A) Graph \( r1 = 10 \sin \theta \) and \( r2 = 10 \cos \theta \) in the same viewing window. Use TRACE on \( r1 \), and estimate the polar coordinates where the two graphs intersect. Do the same thing for \( r2 \). Which intersection point appears to have the same polar coordinates on each curve and consequently represents a simultaneous solution to both equations? Which intersection point appears to have different polar coordinates on each curve and consequently does not represent a simultaneous solution? Solve the system for \( r \) and \( \theta \).

(B) Explain how rectangular coordinate systems differ from polar coordinate systems relative to intersection points and simultaneous solutions of systems of equations in the respective systems.
In a rectangular coordinate system the simplest types of equations to graph are found by setting the rectangular variables \( x \) and \( y \) equal to constants:

\[
x = a \quad \text{and} \quad y = b
\]

The graphs are straight lines: The graph of \( x = a \) is a vertical line, and the graph of \( y = b \) is a horizontal line. A glance at Table 1 on the next page shows that horizontal and vertical lines do not have simple equations in polar coordinates.

Two of the simplest types of polar equations to graph in a polar coordinate system are found by setting the polar variables \( r \) and \( \theta \) equal to constants:

\[
r = a \quad \text{and} \quad \theta = b
\]

Figure 10 illustrates the graphs of \( \theta = \pi/4 \) and \( r = 5 \).

**FIGURE 10**

![Figure 10](image)

Table 1 illustrates a number of standard polar graphs and their equations. Polar graphing is often made easier if you have some idea of the final form.

**Application**

Serious sailboat racers make polar plots of boat speed at various angles to the wind with various sail combinations at different wind speeds. With many polar plots for different sizes and types of sails at different wind speeds, they are able to accurately choose a sail for the optimum performance for different points of sail relative to any given wind strength. Figure 11 illustrates one such polar plot, where the maximum speed appears to be about 7.5 knots at 105° off the wind (with spinnaker sail set).

**FIGURE 11** Polar diagram showing optimum sailing speed at different sailing angles to the wind.
### TABLE 1 Standard Polar Graphs

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = a$</td>
<td>Line through origin: $\theta = a$</td>
</tr>
<tr>
<td>$r = a \cos \theta$</td>
<td>Circle: $r = a \cos \theta$</td>
</tr>
<tr>
<td>$r = a \sin \theta$</td>
<td>Circle: $r = a \sin \theta$</td>
</tr>
<tr>
<td>$r = a + a \cos \theta$</td>
<td>Cardioid: $r = a + a \cos \theta$</td>
</tr>
<tr>
<td>$r = a + a \sin \theta$</td>
<td>Cardioid: $r = a + a \sin \theta$</td>
</tr>
<tr>
<td>$r = a \cos 3\theta$</td>
<td>Three-leafed rose: $r = a \cos 3\theta$</td>
</tr>
<tr>
<td>$r = a \cos 2\theta$</td>
<td>Four-leafed rose: $r = a \cos 2\theta$</td>
</tr>
<tr>
<td>$r^2 = a^2 \cos 2\theta$</td>
<td>Lemniscate: $r^2 = a^2 \cos 2\theta$</td>
</tr>
<tr>
<td>$r = \alpha \theta$, $\alpha &gt; 0$</td>
<td>Archimedes' spiral: $r = \alpha \theta$, $\alpha &gt; 0$</td>
</tr>
</tbody>
</table>

---

**Answers to Matched Problems**

1. (A)

---

**Table 1 Standard Polar Graphs**

- **Line through origin:** $\theta = a$
- **Vertical line:** $r = a / \cos \theta = a \sec \theta$
- **Horizontal line:** $r = a / \sin \theta = a \csc \theta$
- **Circle:** $r = a$
- **Circle:** $r = a \cos \theta$
- **Circle:** $r = a \sin \theta$
- **Cardioid:** $r = a + a \cos \theta$
- **Cardioid:** $r = a + a \sin \theta$
- **Three-leafed rose:** $r = a \cos 3\theta$
- **Four-leafed rose:** $r = a \cos 2\theta$
- **Lemniscate:** $r^2 = a^2 \cos 2\theta$
- **Archimedes' spiral:** $r = \alpha \theta$, $\alpha > 0$

**Answers to Matched Problems**

1. (A)
2. (A) $(1.603, -8.528)$  (B) $(7.588, 147.98^\circ)$
3. $r = 6 \cos \theta$
4. $x^2 + y^2 + 2y = 0$
5. (A)

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>0.0</td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>4.0</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>6.9</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>8.0</td>
</tr>
<tr>
<td>$120^\circ$</td>
<td>6.9</td>
</tr>
<tr>
<td>$150^\circ$</td>
<td>4.0</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Graph Repeats

(B) $x^2 + (y - 4)^2 = 4^2$
A circle with center at $(0, 4)$ and radius 4.

6. $r = 5 + 5 \sin \theta$
Cardioid

7. $r = 6 \sin 2\theta$
Four-leafed rose
Graph Problems 11 and 12 in a polar coordinate system.

8. (A) \( r = 20, 0 \leq \theta \leq 2\pi \)
8. (B) \( r = 5 + 5 \sin \theta \)

- \( \theta = 0 \)
- \( r = 6 \sin 20 \)

EXERCISE 7-5

A

Plot A, B, and C in Problems 1–8 in a polar coordinate system.

1. \( A(4, 0^\circ), B(7, 180^\circ), C(9, 45^\circ) \)
2. \( A(8, 0^\circ), B(5, 90^\circ), C(6, 30^\circ) \)
3. \( A(-4, 0^\circ), B(-7, 180^\circ), C(-9, 45^\circ) \)
4. \( A(-8, 0^\circ), B(-5, 90^\circ), C(-6, 30^\circ) \)
5. \( A(8, -\pi/3), B(4, -\pi/4), C(10, -\pi/6) \)
6. \( A(6, -\pi/6), B(5, -\pi/2), C(8, -\pi/4) \)
7. \( A(-6, -\pi/6), B(-5, -\pi/2), C(-8, -\pi/4) \)
8. \( A(-6, -\pi/2), B(-5, -\pi/3), C(-8, -\pi/4) \)

9. A point in a polar coordinate system has coordinates \((-5, 3\pi/4)\). Find all other polar coordinates for the point, \(-2\pi \leq \theta \leq 2\pi\), and verbally describe how the coordinates are associated with the point.

10. A point in a polar coordinate system has coordinates \((6, -30^\circ)\). Find all other polar coordinates for the point, \(-360^\circ \leq \theta \leq 360^\circ\), and verbally describe how the coordinates are associated with the point.

Graph Problems 11 and 12 in a polar coordinate system using point-by-point plotting and the special values 0, \(\pi/6\), \(\pi/4\), \(\pi/3\), \(\pi/2\), 2\(\pi/3\), 3\(\pi/4\), 5\(\pi/6\), and \(\pi\) for \(\theta\).

Verify the graphs of Problems 11 and 12 on a graphing utility.

11. \( r = 10 \sin \theta \)
12. \( r = 10 \cos \theta \)

Graph Problems 13–16 in a polar coordinate system.

13. \( r = 8 \)
14. \( r = 5 \)
15. \( \theta = \pi/3 \)
16. \( \theta = \pi/6 \)

In Problems 17–22, convert the polar coordinates to rectangular coordinates to three decimal places.

17. \((6, \pi/6)\)
18. \((7, 2\pi/3)\)
19. \((-2, 7\pi/8)\)
20. \((3, -3\pi/7)\)
21. \((-4.233, -2.084)\)
22. \((-9.028, -0.663)\)

B

In Problems 23–28, convert the rectangular coordinates to polar coordinates with \(\theta\) in degree measure, \(-180^\circ < \theta \leq 180^\circ\), and \(R \geq 0\).

23. \((-8, 0)\)
24. \((0, -5)\)
25. \((-5, -5)\)
26. \((1, -\sqrt{3})\)
27. \((9.79, 5.13)\)
28. \((-4.26, 31.1)\)

In Problems 29–38, use rapid graphing techniques to sketch the graph of each polar equation.

Verify the graphs of Problems 29–38 on a graphing utility.

29. \( r = 4 \sin \theta \)
30. \( r = 4 \cos \theta \)
31. \( r = 10 \sin 20 \)
32. \( r = 8 \cos 20 \)
33. \( r = 5 \cos 30 \)
34. \( r = 6 \sin 30 \)
35. \( r = 2 + 2 \sin \theta \)
36. \( r = 3 + 3 \cos \theta \)
37. \( r = 2 + 4 \sin \theta \)
38. \( r = 2 + 4 \cos \theta \)
Problems 39–44 are exploratory problems requiring the use of a graphing utility.

39. Graph each polar equation in its own viewing window:
   \( r = 2 + 2 \sin \theta \), \( r = 4 + 2 \sin \theta \), \( r = 2 + 4 \sin \theta \)

40. Graph each polar equation in its own viewing window:
   \( r = 2 + 2 \cos \theta \), \( r = 4 + 2 \cos \theta \), \( r = 2 + 4 \cos \theta \)

41. (A) Graph each polar equation in its own viewing window:
    \( r = 4 \sin \theta \), \( r = 4 \sin 3\theta \), \( r = 4 \sin 5\theta \)
   (B) What would you guess to be the number of leaves for \( r = 4 \sin 7\theta \)?
   (C) What would you guess to be the number of leaves for \( r = a \sin n\theta \), \( a > 0 \) and \( n \) odd?

42. (A) Graph each polar equation in its own viewing window:
    \( r = 4 \cos \theta \), \( r = 4 \cos 3\theta \), \( r = 4 \cos 5\theta \)
   (B) What would you guess to be the number of leaves for \( r = 4 \cos 7\theta \)?
   (C) What would you guess to be the number of leaves for \( r = a \cos n\theta \), \( a > 0 \) and \( n \) odd?

43. (A) Graph each polar equation in its own viewing window:
    \( r = 4 \sin 2\theta \), \( r = 4 \sin 4\theta \), \( r = 4 \sin 6\theta \)
   (B) What would you guess to be the number of leaves for \( r = 4 \sin 8\theta \)?
   (C) What would you guess to be the number of leaves for \( r = a \sin n\theta \), \( a > 0 \) and \( n \) even?

44. (A) Graph each polar equation in its own viewing window:
    \( r = 4 \cos 2\theta \), \( r = 4 \cos 4\theta \), \( r = 4 \cos 6\theta \)
   (B) What would you guess to be the number of leaves for \( r = 4 \cos 8\theta \)?
   (C) What would you guess to be the number of leaves for \( r = a \cos n\theta \), \( a > 0 \) and \( n \) even?

In Problems 45–50, change each rectangular equation to polar form. Identify the graph as a line, circle, etc.

45. \( x^2 + y^2 = 4 \)
46. \( x + y = 0 \)
47. \( x - \sqrt{3}y = 0 \)
48. \( x^2 + y^2 + 8x = 0 \)
49. \( 5y = x^2 \)
50. \( x^2 - y^2 = 1 \)

In Problems 51–56, change each polar equation to rectangular form. Identify the graph as a line, circle, etc.

51. \( r = 3 \cos \theta \)
52. \( \theta + \pi/3 = 0 \)
53. \( r(4 \sin \theta - \cos \theta) = 1 \)
54. \( r + 5 \sin \theta = 0 \)
55. \( r(2 + \cos \theta) = 1 \)
56. \( r(1 + \cos \theta) = 1 \)

In Problems 59–62, graph each system of equations on the same set of polar coordinate axes. Then solve the system simultaneously. [Note: Any solution \((r, \theta, \psi)\) to the system must satisfy each equation in the system and thus identifies a point of intersection of the two graphs. However, there may be other points of intersection of the two graphs that do not have any coordinates that satisfy both equations. This represents a major difference between the rectangular coordinate system and the polar coordinate system.]

59. \( r = 4 \cos \theta \)
60. \( r = 2 \cos \theta \)
   \( r = -4 \sin \theta \)
   \( 0 \leq \theta \leq \pi \)
   \( 0 \leq \theta \leq \pi \)

61. \( r = 6 \cos \theta \)
62. \( r = 8 \sin \theta \)
   \( r = 6 \sin \theta \)
   \( r = 6 \cos \theta \)
   \( 0^\circ \leq \theta \leq 360^\circ \)
   \( 0^\circ \leq \theta \leq 360^\circ \)

63. Analytic Geometry. A distance formula for the distance between two points in a polar coordinate system follows directly from the law of cosines:
   \[
   d^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)
   \]
   \[
   d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}
   \]
   Find the distance (to three decimal places) between the two points \( P_1(4, \pi/4) \) and \( P_2(1, \pi/2) \).

64. Analytic Geometry. Refer to Problem 63. Find the distance (to three decimal places) between the two points \( P_1(2, 30^\circ) \) and \( P_2(3, 60^\circ) \).

Problems 65–66 refer to the polar diagram in the figure. Polar diagrams of this type are used extensively by serious sailboat racers, and this polar diagram represents speeds in knots of a high-performance sailboat sailing at various angles to a wind blowing at 20 knots.
65. Sailboat Racing. Referring to the figure, estimate to the nearest knot the speed of the sailboat sailing at the following angles to the wind: 30°, 75°, 135°, and 180°.

66. Sailboat Racing. Referring to the figure, estimate to the nearest knot the speed of the sailboat sailing at the following angles to the wind: 45°, 90°, 120°, and 150°.

67. Conic Sections. Using a graphing utility, graph the equation

\[ r = \frac{8}{1 - e \cos \theta} \]

for the following values of \( e \) (called the eccentricity of the conic), and identify each curve as a hyperbola, ellipse, or a parabola.

(A) \( e = 0.4 \)  (B) \( e = 1 \)  (C) \( e = 2 \)

(It is instructive to explore the graph for other positive values of \( e \).)

68. Conic Sections. Using a graphing utility, graph the equation

\[ r = \frac{3.442 \times 10^7}{1 - 0.206 \cos \theta} \]

where \( r \) is measured in miles and the sun is at the pole. Graph the orbit. Use TRACE to find the distance from Mercury to the sun at aphelion (greatest distance from the sun) and at perihelion (shortest distance from the sun).

69. Astronomy. (A) The planet Mercury travels around the sun in an elliptical orbit given approximately by

\[ r = \frac{8}{1 - e \cos \theta} \]

for the following values of \( e \), and identify each curve as a hyperbola, ellipse, or a parabola.

(A) \( e = 0.6 \)  (B) \( e = 1 \)  (C) \( e = 2 \)

(B) Johannes Kepler (1571–1630) showed that a line joining a planet to the sun sweeps out equal areas in space in equal intervals in time (see figure). Use this information to determine whether a planet travels faster or slower at aphelion than at perihelion. Explain your answer.

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**SECTION 7-6 Complex Numbers in Rectangular and Polar Forms**

- Rectangular Form
- Polar Form
- Multiplication and Division in Polar Form
- Historical Note

Utilizing polar concepts studied in the last section, we now show how complex numbers can be written in polar form, which can be very useful in many applications. A brief review of Section 1-5 on complex numbers should prove helpful before proceeding further.